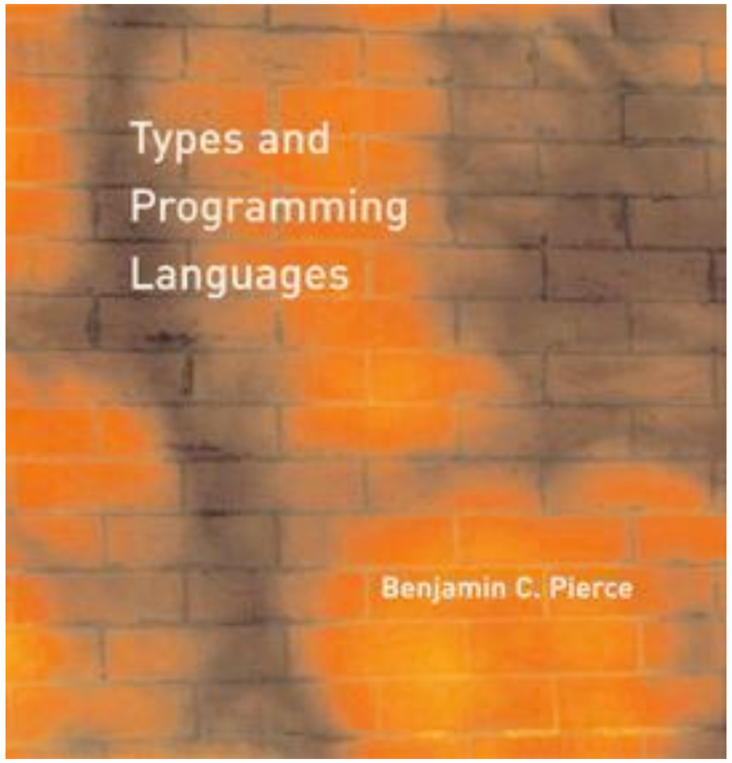
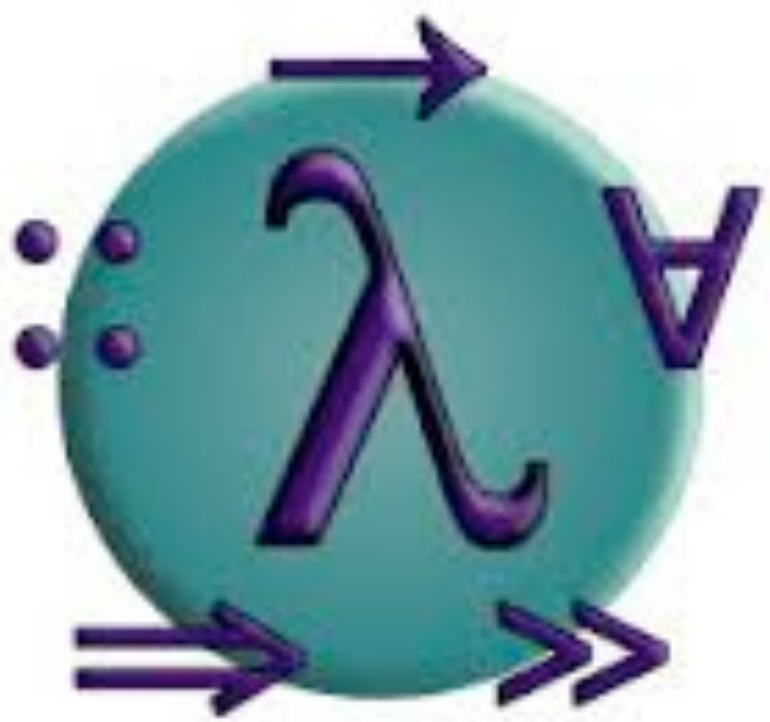
Programming Languages
Fall 2014





Lecture 7: Simple Types and Simply-Typed Lambda Calculus

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Types

```
t ::=
        true
        false
        if t then t else t
        pair t t
        fst t
        snd t
v ::=
        true
        false
        pair v v
```

stuck terms?

how to fix it?

S	$\longrightarrow$	NP VP
S	$\longrightarrow$	S conj S
NP	$\longrightarrow$	Noun
NP	$\longrightarrow$	Det Noun
NP	$\longrightarrow$	NP PP
NP	$\longrightarrow$	NP conj NP
VP	$\longrightarrow$	Verb
VP	$\longrightarrow$	Verb NP
VP	$\longrightarrow$	Verb NP NP
VP	$\longrightarrow$	VP PP
PP	$\longrightarrow$	P NP

Plan

- **First** For today, we'll go back to the simple language of arithmetic and boolean expressions and show how to equip it with a (very simple) type system
- The key property of this type system will be *soundness*: Well-typed programs do not get stuck
- **Next** Next time, we'll develop a simple type system for the lambda-calculus
- We'll spend a good part of the rest of the semester adding features to this type system

Outline

1. begin with a set of terms, a set of values, and an evaluation relation
2. define a set of *types* classifying values according to their "shapes"
3. define a *typing relation* `t : T` that classifies terms according to the shape of the values that result from evaluating them
4. check that the typing relation is *sound* in the sense that,
   4.1 if `t : T` and $\texttt{t} \longrightarrow^{*} \texttt{v}$, then `v : T`
   4.2 if `t : T`, then evaluation of `t` will not get stuck

Review: Arithmetic Expressions – Syntax

`t ::=`		*terms*
	`true`	*constant true*
	`false`	*constant false*
	`if t then t else t`	*conditional*
	`0`	*constant zero*
	`succ t`	*successor*
	`pred t`	*predecessor*
	`iszero t`	*zero test*
`v ::=`		*values*
	`true`	*true value*
	`false`	*false value*
	`nv`	*numeric value*
`nv ::=`		*numeric values*
	`0`	*zero value*
	`succ nv`	*successor value*

Evaluation Rules

$$\texttt{if true then } t_2 \texttt{ else } t_3 \longrightarrow t_2 \qquad \text{(E-IfTrue)}$$

$$\texttt{if false then } t_2 \texttt{ else } t_3 \longrightarrow t_3 \qquad \text{(E-IfFalse)}$$

$$\frac{t_1 \longrightarrow t_1'}{\texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3 \longrightarrow \texttt{if } t_1' \texttt{ then } t_2 \texttt{ else } t_3} \qquad \text{(E-If)}$$

$$\frac{\texttt{t}_1 \longrightarrow \texttt{t}_1'}{\texttt{succ t}_1 \longrightarrow \texttt{succ t}_1'} \qquad \text{(E-Succ)}$$

$$\texttt{pred 0} \longrightarrow \texttt{0} \qquad \text{(E-PredZero)}$$

$$\texttt{pred (succ nv}_1\texttt{)} \longrightarrow \texttt{nv}_1 \qquad \text{(E-PredSucc)}$$

$$\frac{\texttt{t}_1 \longrightarrow \texttt{t}_1'}{\texttt{pred t}_1 \longrightarrow \texttt{pred t}_1'} \qquad \text{(E-Pred)}$$

$$\texttt{iszero 0} \longrightarrow \texttt{true} \qquad \text{(E-IszeroZero)}$$

$$\texttt{iszero (succ nv}_1\texttt{)} \longrightarrow \texttt{false} \qquad \text{(E-IszeroSucc)}$$

$$\frac{\texttt{t}_1 \longrightarrow \texttt{t}_1'}{\texttt{iszero t}_1 \longrightarrow \texttt{iszero t}_1'} \qquad \text{(E-IsZero)}$$

Types

In this language, values have two possible "shapes": they are either booleans or numbers.

`T`	::=	*types*
	`Bool`	*type of booleans*
	`Nat`	*type of numbers*

Typing Rules

$$\texttt{true} : \texttt{Bool} \qquad \text{(T-TRUE)}$$

$$\texttt{false} : \texttt{Bool} \qquad \text{(T-FALSE)}$$

$$\frac{\texttt{t}_1 : \texttt{Bool} \qquad \texttt{t}_2 : \texttt{T} \qquad \texttt{t}_3 : \texttt{T}}{\texttt{if } \texttt{t}_1 \texttt{ then } \texttt{t}_2 \texttt{ else } \texttt{t}_3 : \texttt{T}} \qquad \text{(T-IF)}$$

$$\texttt{0} : \texttt{Nat} \qquad \text{(T-ZERO)}$$

$$\frac{\texttt{t}_1 : \texttt{Nat}}{\texttt{succ } \texttt{t}_1 : \texttt{Nat}} \qquad \text{(T-SUCC)}$$

$$\frac{\texttt{t}_1 : \texttt{Nat}}{\texttt{pred } \texttt{t}_1 : \texttt{Nat}} \qquad \text{(T-PRED)}$$

$$\frac{\texttt{t}_1 : \texttt{Nat}}{\texttt{iszero } \texttt{t}_1 : \texttt{Bool}} \qquad \text{(T-ISZERO)}$$

Typing Derivations

Every pair $(\mathtt{t}, \mathtt{T})$ in the typing relation can be justified by a *derivation tree* built from instances of the inference rules.

$$\dfrac{\dfrac{\dfrac{}{\mathtt{0 : Nat}}\ \textsc{T-Zero}}{\mathtt{iszero\ 0 : Bool}}\ \textsc{T-IsZero} \qquad \dfrac{}{\mathtt{0 : Nat}}\ \textsc{T-Zero} \qquad \dfrac{\dfrac{}{\mathtt{0 : Nat}}\ \textsc{T-Zero}}{\mathtt{pred\ 0 : Nat}}\ \textsc{T-Pred}}{\mathtt{if\ iszero\ 0\ then\ 0\ else\ pred\ 0 : Nat}}\ \textsc{T-If}$$

Proofs of properties about the typing relation often proceed by induction on typing derivations.

Imprecision of Typing

Like other static program analyses, type systems are generally *imprecise*: they do not predict exactly what kind of value will be returned by every program, but just a conservative (safe) approximation.

$$\frac{\texttt{t}_1 : \texttt{Bool} \qquad \texttt{t}_2 : \texttt{T} \qquad \texttt{t}_3 : \texttt{T}}{\texttt{if } \texttt{t}_1 \texttt{ then } \texttt{t}_2 \texttt{ else } \texttt{t}_3 : \texttt{T}} \quad \text{(T-If)}$$

Using this rule, we cannot assign a type to

```
if true then 0 else false
```

even though this term will certainly evaluate to a number.

Properties of the Typing Relation

Type Safety

The safety (or soundness) of this type system can be expressed by two properties:

1. *Progress:* A well-typed term is not stuck

   If $t : T$*, then either* t *is a value or else* $t \longrightarrow t'$ *for some* t'*.*

2. *Preservation:* Types are preserved by one-step evaluation

   If $t : T$ *and* $t \longrightarrow t'$*, then* $t' : T$*.*

Inversion

Lemma:

1. If $\texttt{true} : \texttt{R}$, then $\texttt{R} = \texttt{Bool}$.
2. If $\texttt{false} : \texttt{R}$, then $\texttt{R} = \texttt{Bool}$.
3. If $\texttt{if}\ \texttt{t}_1\ \texttt{then}\ \texttt{t}_2\ \texttt{else}\ \texttt{t}_3 : \texttt{R}$, then $\texttt{t}_1 : \texttt{Bool}$, $\texttt{t}_2 : \texttt{R}$, and $\texttt{t}_3 : \texttt{R}$.
4. If $\texttt{0} : \texttt{R}$, then $\texttt{R} = \texttt{Nat}$.
5. If $\texttt{succ}\ \texttt{t}_1 : \texttt{R}$, then $\texttt{R} = \texttt{Nat}$ and $\texttt{t}_1 : \texttt{Nat}$.
6. If $\texttt{pred}\ \texttt{t}_1 : \texttt{R}$, then $\texttt{R} = \texttt{Nat}$ and $\texttt{t}_1 : \texttt{Nat}$.
7. If $\texttt{iszero}\ \texttt{t}_1 : \texttt{R}$, then $\texttt{R} = \texttt{Bool}$ and $\texttt{t}_1 : \texttt{Nat}$.

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Proof: ...

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6. If $\texttt{pred } \texttt{t}_1 : \texttt{R}$, then $\texttt{R} = \texttt{Nat}$ and $\texttt{t}_1 : \texttt{Nat}$.
7. If $\texttt{iszero } \texttt{t}_1 : \texttt{R}$, then $\texttt{R} = \texttt{Bool}$ and $\texttt{t}_1 : \texttt{Nat}$.

Proof: ...

This leads directly to a recursive algorithm for calculating the type of a term...

Typechecking Algorithm

```
typeof(t) = if t = true then Bool
            else if t = false then Bool
            else if t = if t1 then t2 else t3 then
              let T1 = typeof(t1) in
              let T2 = typeof(t2) in
              let T3 = typeof(t3) in
              if T1 = Bool and T2=T3 then T2
              else "not typable"
            else if t = 0 then Nat
            else if t = succ t1 then
              let T1 = typeof(t1) in
              if T1 = Nat then Nat else "not typable"
            else if t = pred t1 then
              let T1 = typeof(t1) in
              if T1 = Nat then Nat else "not typable"
            else if t = iszero t1 then
              let T1 = typeof(t1) in
              if T1 = Nat then Bool else "not typable"
```

Canonical Forms

Lemma:

1. If `v` is a value of type `Bool`, then `v` is either `true` or `false`.
2. If `v` is a value of type `Nat`, then `v` is a numeric value.

Proof:

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Proof: Recall the syntax of values:

`v` ::=		*values*
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	`nv`	*numeric value*
`nv` ::=		*numeric values*
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For part 1,

Canonical Forms

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For part 1, if `v` is `true` or `false`, the result is immediate.

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For part 1, if `v` is `true` or `false`, the result is immediate. But `v` cannot be `0` or `succ nv`, since the inversion lemma tells us that `v` would then have type `Nat`, not `Bool`.

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For part 1, if `v` is `true` or `false`, the result is immediate. But `v` cannot be `0` or `succ nv`, since the inversion lemma tells us that `v` would then have type `Nat`, not `Bool`. Part 2 is similar.

Progress

Theorem: Suppose $\mathtt{t}$ is a well-typed term (that is, $\mathtt{t} : \mathtt{T}$ for some $\mathtt{T}$). Then either $\mathtt{t}$ is a value or else there is some $\mathtt{t}'$ with $\mathtt{t} \longrightarrow \mathtt{t}'$.

Progress

Theorem: Suppose $\texttt{t}$ is a well-typed term (that is, $\texttt{t : T}$ for some $\texttt{T}$). Then either $\texttt{t}$ is a value or else there is some $\texttt{t}'$ with $\texttt{t} \longrightarrow \texttt{t}'$.

Proof:

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Proof: By induction on a derivation of $\mathtt{t : T}$.

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The T-True, T-False, and T-Zero cases are immediate, since $\mathtt{t}$ in these cases is a value.

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Case T-If: $\mathtt{t} = \mathtt{if}\ \mathtt{t_1}\ \mathtt{then}\ \mathtt{t_2}\ \mathtt{else}\ \mathtt{t_3}$

$\mathtt{t_1 : Bool} \qquad \mathtt{t_2 : T} \qquad \mathtt{t_3 : T}$

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Theorem: Suppose $\mathtt{t}$ is a well-typed term (that is, $\mathtt{t} : \mathtt{T}$ for some $\mathtt{T}$). Then either $\mathtt{t}$ is a value or else there is some $\mathtt{t}'$ with $\mathtt{t} \longrightarrow \mathtt{t}'$.

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The T-True, T-False, and T-Zero cases are immediate, since $\mathtt{t}$ in these cases is a value.

Case T-If: $\mathtt{t} = \mathtt{if}\ \mathtt{t}_1\ \mathtt{then}\ \mathtt{t}_2\ \mathtt{else}\ \mathtt{t}_3$

$\mathtt{t}_1 : \mathtt{Bool} \qquad \mathtt{t}_2 : \mathtt{T} \qquad \mathtt{t}_3 : \mathtt{T}$

By the induction hypothesis, either $\mathtt{t}_1$ is a value or else there is some $\mathtt{t}_1'$ such that $\mathtt{t}_1 \longrightarrow \mathtt{t}_1'$. If $\mathtt{t}_1$ is a value, then the canonical forms lemma tells us that it must be either $\mathtt{true}$ or $\mathtt{false}$, in which case either E-IfTrue or E-IfFalse applies to $\mathtt{t}$. On the other hand, if $\mathtt{t}_1 \longrightarrow \mathtt{t}_1'$, then, by E-If,
$\mathtt{t} \longrightarrow \mathtt{if}\ \mathtt{t}_1'\ \mathtt{then}\ \mathtt{t}_2\ \mathtt{else}\ \mathtt{t}_3$.

Preservation

Theorem: If $\texttt{t} : \texttt{T}$ and $\texttt{t} \longrightarrow \texttt{t}'$, then $\texttt{t}' : \texttt{T}$.

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Case T-True: $\quad \mathtt{t} = \mathtt{true} \qquad \mathtt{T} = \mathtt{Bool}$

Then $\mathtt{t}$ is a value, so it cannot be that $\mathtt{t} \longrightarrow \mathtt{t}'$ for any $\mathtt{t}'$, and the theorem is vacuously true.

Preservation

Theorem: If $\mathtt{t} : \mathtt{T}$ and $\mathtt{t} \longrightarrow \mathtt{t}'$, then $\mathtt{t}' : \mathtt{T}$.

Proof: By induction on the given typing derivation.

Case T-If:

$$\mathtt{t} = \mathtt{if}\ \mathtt{t}_1\ \mathtt{then}\ \mathtt{t}_2\ \mathtt{else}\ \mathtt{t}_3 \quad \mathtt{t}_1 : \mathtt{Bool} \quad \mathtt{t}_2 : \mathtt{T} \quad \mathtt{t}_3 : \mathtt{T}$$

There are three evaluation rules by which $\mathtt{t} \longrightarrow \mathtt{t}'$ can be derived: E-IfTrue, E-IfFalse, and E-If. Consider each case separately.

Preservation

Theorem: If $\mathtt{t} : \mathtt{T}$ and $\mathtt{t} \longrightarrow \mathtt{t}'$, then $\mathtt{t}' : \mathtt{T}$.

Proof: By induction on the given typing derivation.

Case T-If:

$$\mathtt{t} = \mathtt{if}\ \mathtt{t}_1\ \mathtt{then}\ \mathtt{t}_2\ \mathtt{else}\ \mathtt{t}_3 \quad \mathtt{t}_1 : \mathtt{Bool} \quad \mathtt{t}_2 : \mathtt{T} \quad \mathtt{t}_3 : \mathtt{T}$$

There are three evaluation rules by which $\mathtt{t} \longrightarrow \mathtt{t}'$ can be derived: E-IfTrue, E-IfFalse, and E-If. Consider each case separately.

Subcase E-IfTrue: $\mathtt{t}_1 = \mathtt{true}$ $\quad$ $\mathtt{t}' = \mathtt{t}_2$

Immediate, by the assumption $\mathtt{t}_2 : \mathtt{T}$.

(E-IfFalse subcase: Similar.)

Preservation

Theorem: If $\mathtt{t} : \mathtt{T}$ and $\mathtt{t} \longrightarrow \mathtt{t}'$, then $\mathtt{t}' : \mathtt{T}$.

Proof: By induction on the given typing derivation.

Case T-IF:

$$\mathtt{t} = \mathtt{if}\ \mathtt{t}_1\ \mathtt{then}\ \mathtt{t}_2\ \mathtt{else}\ \mathtt{t}_3 \quad \mathtt{t}_1 : \mathtt{Bool} \quad \mathtt{t}_2 : \mathtt{T} \quad \mathtt{t}_3 : \mathtt{T}$$

There are three evaluation rules by which $\mathtt{t} \longrightarrow \mathtt{t}'$ can be derived: E-IFTRUE, E-IFFALSE, and E-IF. Consider each case separately.

Subcase E-IF: $\mathtt{t}_1 \longrightarrow \mathtt{t}_1' \qquad \mathtt{t}' = \mathtt{if}\ \mathtt{t}_1'\ \mathtt{then}\ \mathtt{t}_2\ \mathtt{else}\ \mathtt{t}_3$

Applying the IH to the subderivation of $\mathtt{t}_1 : \mathtt{Bool}$ yields $\mathtt{t}_1' : \mathtt{Bool}$. Combining this with the assumptions that $\mathtt{t}_2 : \mathtt{T}$ and $\mathtt{t}_3 : \mathtt{T}$, we can apply rule T-IF to conclude that $\mathtt{if}\ \mathtt{t}_1'\ \mathtt{then}\ \mathtt{t}_2\ \mathtt{else}\ \mathtt{t}_3 : \mathtt{T}$, that is, $\mathtt{t}' : \mathtt{T}$.

Recap: Type Systems

- Very successful example of a *lightweight formal method*
- big topic in PL research
- enabling technology for all sorts of other things, e.g. language-based security
- the skeleton around which modern programming languages are designed

The Simply Typed Lambda-Calculus

The simply typed lambda-calculus

The system we are about to define is commonly called the *simply typed lambda-calculus*, or $\lambda_{\rightarrow}$ for short.

Unlike the untyped lambda-calculus, the "pure" form of $\lambda_{\rightarrow}$ (with no primitive values or operations) is not very interesting; to talk about $\lambda_{\rightarrow}$, we always begin with some set of "base types."

- So, strictly speaking, there are *many* variants of $\lambda_{\rightarrow}$, depending on the choice of base types.
- For now, we'll work with a variant constructed over the booleans.

Untyped lambda-calculus with booleans

`t ::=`		*terms*
	`x`	*variable*
	`λx.t`	*abstraction*
	`t t`	*application*
	`true`	*constant true*
	`false`	*constant false*
	`if t then t else t`	*conditional*
`v ::=`		*values*
	`λx.t`	*abstraction value*
	`true`	*true value*
	`false`	*false value*

“Simple Types”

T ::=		*types*
	Bool	*type of booleans*
	T→T	*types of functions*

Type Annotations

We now have a choice to make. Do we...

- annotate lambda-abstractions with the expected type of the argument

$$\lambda \texttt{x:T}_1\texttt{. t}_2$$

  (as in most mainstream programming languages), or

- continue to write lambda-abstractions as before

$$\lambda \texttt{x. t}_2$$

  and ask the typing rules to "guess" an appropriate annotation (as in OCaml)?

Both are reasonable choices, but the first makes the job of defining the typin rules simpler. Let's take this choice for now.

Typing rules

$$\texttt{true} : \texttt{Bool} \qquad \text{(T-True)}$$

$$\texttt{false} : \texttt{Bool} \qquad \text{(T-False)}$$

$$\frac{\texttt{t}_1 : \texttt{Bool} \qquad \texttt{t}_2 : \texttt{T} \qquad \texttt{t}_3 : \texttt{T}}{\texttt{if } \texttt{t}_1 \texttt{ then } \texttt{t}_2 \texttt{ else } \texttt{t}_3 : \texttt{T}} \qquad \text{(T-If)}$$

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$$\frac{???}{\lambda \texttt{x}:\texttt{T}_1.\texttt{t}_2 : \texttt{T}_1 \to \texttt{T}_2} \qquad \text{(T-Abs)}$$

Typing rules

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$$\frac{\Gamma, \texttt{x}{:}\texttt{T}_1 \vdash \texttt{t}_2 : \texttt{T}_2}{\Gamma \vdash \lambda\texttt{x}{:}\texttt{T}_1\texttt{.}\texttt{t}_2 : \texttt{T}_1 \rightarrow \texttt{T}_2} \qquad \text{(T-Abs)}$$

$$\frac{\texttt{x}{:}\texttt{T} \in \Gamma}{\Gamma \vdash \texttt{x} : \texttt{T}} \qquad \text{(T-Var)}$$

Typing rules

$$\Gamma \vdash \texttt{true} : \texttt{Bool} \qquad \text{(T-True)}$$

$$\Gamma \vdash \texttt{false} : \texttt{Bool} \qquad \text{(T-False)}$$

$$\frac{\Gamma \vdash \texttt{t}_1 : \texttt{Bool} \qquad \Gamma \vdash \texttt{t}_2 : \texttt{T} \qquad \Gamma \vdash \texttt{t}_3 : \texttt{T}}{\Gamma \vdash \texttt{if } \texttt{t}_1 \texttt{ then } \texttt{t}_2 \texttt{ else } \texttt{t}_3 : \texttt{T}} \qquad \text{(T-If)}$$

$$\frac{\Gamma, \texttt{x}{:}\texttt{T}_1 \vdash \texttt{t}_2 : \texttt{T}_2}{\Gamma \vdash \lambda \texttt{x}{:}\texttt{T}_1 . \texttt{t}_2 : \texttt{T}_1 {\to} \texttt{T}_2} \qquad \text{(T-Abs)}$$

$$\frac{\texttt{x}{:}\texttt{T} \in \Gamma}{\Gamma \vdash \texttt{x} : \texttt{T}} \qquad \text{(T-Var)}$$

$$\frac{\Gamma \vdash \texttt{t}_1 : \texttt{T}_{11} {\to} \texttt{T}_{12} \qquad \Gamma \vdash \texttt{t}_2 : \texttt{T}_{11}}{\Gamma \vdash \texttt{t}_1 \ \texttt{t}_2 : \texttt{T}_{12}} \qquad \text{(T-App)}$$

Typing Derivations

What derivations justify the following typing statements?

- $\vdash (\lambda \texttt{x:Bool.x})\ \texttt{true} : \texttt{Bool}$
- $\texttt{f:Bool}\rightarrow\texttt{Bool} \vdash \texttt{f (if false then true else false)} : \texttt{Bool}$
- $\texttt{f:Bool}\rightarrow\texttt{Bool} \vdash$
  $\lambda \texttt{x:Bool. f (if x then false else x)} : \texttt{Bool}\rightarrow\texttt{Bool}$

→ (typed) Based on λ (5-3)

Syntax

t	::=		*terms:*
		x	*variable*
		$\lambda x{:}T.t$	*abstraction*
		t t	*application*
v	::=		*values:*
		$\lambda x{:}T.t$	*abstraction value*
T	::=		*types:*
		$T \to T$	*type of functions*
Γ	::=		*contexts:*
		∅	*empty context*
		$\Gamma, x{:}T$	*term variable binding*

Evaluation $\boxed{t \longrightarrow t'}$

$$\frac{t_1 \longrightarrow t_1'}{t_1\ t_2 \longrightarrow t_1'\ t_2} \quad \text{(E-App1)}$$

$$\frac{t_2 \longrightarrow t_2'}{v_1\ t_2 \longrightarrow v_1\ t_2'} \quad \text{(E-App2)}$$

$$(\lambda x{:}T_{11}.t_{12})\ v_2 \longrightarrow [x \mapsto v_2]t_{12} \quad \text{(E-AppAbs)}$$

Typing $\boxed{\Gamma \vdash t : T}$

$$\frac{x{:}T \in \Gamma}{\Gamma \vdash x : T} \quad \text{(T-Var)}$$

$$\frac{\Gamma, x{:}T_1 \vdash t_2 : T_2}{\Gamma \vdash \lambda x{:}T_1.t_2 : T_1 \to T_2} \quad \text{(T-Abs)}$$

$$\frac{\Gamma \vdash t_1 : T_{11} \to T_{12} \qquad \Gamma \vdash t_2 : T_{11}}{\Gamma \vdash t_1\ t_2 : T_{12}} \quad \text{(T-App)}$$

Properties of $\lambda_{\rightarrow}$

The fundamental property of the type system we have just defined is *soundness* with respect to the operational semantics.

1. *Progress:* A closed, well-typed term is not stuck

   If $\vdash t : T$, *then either* t *is a value or else* $t \longrightarrow t'$ *for some* t'.

2. *Preservation:* Types are preserved by one-step evaluation

   If $\Gamma \vdash t : T$ *and* $t \longrightarrow t'$, *then* $\Gamma \vdash t' : T$.

Proving progress

Same steps as before...

Proving progress

Same steps as before...

- inversion lemma for typing relation
- canonical forms lemma
- progress theorem

Inversion

Lemma:

1. If $\Gamma \vdash \texttt{true} : \texttt{R}$, then $\texttt{R} = \texttt{Bool}$.
2. If $\Gamma \vdash \texttt{false} : \texttt{R}$, then $\texttt{R} = \texttt{Bool}$.
3. If $\Gamma \vdash \texttt{if } \texttt{t}_1 \texttt{ then } \texttt{t}_2 \texttt{ else } \texttt{t}_3 : \texttt{R}$, then $\Gamma \vdash \texttt{t}_1 : \texttt{Bool}$ and $\Gamma \vdash \texttt{t}_2, \texttt{t}_3 : \texttt{R}$.

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5. If $\Gamma \vdash \lambda\texttt{x}{:}\texttt{T}_1\texttt{.t}_2 : \texttt{R}$, then $\texttt{R} = \texttt{T}_1 \rightarrow \texttt{R}_2$ for some $\texttt{R}_2$ with $\Gamma, \texttt{x}{:}\texttt{T}_1 \vdash \texttt{t}_2 : \texttt{R}_2$.

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6. If $\Gamma \vdash \texttt{t}_1\ \texttt{t}_2 : \texttt{R}$, then

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5. If $\Gamma \vdash \lambda\texttt{x}{:}\texttt{T}_1\texttt{.t}_2 : \texttt{R}$, then $\texttt{R} = \texttt{T}_1 \to \texttt{R}_2$ for some $\texttt{R}_2$ with $\Gamma, \texttt{x}{:}\texttt{T}_1 \vdash \texttt{t}_2 : \texttt{R}_2$.
6. If $\Gamma \vdash \texttt{t}_1\ \texttt{t}_2 : \texttt{R}$, then there is some type $\texttt{T}_{11}$ such that $\Gamma \vdash \texttt{t}_1 : \texttt{T}_{11} \to \texttt{R}$ and $\Gamma \vdash \texttt{t}_2 : \texttt{T}_{11}$.

Canonical Forms

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2. If $\mathtt{v}$ is a value of type $\mathtt{T}_1 \rightarrow \mathtt{T}_2$, then $\mathtt{v}$ has the form $\lambda \mathtt{x}:\mathtt{T}_1.\mathtt{t}_2$.

Progress

Theorem: Suppose $\mathtt{t}$ is a closed, well-typed term (that is, $\vdash \mathtt{t} : \mathtt{T}$ for some $\mathtt{T}$). Then either $\mathtt{t}$ is a value or else there is some $\mathtt{t}'$ with $\mathtt{t} \longrightarrow \mathtt{t}'$.

Proof: By induction

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Consider the case for application, where $\texttt{t} = \texttt{t}_1\ \texttt{t}_2$ with $\vdash \texttt{t}_1 : \texttt{T}_{11} \to \texttt{T}_{12}$ and $\vdash \texttt{t}_2 : \texttt{T}_{11}$.

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Theorem: Suppose $\texttt{t}$ is a closed, well-typed term (that is, $\vdash \texttt{t} : \texttt{T}$ for some $\texttt{T}$). Then either $\texttt{t}$ is a value or else there is some $\texttt{t}'$ with $\texttt{t} \longrightarrow \texttt{t}'$.

Proof: By induction on typing derivations. The cases for boolean constants and conditions are the same as before. The variable case is trivial (because $\texttt{t}$ is closed). The abstraction case is immediate, since abstractions are values.
Consider the case for application, where $\texttt{t} = \texttt{t}_1\ \ \texttt{t}_2$ with $\vdash \texttt{t}_1 : \texttt{T}_{11}{\rightarrow}\texttt{T}_{12}$ and $\vdash \texttt{t}_2 : \texttt{T}_{11}$. By the induction hypothesis, either $\texttt{t}_1$ is a value or else it can make a step of evaluation, and likewise $\texttt{t}_2$. If $\texttt{t}_1$ can take a step, then rule E-App1 applies to $\texttt{t}$. If $\texttt{t}_1$ is a value and $\texttt{t}_2$ can take a step, then rule E-App2 applies. Finally, if both $\texttt{t}_1$ and $\texttt{t}_2$ are values, then the canonical forms lemma tells us that $\texttt{t}_1$ has the form $\lambda\texttt{x}{:}\texttt{T}_{11}\texttt{.t}_{12}$, and so rule E-AppAbs applies to $\texttt{t}$.

Preservation

Theorem: If $\Gamma \vdash \mathtt{t} : \mathtt{T}$ and $\mathtt{t} \longrightarrow \mathtt{t}'$, then $\Gamma \vdash \mathtt{t}' : \mathtt{T}$.

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Which case is the hard one??

Preservation

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Case T-APP: Given

$$\mathtt{t} = \mathtt{t}_1\ \mathtt{t}_2$$
$$\Gamma \vdash \mathtt{t}_1 : \mathtt{T}_{11} \rightarrow \mathtt{T}_{12}$$
$$\Gamma \vdash \mathtt{t}_2 : \mathtt{T}_{11}$$
$$\mathtt{T} = \mathtt{T}_{12}$$

Show

$$\Gamma \vdash \mathtt{t}' : \mathtt{T}_{12}$$

Preservation

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By the inversion lemma for evaluation, there are three subcases...

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Theorem: If $\Gamma \vdash \mathtt{t} : \mathtt{T}$ and $\mathtt{t} \longrightarrow \mathtt{t}'$, then $\Gamma \vdash \mathtt{t}' : \mathtt{T}$.

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Case T-App: Given $\mathtt{t} = \mathtt{t}_1\ \mathtt{t}_2$

$\Gamma \vdash \mathtt{t}_1 : \mathtt{T}_{11} \to \mathtt{T}_{12}$

$\Gamma \vdash \mathtt{t}_2 : \mathtt{T}_{11}$

$\mathtt{T} = \mathtt{T}_{12}$

Show $\Gamma \vdash \mathtt{t}' : \mathtt{T}_{12}$

By the inversion lemma for evaluation, there are three subcases...

Subcase: $\mathtt{t}_1 = \lambda \mathtt{x}{:}\mathtt{T}_{11}.\ \mathtt{t}_{12}$

$\mathtt{t}_2$ a value $\mathtt{v}_2$

$\mathtt{t}' = [\mathtt{x} \mapsto \mathtt{v}_2]\mathtt{t}_{12}$

Preservation

Theorem: If $\Gamma \vdash \mathtt{t} : \mathtt{T}$ and $\mathtt{t} \longrightarrow \mathtt{t}'$, then $\Gamma \vdash \mathtt{t}' : \mathtt{T}$.

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Show $\Gamma \vdash \mathtt{t}' : \mathtt{T}_{12}$

By the inversion lemma for evaluation, there are three subcases...

Subcase: $\mathtt{t}_1 = \lambda \mathtt{x}{:}\mathtt{T}_{11}.\ \mathtt{t}_{12}$

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Uh oh.

The "Substitution Lemma"

Lemma: Types are preserved under substititition.

That is, if $\Gamma, \mathtt{x}{:}\mathtt{S} \vdash \mathtt{t} : \mathtt{T}$ and $\Gamma \vdash \mathtt{s} : \mathtt{S}$, then $\Gamma \vdash [\mathtt{x} \mapsto \mathtt{s}]\mathtt{t} : \mathtt{T}$.

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Proof: ...

Preservation

Recommended: Complete the proof of preservation