CS450/550 — Homework 1 Solutions — Due Tuesday, January 16

The following questions are intended to help you refresh your memory of analytic geometry in the plane and in space. Consult Appendix A in the textbook. You may also find it helpful to consult a textbook on analytic geometry.

1. What is the slope of the line connecting $(-2, 4)$ with $(6, -2)$?

We compute the slope as

$$\frac{\Delta y}{\Delta x} = \frac{(-2) - 4}{6 - (-2)} = \frac{-6}{8} = -\frac{3}{4}$$

2. Write down a parametric vector equation for the line from problem 1. This equation should have the form:

$$(x, y) = \ldots s \ldots$$

where s is the free parameter.

$$(x, y) = (-2, 4) + s(8, -6)$$

3. Express the line from problem 1 in slope-intercept form (so that y is a function of x).

We need to compute the point where the line intercepts the y axis. We start with the solution to Problem 2. This gives us the parametric equation

$$(x, y) = (-2, 4) + s(8, -6)$$

which we expand as two simultaneous equations

$$x = -2 + 8s \qquad y = 4 - 6s$$

Plugging $x = 0$ into the first equation, we get $s = \frac{1}{4}$. Plugging this into the second equation gives $y = \frac{5}{2}$. The equation is then

$$y = -\frac{3}{4}x + \frac{5}{2}$$

3. Write down an equation for a line that is perpendicular to the line in problem 1 and passes through the point $(-2, 4)$.

If two lines are perpendicular, the dot product of their direction vectors is zero. The direction vector for the line in Problem 1 is $(8, -6)$, so we can compute the new slope as any solution to the equation

$$(x, y) \cdot (8, -6) = 0$$

One such solution is $(x, y) = (3, 4)$. This gives us the parametric equation

$$(x, y) = (-2, 4) + s(3, 4).$$

Converting this to slope-intercept form gives us

$$y = \frac{4}{3}x + \frac{20}{3}$$

5. What is the cosine of the angle between the two vectors $(2, 2)$ and $(1, 6)$?

Normalize them and take the dot product. The first vector normalized is $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. The second vector normalized is $(\frac{1}{\sqrt{37}}, \frac{6}{\sqrt{37}})$. Numerically, this evaluates to:

$$(0.707140.7071) \cdot (0.16439, 0.98639) = 0.81373.$$

This is the cosine of the angle between the two vectors.

6. Write down a parametric vector equation for the plane that passes through the three points (0,0,0), (2,1,3), and (-8,2,0).

$$(x, y, z) = (0, 0, 0) + u(2, 1, 3) + v(-8, 2, 0)$$

7. Write down an equation for the plane from problem 6 in the form $Ax + By + Cz - D = 0$.

$$x + 4y - 2z = 0$$

8. Write down a vector that is orthogonal to the plane from problem 6. (Hint: Use the cross-product.)

This is just the coefficients of x, y, and z from the answer to problem 7: $(1, 4, -2)$.

Alternatively, we can take $(2, 1, 3) \times (-8, 2, 0) = (1 * 0 - 3 * 2, 3 * (-8) - 2 * 0, 2 * 2 - 1 * (-8)) = (-6, -24, 12) = (-1, -4, 2)$

9. Compute the matrix product $C = AB$ for the following two matrices:

$$A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & -4 & 0 \\ 4 & -1 & 2 \end{pmatrix}$$

$$B = \begin{pmatrix} -1 & -2 & 2 \\ -2 & 0 & 4 \\ 2 & 3 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 4 & 8 \\ 9 & 2 & -18 \\ 2 & -2 & 6 \end{pmatrix}$$

10. Write down the transpose of matrix A from problem 9.

$$A^T = \begin{pmatrix} 1 & -1 & 4 \\ 1 & -4 & -1 \\ 2 & 0 & 2 \end{pmatrix}$$

11. Compute the determinant of matrix B from problem 9.

The determinant is -20.

12. Compute the matrix sum $S = A + B$ using the matrices from problem 9.

$$S = \begin{pmatrix} 0 & -1 & 4 \\ -3 & -4 & 4 \\ 6 & 2 & 3 \end{pmatrix}$$

For Students Enrolled in CS550:

13. Given an equation for a plane in 3-D space, $Ax + By + Cz - D = 0$, what is the formula for the unit vector that is normal to this plane?

$$\frac{(A, B, C)}{\sqrt{A^2 + B^2 + C^2}}$$

14. What is the form of an equation (or set of equations) for a line in 3-D space?

$$\frac{x - x_0}{\alpha} = \frac{y - y_0}{\beta} = \frac{z - z_0}{\gamma}$$

Alternatively, express the line parametrically as

$$(x, y, z) = P + sQ$$

where P is a point on the line and Q is the direction vector of the line.

15. Given an equation or equations for a line in 3-D space and an equation for a plane in 3-D space, write down an expression for the point where the line intersects the plane.

We start with a line in parametric form passing through point (x_0, y_0, z_0) along the direction (dx, dy, dz):

$$x = x_0 + tdx$$
$$y = y_0 + tdy$$
$$z = z_0 + tdz$$

We also start with the equation for the plane:

$$Ax + By + Cz + D = 0$$

Let the intersection point be (x, y, z), then our plan is to plug the three parametric equations into the plane equation and solve for t in terms of $A, B, C, D, X_0, y_0, z_0, dx, dy$, and dz. Then we can go back and substitute this into the line equations to solve for x, y, and z.

Step 1: Substitute the parametric equations into the plane equation:

$$A(x_0 + tdx) + B(y_0 + tdy) + C(z_0 + tdz) + D = 0$$

Gathering terms, we get

$$(Ax_0 + By_0 + Cz_0 + D) + t(Adx + Bdy + Cdz) = 0$$

Solving for t we get

$$t = -\left(\frac{Ax_0 + By_0 + Cz_0 + D}{Adx + Bdy + Cdz}\right)$$

Backsubstituting we get:

$$\begin{aligned} x &= x_0 - dx\left(\frac{Ax_0+By_0+Cz_0+D}{Adx+Bdy+Cdz}\right) \\ y &= y_0 - dy\left(\frac{Ax_0+By_0+Cz_0+D}{Adx+Bdy+Cdz}\right) \\ z &= z_0 - dz\left(\frac{Ax_0+By_0+Cz_0+D}{Adx+Bdy+Cdz}\right) \end{aligned}$$

16. Given equations for two planes in 3-D space, give an equation for the line that is their intersection.

Let the two planes be:

$$\begin{aligned} A_1x + B_1y + C_1z + D_1 &= 0 \\ A_2x + B_2y + C_2z + D_2 &= 0 \end{aligned}$$

The key insight for this problem is that the direction vector of the line will be orthogonal to the normal vectors of the two planes. The reason is that the line will lie in each plane, and any line inside a plane is orthogonal to the normal vector for the plane. The normal vectors for these two planes are (A_1, B_1, C_1) and (A_2, B_2, C_2), so we can take their cross product to get

$$(A_3, B_3, C_3) = (B_1C_2 - B_2C_1, C_1A_2 - C_2A_1, A_1B_2 - A_2B_1)$$

Given this direction vector, we can define the line if we can find one point that lies on the line. We will find the point on the line that is closest to the origin. This point will be the intersection between the line and the plane that is normal to the line and passes through the origin. That plane is

$$A_3x + B_3y + C_3z = 0$$

So we have three equations in three unknowns. These can be solved by Gaussian elimination.

Let (x_0, y_0, z_0) be the solution to these three equations for (x, y, z). Then the equation for the line will be

$$(x, y, z) = (x_0, y_0, z_0) + s(A_3, B_3, C_3)$$