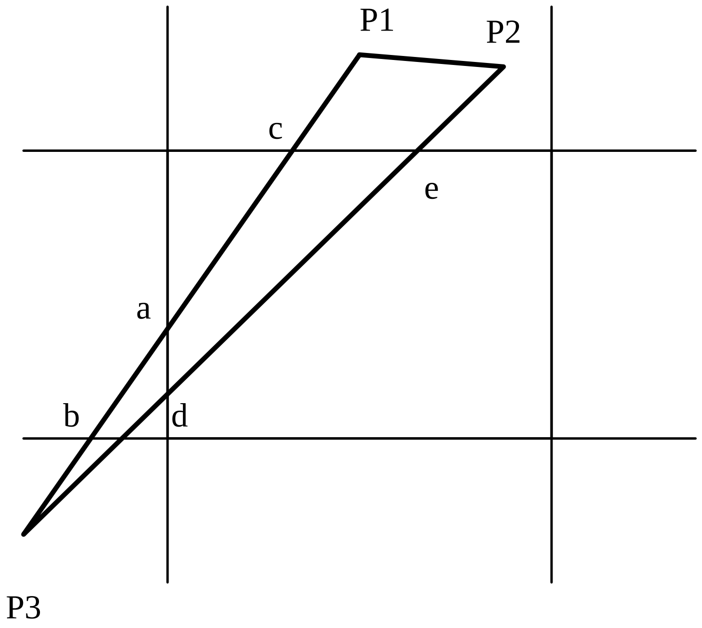
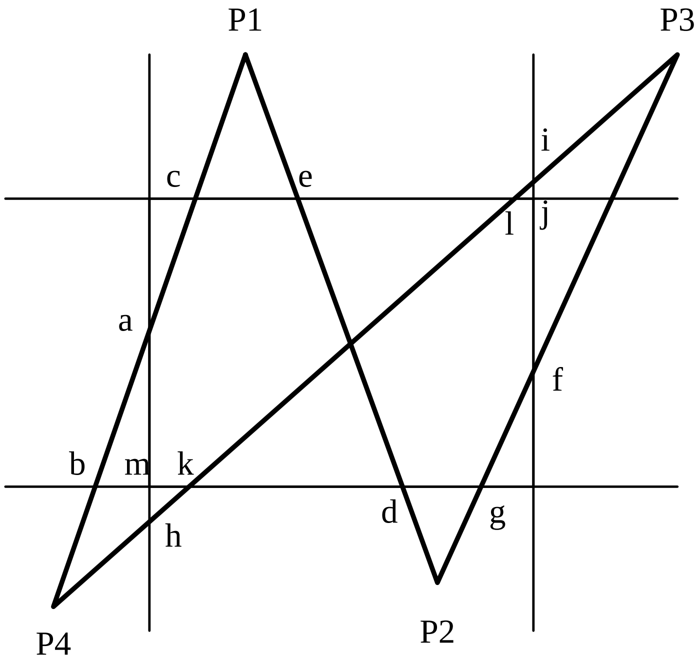
CS450/550 — Written Homework 3 — Solutions

Chapter 4, problem 3.




```
First Pass:                                       Second Pass
    P3     P3     P3     P3                           P3      d      d      d
    P1  L  a   R   a  B   b  T  b                     P1  L   a  R   a  B   a  T  a
                          a  T  a                             P1  R  P1  B  P1  T  c
           P1  R  P1  B  P1  T  c                     P2  L  P2  R  P2  B  P2  T XXX
    P2  L  P2  R  P2  B  P2  T XXX                    P3  L   d  R   d  B   d  T  e
    P3  L   d  R   d  B   d  T  e                                                 d
                                d
```

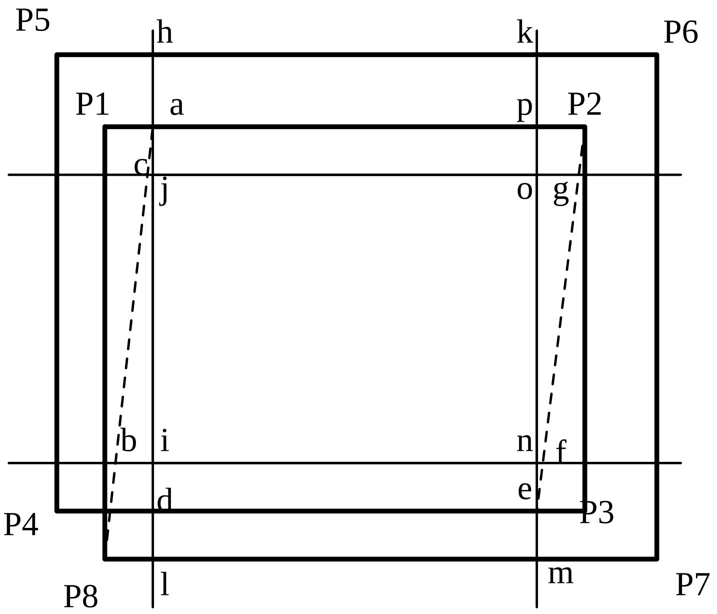
Chapter 4, problem 4.




```
First Pass:
    P4      P4      P4      P4
    P1  L    a  R    a  B    b  T  b
                             a  T  a
            P1  R   P1  B   P1  T  c
    P2  L   P2  R   P2  B    d  T  e
                                   d
    P3  L   P3  R    f  B    g  T  g
                             f  T  f
    P4  L    h  R    i  B    i  T  j
                     h  B    k  T  l
                                   k
```

```
Second Pass:
    P4       h       h       k
    P1  L    a  R    a  B    m  T  m
                             a  T  a
            P1  R   P1  B   P1  T  c
    P2  L   P2  R   P2  B    d  T  e
                                   d
    P3  L   P3  R    f  B    g  T  g
                             f  T  f
    P4  L    h  R    i  B    i  T  j
                     h  B    k  T  l
                                   k
```

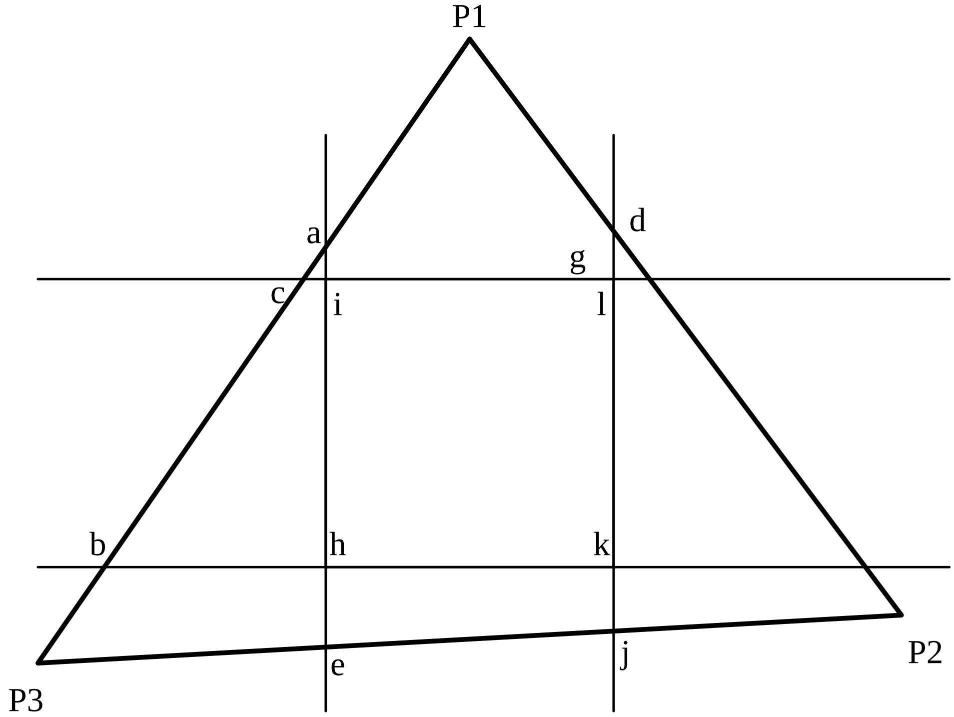
Chapter 4, problem 5.




```
First Pass:
    P8      P8      P8      P8
    P1  L   XXX
    P2  L    a  R    a  B    b  T  b
                             a  T  c
           P2  R  P2  B  P2  T XXX
    P3  L  P3  R XXX
    P4  L    d  R    e  B    f  T  g
                                   f
                     d  B  XXX
    P5  L XXX
    P6  L    h  R    h  B    i  T  i
                             h  T  j
           P6  R    k  B    k  T XXX
    P7  L  P7  R   XXX
    P8  L    l  R    m  B    n  T  o
                                   n
                     l  B  XXX
```

```
Second Pass
    P8       l       l       n
    P1  L   XXX
    P2  L    a  R    a  B    i  T  i
                             a  T  j
           P2  R    p  B    p  T XXX
    P3  L  P3  R  XXX
    P4  L    d  R    e  B    n  T  o
                                   n
                     d  B  XXX
    P5  L XXX
    P6  L    h  R    h  B    i  T  i
                             h  T  j
           P6  R    k  B    k  T XXX
    P7  L  P7  R  XXX
    P8  L    l  R    m  B    n  T  o
                                   n
                     l  B  XXX
```

Chapter 4, problem 8.




```
First Pass:
    P3     P3     P3     P3
    P1  L    a  R    a  B    b  T  b
                             a  T  c
           P1  R  P1  B  P1  T XXX
    P2  L  P2  R    d  B    d  T XXX
    P3  L    e  R    j  B    k  T  l
                                   k
                  e  B XXXX
```

```
Second Pass:
    P3      e      e      k
    P1  L    a  R    a  B    h  T  h
                             a  T  i
           P1  R  P1  B  P1  T XXX
    P2  L  P2  R    d  B    d  T XXX
    P3  L    e  R    j  B    k  T  l
                                   k
                  e  B  XXX
```

Chapter 4, problem 10.

10. Find the matrix that expresses scaling by (S_x, S_y) relative to the point (c_x, c_y).

We do this by composing three transformations: (a) translate (c_x, c_y) to the origin, (b) scale by (S_x, S_y), (c) translate back to (c_x, c_y):

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -c_x & -c_y & 1 \end{pmatrix} \begin{pmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c_x & c_y & 1 \end{pmatrix} = \begin{pmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ c_x(1-S_x) & c_y(1-S_y) & 1 \end{pmatrix}$$

Chapter 4, Problem 12. Find the matrix for reflection about the x axis.

We can do this by inspection. To reflect a point (x, y) about the x axis, we get the point $(x, -y)$. Hence, we just need the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Chapter 5, Problem 1.

1. Compute composite transformation matrix for the following transformations:

Translate by (3,2,4).

Construct a matrix for each transformation and then multiply them out.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 4 & 1 \end{pmatrix}$$

Rotate about x by 60 degrees:

$\sin 60 = 0.866 \quad \cos 60 = 0.5$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & 0.866 & 0 \\ 0 & -0.866 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Scale by (1.5, -2, 2):

$$\begin{pmatrix} 1.5 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rotate about y by 25 degrees:

$\sin 25 = 0.423 \quad \cos 25 = 0.906$

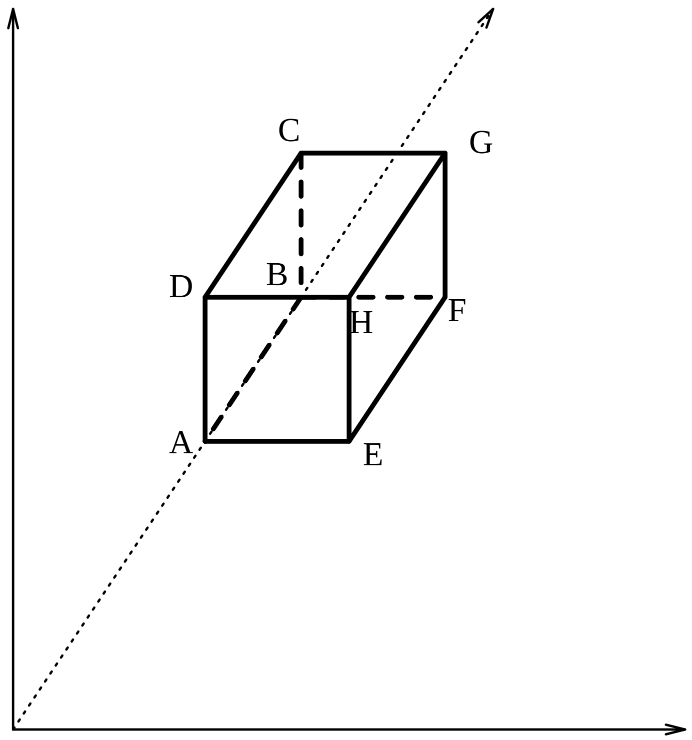
$$\begin{pmatrix} 0.906 & 0 & 0.423 & 0 \\ 0 & 1 & 0 & 0 \\ -0.423 & 0 & 0.906 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The product of these matrices is:

$$\begin{pmatrix} 1.359 & 0 & 0.6345 & 0 \\ -0.733 & -1 & 1.569 & 0 \\ -0.423 & 1.732 & 0.906 & 0 \\ 0.920 & 4.928 & 8.667 & 1 \end{pmatrix}$$

8. Consider a cube with vertexes at (0,0,2) (0,0,3) (0,1,3) (0,1,2) (1,0,2) (1,0,3) (1,1,3) (1,1,2). Suppose the viewpoint is at (0,0,-4).

Here is a picture:



For each of the vertexes,

a Compute the orthogonal projection onto the XY plane in the direction (0,0,-1).

By inspection, the orthogonal projection is the square with vertexes at (0,0,0), (1,0,0), (1,1,0), and (0,1,0). By algebra, the direction of the orthographic projection is $(x_p, y_p, z_p) = (0, 0, -1)$. The projection is computed as

$$\begin{aligned} x_{pl} &= x - \frac{z}{z_p} x_p \\ y_{pl} &= y - \frac{z}{z_p} y_p \\ z_{pl} &= 0 \end{aligned}$$

Since x_p and y_p are zero, the second terms are zero, so we just take the x and y coordinates of each point.

b Compute the perspective projection onto the XY plane.

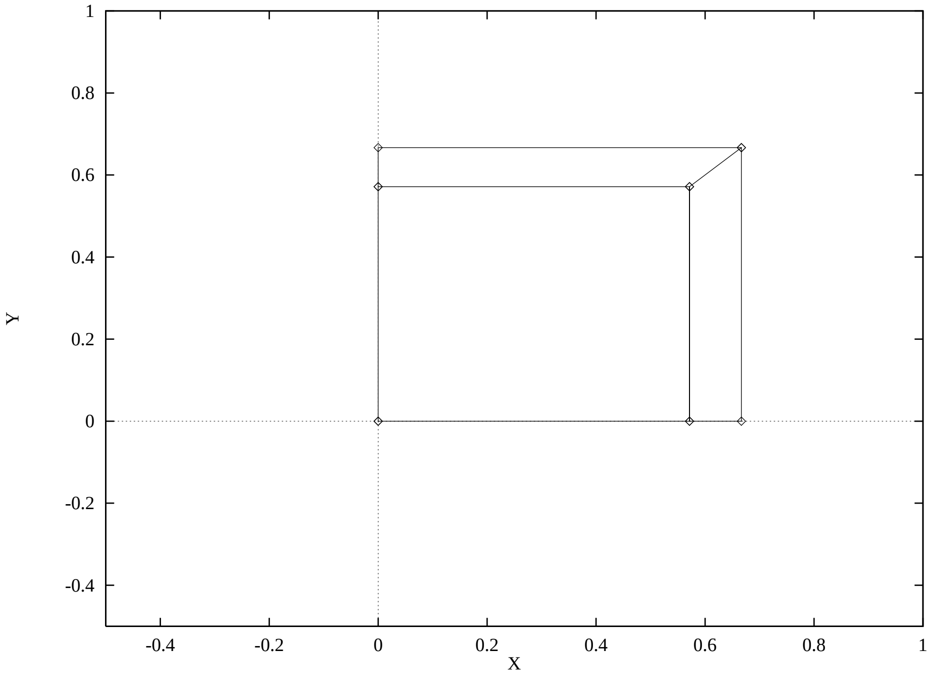
The perspective transformation is

$$x_{pl} = \frac{d \cdot x}{d + z} \qquad y_{pl} = \frac{d \cdot y}{d + z} \qquad z_{pl} = 0$$

Here are the results of transforming each of the vertexes:

Original	Transformed
(0,0,2)	(0,0,0)
(0,0,3)	(0,0,0)
(0,1,3)	(0,0.57,0)
(0,1,2)	(0,0.67,0)
(1,0,2)	(0,67,0,0)
(1,0,3)	(0.57,0,0)
(1,1,3)	(0.57,0.57,0)
(1,1,2)	(0.67,0.67,0)

Here is a visual display. Note that the larger square is the front face and the smaller square is the backface, so with backface elimination, only the front face will be visible.



c Compute the perspective depth transformation.

We just fix the z values from the previous part of this problem.

Original	Transformed
(0,0,2)	(0,0,0.33)
(0,0,3)	(0,0,0.43)
(0,1,3)	(0,0.57,0.43)
(0,1,2)	(0,0.67,0.33)
(1,0,2)	(0,67,0,0.33)
(1,0,3)	(0.57,0,0.43)
(1,1,3)	(0.57,0.57,0.43)
(1,1,2)	(0.67,0.67,0.33)

9. Compute the $VMAT$ view transformation matrix for a camera viewpoint specified as follows: Viewpoint $E = (5,5,5)$, Up direction $V = (0,1,0)$, Object reference point $R = (0,0,1)$, distance $d = 2$.

$$
\begin{aligned}
O &= E - d\frac{R-E}{||R-E||} \\
&= (3.769, 3.769, 4.015) \\
VMAT &= T \cdot M \\
T &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3.769 & -3.769 & -4.015 & 1) \end{pmatrix} \\
N &= R - E = (-5,-5,-4)
\end{aligned}
$$

$$
\begin{aligned}
Z &= \frac{N}{||N||} = (-0.615, -0.615, -0.492) \\
V' &= V - (Z \cdot V)Z \\
&= (0,1,0) - (-0.615)(-0.615, -0.615, -0.492) \\
&= (-0.378, 0.622, -0.303) \\
Y &= \frac{V'}{||V'||} = (-0.480, 0.788, -0.384) \\
X &= Y \times Z = (-0.625, 0, 0.781) \\
M &= \begin{pmatrix} -0.625 & -0.480 & -0.615 & 0 \\ 0.0 & 0.788 & -0.615 & 0 \\ 0.781 & -0.384 & -0.492 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
VMAT &= \begin{pmatrix} -0.625 & -0.480 & -0.615 & 0 \\ 0.0 & 0.788 & -0.615 & 0 \\ 0.781 & -0.384 & -0.492 & 0 \\ -0.780 & 0.384 & 6.616 & 1 \end{pmatrix}
\end{aligned}
$$

Additional Problems for Students in CS550:

Chapter 4, Problem 17. Prove that the transformation of the straight line from point P_1 to point P_2 is identical to the straight line between the transformations of P_1 and P_2.

A general transformation matrix (in homogeneous coordinates) has the form

$$
T = \begin{pmatrix} a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \\ c_1 & c_2 & 1 \end{pmatrix}
$$

An arbitrary point along the line from P_1 to P_2 has the form

$$
(x, y, 1) = P_1 + s(P_2 - P_1)
$$

for some parameter s and where P_1 and P_2 are given in homogeneous coordinates.

Our goal is to prove that

$$
T \cdot [P_1 + s(P_2 - P_1)] = T \cdot P_1 + s(T \cdot P_1 - T \cdot P_2)
$$

This is trivial, because T is a linear transformation. Hence

$$
\begin{aligned}
T \cdot [P_1 + s(P_2 - P_1)] &= T \cdot P_1 + T \cdot [s(P_2 - P_1)] \\
&= T \cdot P_1 + s[T \cdot (P_2 - P_1)] \\
&= T \cdot P_1 + s[(T \cdot P_2 - T \cdot P_1)]
\end{aligned}
$$

Chapter 4, Problem 22

22. Show analytically that when only translations are performed, their order is insignificant.

$$
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & d & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & d & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a+c & b+d & 1 \end{pmatrix}
$$

Chapter 5 Problem 8. Show that the perspective depth transformation transforms lines into lines.

Let $PDT[]$ be the perspective depth transformation. We wish to show that

$$PDT[A + t(B - A)] = PDT[A] + \alpha(PDT[B] - PDT[A])$$

such that $0 \leq t \leq 1$ maps 1:1 and onto $0 \leq \alpha \leq 1$. To do this, we will write down the x, y, and z components of this equation and solve for the various α's. If they are all the same, and if t maps properly to α, then the proof will be complete.

Answer to part a.

$$\begin{aligned}
\frac{[x_1 + t(x_2 - x_1)]d}{z_1 + t(z_2 - z_1) + d} &= \frac{x_1 d}{z_1 + d} + \alpha_x \left(\frac{x_2 d}{z_2 + d} - \frac{x_1 d}{z_1 + d} \right) \\
\frac{[x_1 + t(x_2 - x_1)]d}{z_1 + t(z_2 - z_1) + d} - \frac{x_1 d}{z_1 + d} &= \alpha_x \left(\frac{x_2 d}{z_2 + d} - \frac{x_1 d}{z_1 + d} \right) \\
\frac{[x_1 + t(x_2 - x_1)]d(z_1 + d) - x_1 d[z_1 + t(z_2 - z_1) + d]}{[z_1 + t(z_2 - z_1) + d][z_1 + d]} &= \alpha_x \left(\frac{x_2 d}{z_2 + d} - \frac{x_1 d}{z_1 + d} \right) \\
\frac{[x_1 + t(x_2 - x_1)]d(z_1 + d) - x_1 d[z_1 + t(z_2 - z_1) + d]}{[z_1 + t(z_2 - z_1) + d][z_1 + d]} &= \alpha_x \left(\frac{x_2 d}{z_2 + d} - \frac{x_1 d}{z_1 + d} \right) \\
\frac{td(z_1 x_2 - x_1 z_2 + d(x_2 - x_1)}{[z_1 + t(z_2 - z_1) + d][z_1 + d]} &= \alpha_x \left(\frac{x_2 d}{z_2 + d} - \frac{x_1 d}{z_1 + d} \right) \\
\frac{td(z_1 x_2 - x_1 z_2 + d(x_2 - x_1)}{[z_1 + t(z_2 - z_1) + d][z_1 + d]} &= \alpha_x \left(\frac{(x_2 d)(z_1 + d) - (x_1 d)(z_2 + d)}{(z_2 + d)(z_1 + d)} \right) \\
\frac{td(z_1 x_2 - x_1 z_2 + d(x_2 - x_1)}{[z_1 + t(z_2 - z_1) + d][z_1 + d]} &= \alpha_x \left(\frac{d(x_2 z_1 - x_1 z_2 + d(x_2 - x_1))}{(z_2 + d)(z_1 + d)} \right) \\
\frac{[td(z_1 x_2 - x_1 z_2 + d(x_2 - x_1)](z_2 + d)(z_1 + d)}{[z_1 + t(z_2 - z_1) + d][z_1 + d]d(x_2 z_1 - x_1 z_2 + d(x_2 - x_1))} &= \alpha_x \\
t\frac{z_2 + d}{[z_1 + t(z_2 - z_1) + d]} &= \alpha_x
\end{aligned}$$

Answer to part b.

$$\begin{aligned}
\frac{z_1 + t(z_2 - z_1)}{z_1 + t(z_2 - z_1) + d} &= \frac{z_1}{z_1 + d} + \alpha_z \left(\frac{z_2}{z_2 + d} - \frac{z_1}{z_1 + d} \right) \\
\frac{z_1 + t(z_2 - z_1)}{z_1 + t(z_2 - z_1) + d} - \frac{z_1}{z_1 + d} &= \alpha_z \left(\frac{z_2}{z_2 + d} - \frac{z_1}{z_1 + d} \right) \\
\frac{[z_1 + t(z_2 - z_1)](z_1 + d) - z_1[z_1 + t(z_2 - z_1) + d]}{[z_1 + t(z_2 - z_1) + d][z_1 + d]} &= \alpha_z \left(\frac{z_2}{z_2 + d} - \frac{z_1}{z_1 + d} \right) \\
\frac{td(z_2 - z_1)}{[z_1 + t(z_2 - z_1) + d][z_1 + d]} &= \alpha_z \left(\frac{z_2}{z_2 + d} - \frac{z_1}{z_1 + d} \right) \\
\frac{td(z_2 - z_1)}{[z_1 + t(z_2 - z_1) + d][z_1 + d]} &= \alpha_z \frac{z_2(z_1 + d) - z_1(z_2 + d)}{(z_2 + d)(z_1 + d)} \\
\frac{td(z_2 - z_1)}{[z_1 + t(z_2 - z_1) + d][z_1 + d]} &= \alpha_z \frac{d(z_2 - z_1)}{(z_2 + d)(z_1 + d)} \\
\frac{td(z_2 - z_1)(z_2 + d)(z_1 + d)}{[z_1 + t(z_2 - z_1) + d][z_1 + d]d(z_2 - z_1)} &= \alpha_z \\
\frac{t(z_2 + d)}{[z_1 + t(z_2 - z_1) + d]} &= \alpha_z
\end{aligned}$$

Answer to part c. This will be the same as part a, except that y should be substituted whereever x occurs.

Answer to part d. We can see that all three α values are identical, so we must now show that t maps to α smoothly. Specifically, we want to show that $t = 0$ implies $\alpha = 0$, $t = 1$ implies $\alpha = 1$, and the first derivative $\frac{d\alpha}{dt} \geq 0$ so that α increases monotonically with increasing t. It is easy to check the first two conditions. When $t = 0$, the left-hand-side is zero, so $\alpha = 0$. When $t = 1$, the denominator on the left becomes $z_2 + d$, so that it cancels the numerator, and we get $\alpha = 1$.

The first derivative

$$\frac{d\alpha}{dt} = \frac{(z_1 + d)(z_2 + d)}{[t(z_2 - z_1) + z_1 + d]^2}.$$

Without loss of generality, let us assume that $z_2 > z_1 > -d$ (if $z_1 > z_2$, we can interchange P_1 and P_2 and reverse the direction in which we parameterize the line). Then all terms in both the numerator and denominator are positive, so the first derivative is positive for $0 < t < 1$.

Chapter 5 Problem 9. Show that the perspective depth transformation transforms planes into planes.

This is trivial given the results from problem 8. In parametric form, a plane equation is just

$$P = P_1 + s(P_2 - P_1) + t(P_3 - P_1)$$

We have already shown that

$$PDT[A + u(B - A)] = PDT[A] + \alpha_1(PDT[B] - PDT[A])$$

Now let $A = C + v(D - C)$, Then

$$\begin{aligned} A + u(B - A) &= C + v(D - C) + u(B - C - v(D - C)) \\ &= C + v(D - C) + u(B - C) - uv(D - C)) \\ &= C + v(1 - u)(D - C) + u(B - C) \end{aligned}$$

And

$$\begin{aligned} PDT[A] + \alpha_1(PDT[B] - PDT[A]) &= PDT[C + v(D - C)] + \alpha_1(PDT[B] - PDT[C + v(D - C)]) \\ &= PDT[C] + \alpha_2(PDT[D] - PDT[C]) + \\ &\qquad \alpha_1(PDT[B] - (PDT[C] + \alpha_2(PDT[D] - PDT[C]))) \\ &= PDT[C] + \alpha_2(PDT[D] - PDT[C]) + \alpha_1(PDT[B] - PDT[C]) - \\ &\qquad \alpha_1\alpha_2(PDT[D] - PDT[C]) \\ &= PDT[C] + \alpha_2(1 - \alpha_1)(PDT[D] - PDT[C]) + \alpha_1(PDT[B] - PDT[C]) \end{aligned}$$

Now, let $C = P_1$, $D = P_2$, $B = P_3$, $s = v(1-u)$ and $t = u$. Also, let $\beta_1 = \alpha_2(1 - \alpha_1)$ and $\beta_2 = \alpha_1$, then we have

$$P_1 + s(P_2 - P_1) + t(P_3 - P_1) = PDT[P_1] + \beta_1(PDT[P_2] - PDT[P_1]) + \beta_2(PDT[P_3] - PDT[P_1])$$

and this is a parametric equation for a plane.