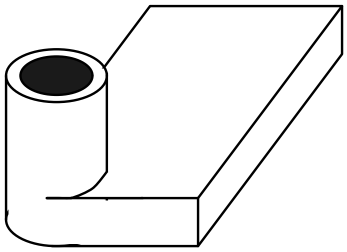
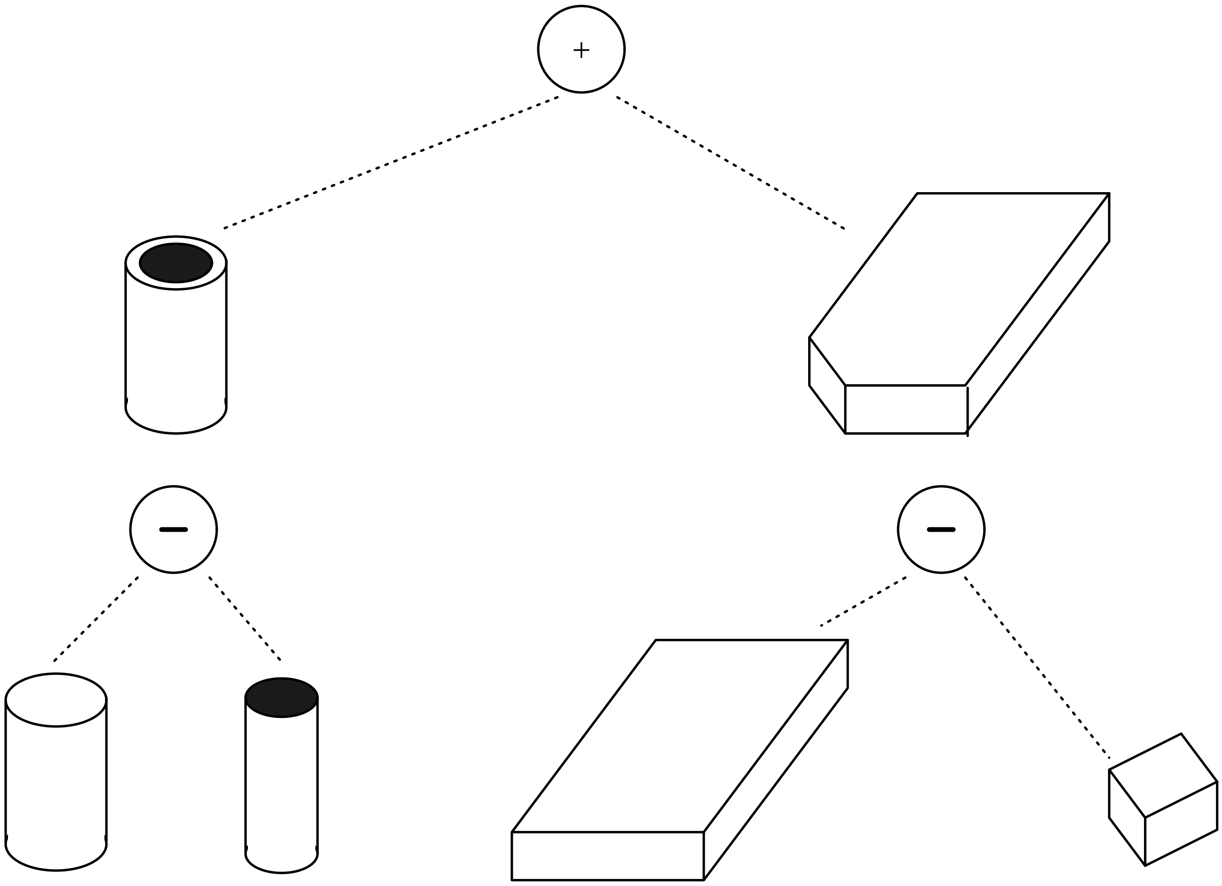
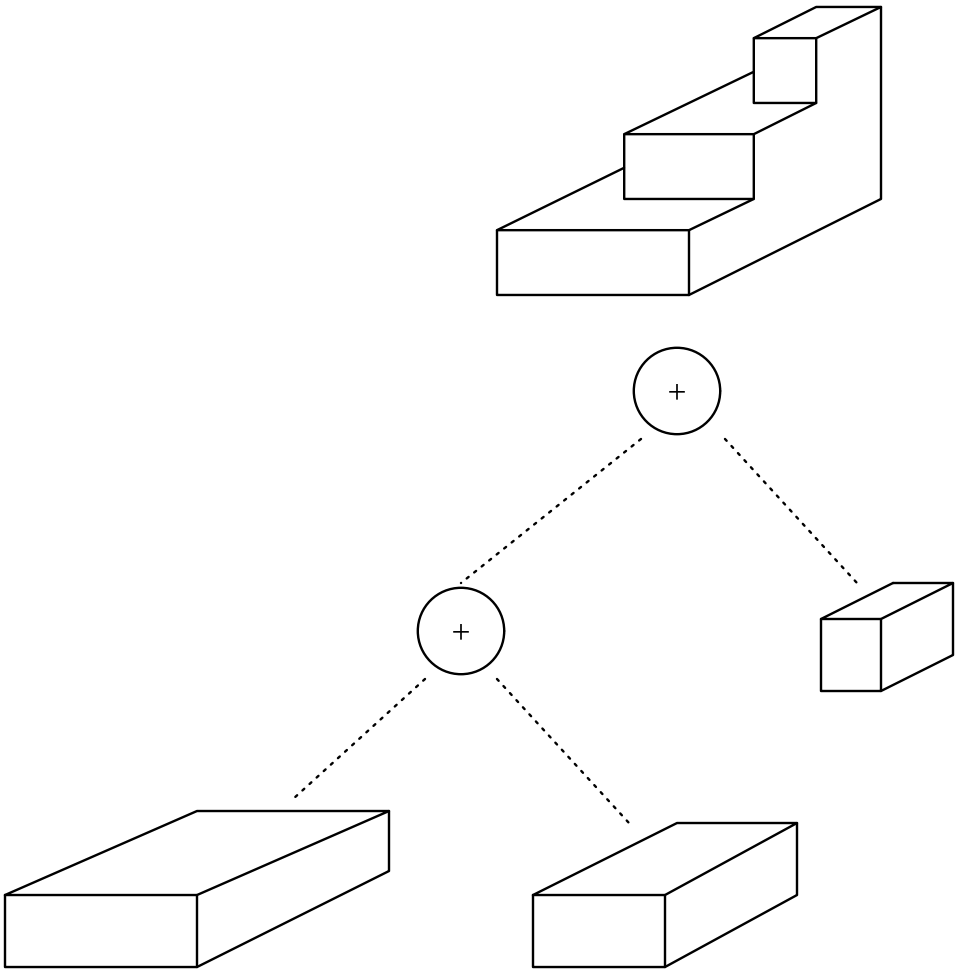
CS450/550 — Written Homework 7 — Solutions

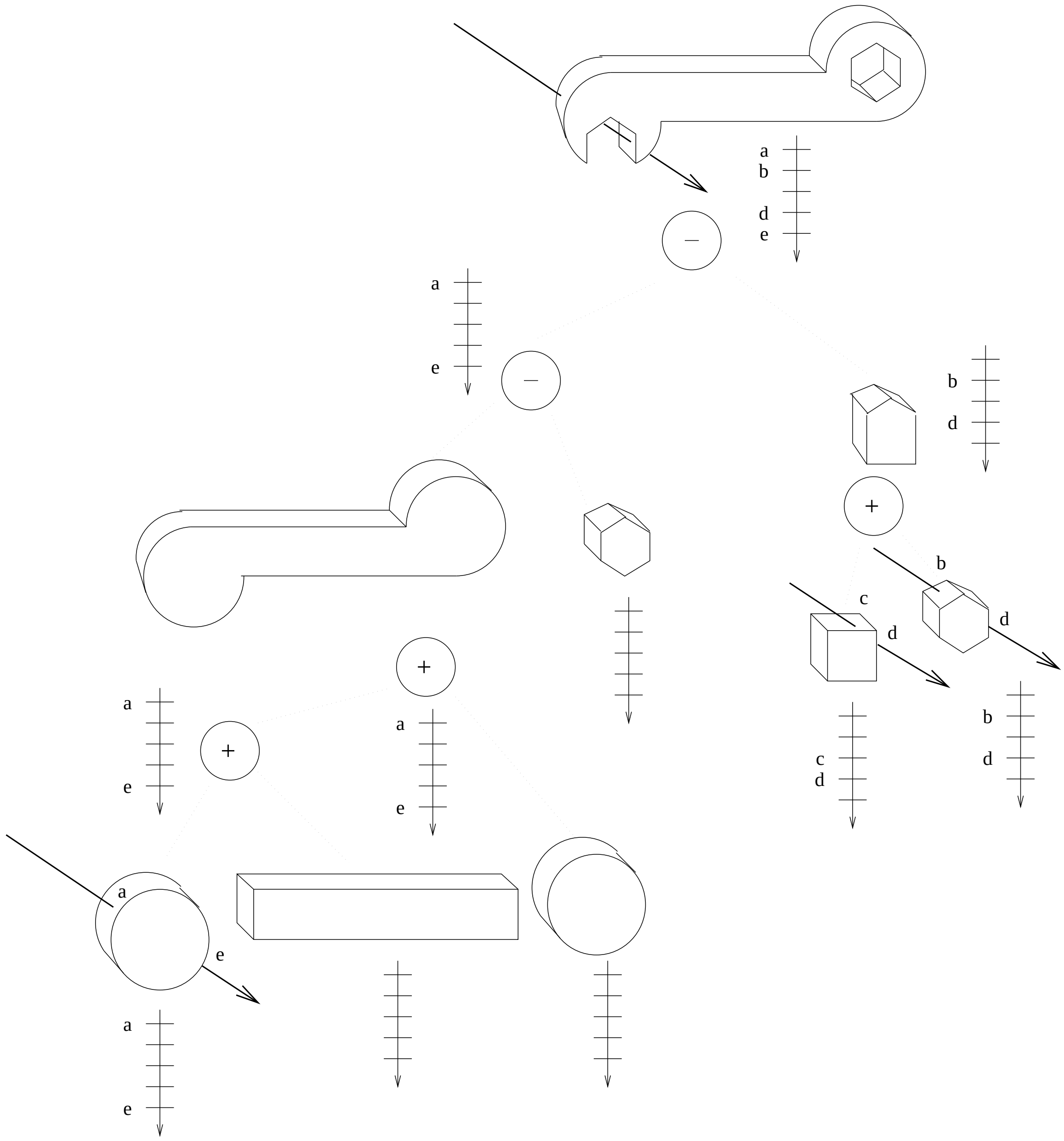
1. For each of the following objects, show a CSG tree that would generate that object.



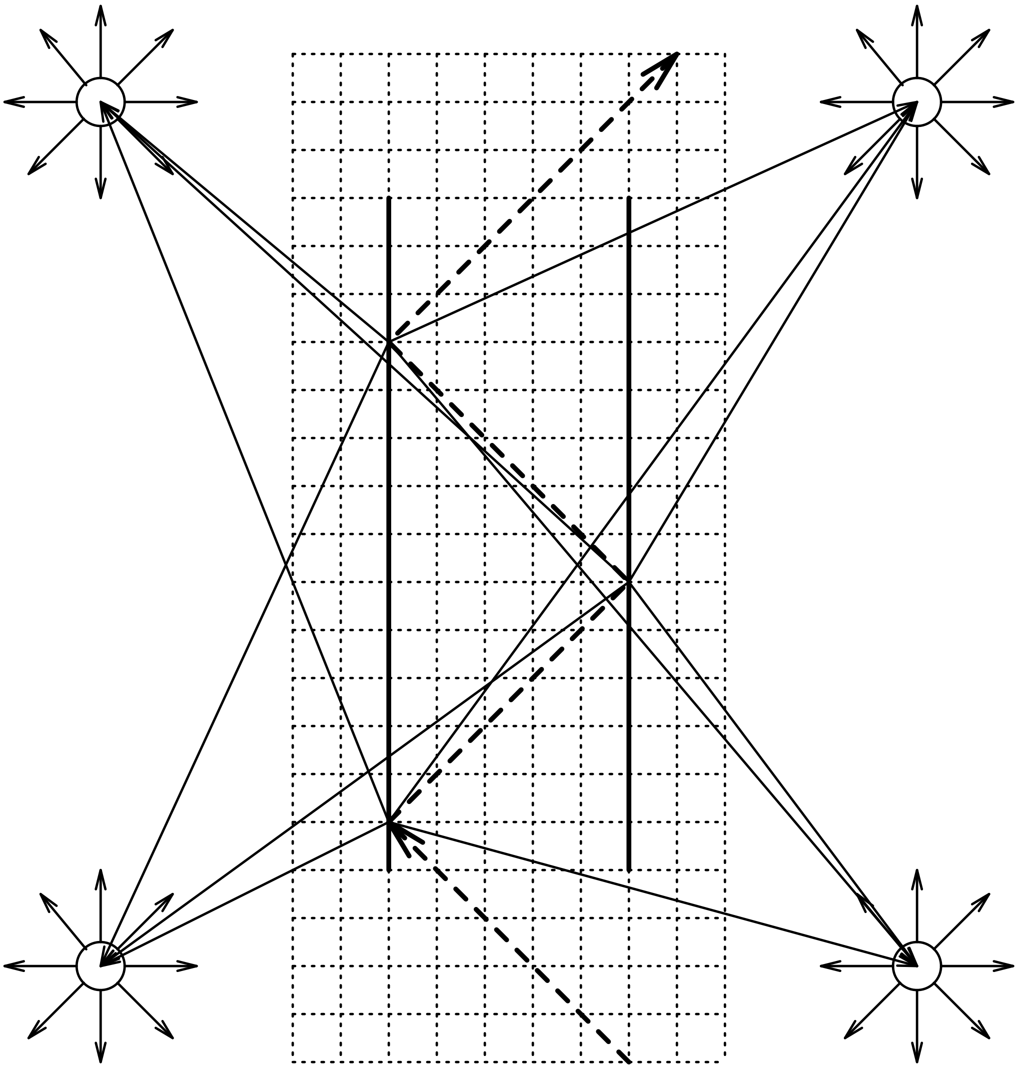




2. Show the hit-lists constructed for each node of this CSG tree as the ray intersects the object. This will include the hit lists for the leaf nodes and the "woven" hit lists for the internal nodes all the way up to the root node. Use consecutive letters (a, b, c, ...) to name each intersection point.


a
b
d
e
a
e
b
d
b
c
d
d
a
e
a
e
c
d
b
d
a
e
a
e

3. The following scene shows one primary ray, four point light sources, and two vertical mirrors in a 2D world. The mirrors are reflective on all sides. They are not quite perfect—they exhibit some diffuse reflection as well. The ray strikes the first mirror at a 45 degree angle.



(a) How many additional rays will need to be traced to finish tracing this primary ray?

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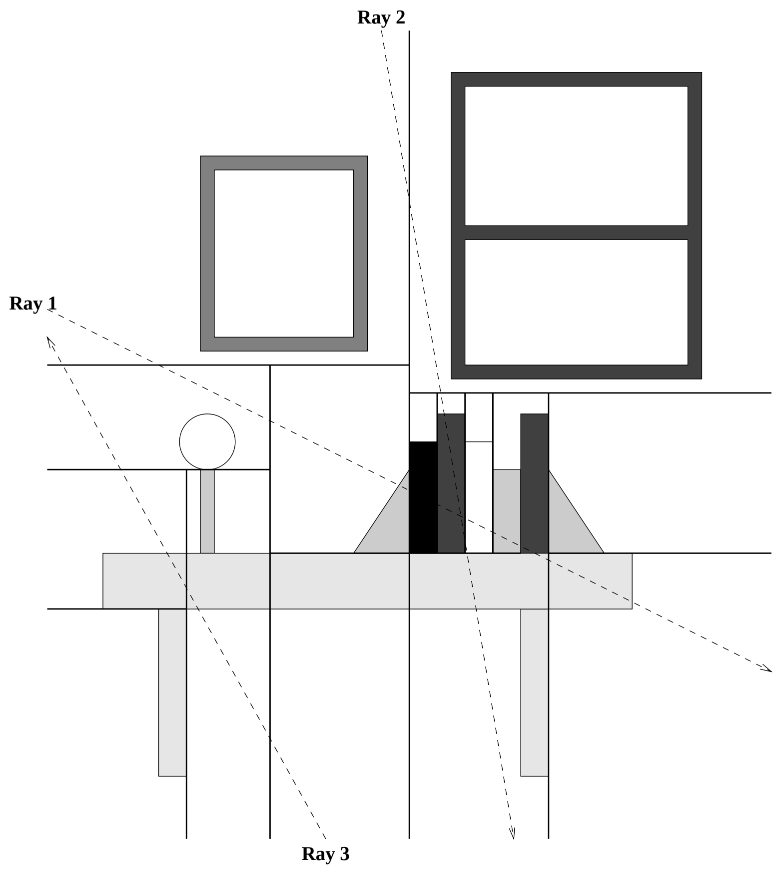
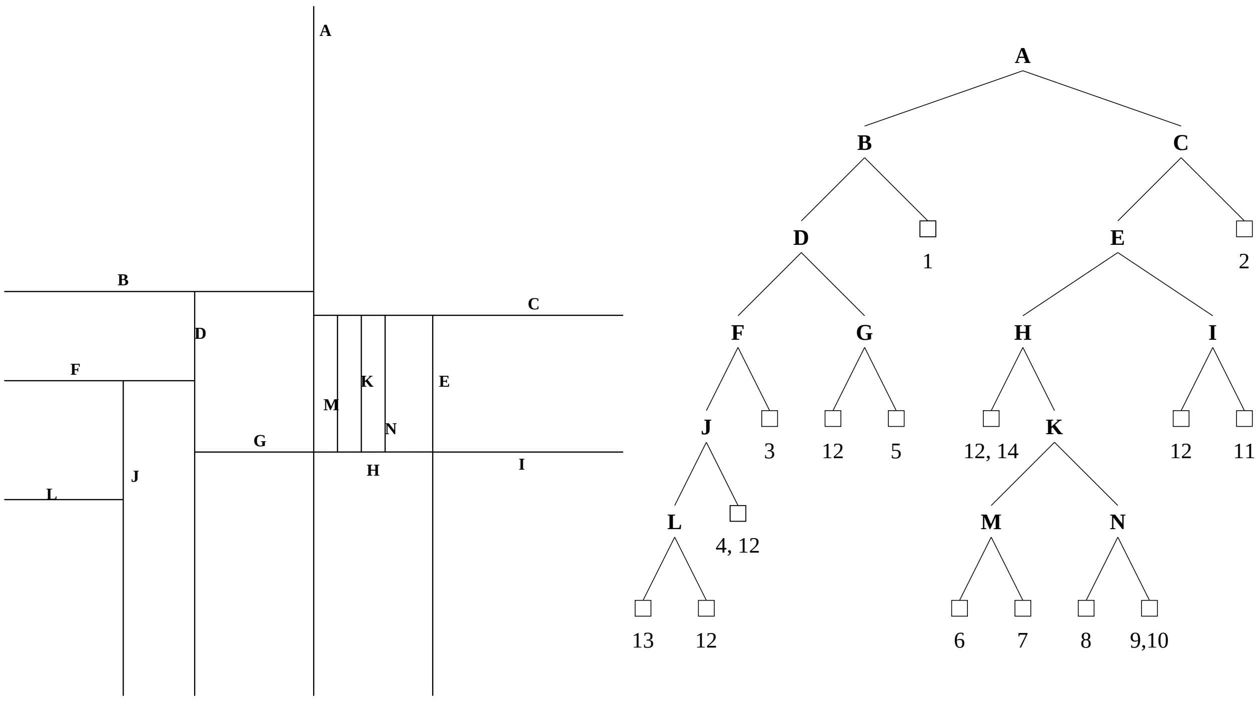
(b) Draw those rays on the figure. The dotted grid lines are there to help you; they are not part of the scene itself.

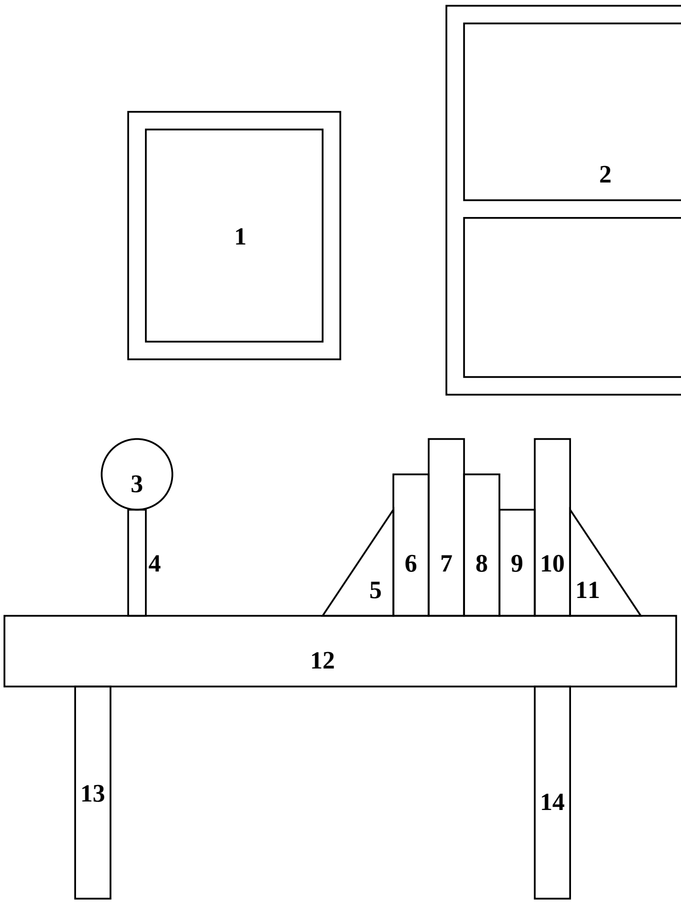
4. Consider the following 2-D scene with a 2-D BSP tree overlaid on it and three rays shown.



The following figures give names to each splitting plane and show the structure of the BSP tree. **NOTE: there was a bug in this figure. I have corrected the bug in the solution.**



The following figure gives the names for the solid objects.



(a) For each leaf in the BSP tree, indicate which solid objects intersect the region corresponding to that leaf.

Note that for any internal node that has two leaves, it is ambiguous which side of the splitting plane is the "front" side. For example, at node G, my solution corresponds to the "front" side facing downward. If the front were to be upward, then the left child leaf should contain object 5 and the right child object 12. A similar problem arises at nodes L, M, N, and I.

(b) For Ray 1, show the order in which Ray 1 is intersected with each of the solid objects and/or splitting planes until the nearest hit point is found.

object 1, plane B, object 3, plane F, plane D, object 5
Notice that the test against plane F is performed because after tracing against object 3, we climb up the BSP tree checking for intersections against each plane.

(c) Same for Ray 2.

object 1, plane B, plane A, object 2, plane C, object 7

(d) Same for Ray 3.

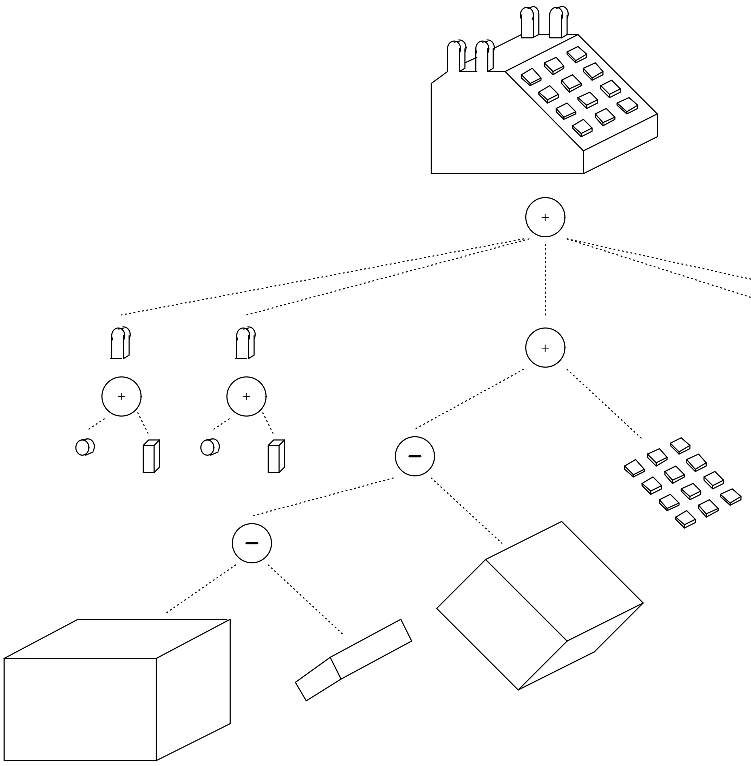
object 12, plane D

This is trickier than I intended. When ray 3 hits object 12, the hit point is not in the region corresponding to the current leaf of the BSP tree. So we must follow the ray until it reaches a region where the hit point is located. In this case, we cross plane D and we are then in that region.

For CS550 students only:

5. Draw a CSG tree for this object.

Obviously my solution isn't quite complete. I should do individual unions for each of the 9 keys on the telephone. Also, obviously there are many many possible trees that would be correct.



6. Consider a cylinder centered at (1,3,5) with height 6 and radius 2 with the central axis of the cylinder along the direction (1,1,1).

(a) What is the unit-to-arbitrary transformation matrix to transform a unit cylinder into this cylinder? What is the required translation vector to move this cylinder to the origin?

We must scale all dimensions by 2 to get radius 2. Then we must scale the y dimension by 3 to get height 6, so the resulting scaling matrix is

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Rotating the cylinder into the proper orientation is tricky. Let us work backwards from the final position. Suppose we consider the point at (0,1,0) in the unit coordinate system (after scaling). Where should this point be in the world coordinate system? It should be distance 1 away from the origin along the ray (1,1,1). Or in other words, it should be at the point we obtain by normalizing the ray (1,1,1) = (0.577, 0.577, 0.577). Suppose we rotate this ray about the y axis so that it now lies in the $x-y$ plane. This is a rotation by -45 degrees about the y axis. The rotation matrix that does this is

$$R_y^T = \begin{pmatrix} \cos -45 & 0 & \sin -45 \\ 0 & 1 & 0 \\ -\sin -45 & 0 & \cos -45 \end{pmatrix}$$

If we apply this to (0.577, 0.577, 0.577) we get (0.816, 0.577, 0). Now we need to figure out what angle to rotate this point about the z axis to get to the point (0, 1, 0). From trigonometry, we can determine that $\sin\theta = 0.816$ is the amount. This is 0.9553 radians = 54.73 degrees.

Now we can construct our rotation matrices by reversing this process: First we rotate about z by -0.9553 radians. Then we rotate about y by $\pi/4$ radians.

$$R_z = \begin{pmatrix} 0.577 & -0.816 & 0 \\ 0.816 & 0.577 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_y = \begin{pmatrix} .707 & 0 & .707 \\ 0 & 1 & 0 \\ -.707 & 0 & .707 \end{pmatrix}$$

The combined transformation matrix is therefore

$$A \cdot R_z \cdot R_y = \begin{pmatrix} 0.816 & -1.633 & 0.816 \\ 1.732 & 1.732 & 1.732 \\ -1.414 & 0 & 1.414 \end{pmatrix}$$

We must translate the scaled cylinder by $(1,3,5)$.

(b) What is the arbitrary-to-unit translation vector and transformation matrix to convert the cylinder into the standard cylinder.

First translate by $(-1,-3,-5)$, then apply the following matrix:

$$R_y^T \cdot R_z^T \cdot A^{-1} = \begin{pmatrix} 0.204 & 0.192 & -0.354 \\ -0.408 & 0.192 & 0.0 \\ 0.204 & 0.192 & 0.354 \end{pmatrix}$$

(c) Consider the ray with start point (0,0,0) and direction (0,0.5,0.5). Apply arbitrary-to-unit transformation and then compute the nearest intersection point with the cylinder. Then compute the corresponding point in the original coordinate system. Also compute the normal vector at the intersection point in the unit coordinate system and in the original coordinate system. Show your work.

First we must normalize the ray direction vector to be (0 0.707 0.707). Next we must apply the arbitrary-to-unit transformation. In the unit coordinate system, the starting point of the ray is $(-2.961 - 1.732 - 1.414)$ and the direction vector is $(-0.144, 0.272, 0.25)$. Note that this ray has a length of only 0.397.

Now we must intersect it with the unit cylinder. First, we intersect it with the infinite cylinder $x^2 + z^2 = 1$. I compute

$$\begin{aligned} a_2 &= dx^2 + dz^2 = 0.08333 \\ a_1 &= sx \cdot dx + sz \cdot dz = -0.354 \\ a_0 &= sx^2 + sz^2 - 1 = 1 \end{aligned}$$

We plug this into the (simplified) quadratic formula $u = \frac{-a_1 \pm \sqrt{a_1^2 - a_0 a_2}}{a_2}$. The discriminant of this formula is $a_1^2 - a_2 a_0 = 0.042$, so the ray intersects the infinite cylinder. The two computed u

values are 1.793 and 6.692. If we plug the smaller value into the equation $P = (-2.961 - 1.732 - 1.414) + u(-0.144, 0.272, 0.25) = (-0.259, -1.244, -0.966)$. This y value (-1.244) lies outside the range $y \in [-1, 1]$, so this ray does not enter the infinite cylinder along its side. However, if we plug the larger value into the equation $P = (-2.961 - 1.732 - 1.414) + u(-0.144, 0.272, 0.25) = (-0.966, 0.089, 0.259)$ we see that the ray does exit the cylinder through the side. Hence, the ray must enter the cylinder through one of the ends.

Hence, we must check whether the ray intersects the ends of the cylinder. To check the top of the cylinder, we seek a value for t such that

$$\begin{aligned} sy + t \cdot dy &= 1 \\ t &= \frac{1 - sy}{dy} \\ &= 10.038 \end{aligned}$$

Plugging this in gives the intersection point $(-1.449, 1.0, 1.095)$, which lies outside of the unit circle in the $x - y$ plane, and hence, the ray does not intersect the top of the cylinder.

Let's try intersecting against the bottom plane. We seek a value for t such that

$$\begin{aligned} sy + t \cdot dy &= -1 \\ t &= \frac{-1 - sy}{dy} \\ &= 2.690 \end{aligned}$$

Plugging this in, we compute the intersection point with the bottom plane in unit coordinates to be $(-0.388, -1, -0.742)$. This is inside the unit circle, because $x^2 + z^2 = 0.701 < 1$. Hence, the ray enters the cylinder through its bottom end.

If we follow the logic in the book, we would do this computation in a slightly simpler way:

$$\begin{aligned} u_1 &= 1.793 \\ u_2 &= 6.692 \\ p_1 &= 10.038 \\ p_2 &= 2.690 \\ u_{enter} &= \max(\min(u_1, u_2), \min(p_1, p_2)) = 2.690 \\ u_{exit} &= \min(\max(u_1, u_2), \max(p_1, p_2)) = 6.692 \end{aligned}$$

The entry point in the original coordinates is $(0, 0, 0) + 2.690(0, 0.5, 0.5) = (0, 1.902, 1.902)$.

Now that we have found the entry and exit points, we must compute the normal. In unit coordinates, the normal at the entry point is $(0, -1, 0)$. We multiply this by $A^{-1} \cdot R_z^T \cdot R_y^T$ to obtain $(-0.192, -0.192, -0.192)$ which must be renormalized to obtain $(-0.577, -0.577, -0.577)$, which is the correct normal.