Location Assisted Coding for FSO Communication

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***Abstract*—Recent years have witnessed an explosive growth in the number of wireless devices. This development has fueled much research in wireless access technologies to efficiently use Radio Frequency (RF) spectrum. On the other hand, recent advances in free space optical (FSO) technologies promise a complementary approach to increase wireless capacity. In this paper, we describe WiFO, a hybrid WiFi and FSO high-speed wireless local area network (WLAN) of femtocells that can provide high bit rates while maintaining seamless mobility. Importantly, we introduce a novel location assisted coding (LAC) technique, based on which, the number of novel rate allocation algorithms are proposed to increase throughput and reduce interference for multiple users in a dense array of overlapped femtocells. Both theoretical analysis and numerical results show orders of magnitude increase in throughput using LAC over existing schemes for various random topologies.**

I. INTRODUCTION

WiFi devices are projected to continue their significant growth, fueled by the emerging markets for smart homes and the Internet of Things (IoT). This leads to a significant increase wireless bandwidth demand which cannot be met by the existing WiFi systems. Consequently, much recent research has focused on the $802.11ad$ wireless standard to increase the WiFi capacity. However, $802.11ad$ requires complex and power hungry circuitry due to sophisticated modulation schemes to obtain high bit rates. On the other hand, recent advances in free space optical (FSO) technologies such as VLC, promise a complementary approach to increase wireless capacity with minimal changes to the existing wireless technologies. The solid state light sources such as Lighting Emitting Diode (LED) and Vertical-cavity Surface-Emitting Laser (VCSEL) are now sufficiently mature that it is possible to transmit data at high bit rates reliably at low power consumption using simple modulation scheme such as ON-OFF Keying. Importantly, the FSO technologies do not interfere with the RF transmissions.

In our previous work, we described a hybrid WiFi-FSO (WiFO) WLAN [1],[2] that can provide orders of magnitude increase in throughput over existing WiFi systems while maintaining seamless mobility. The proposed WiFO architecture was based on the femtocell architecture [3],[4] in which transmissions take place in confined areas (non-overlapped cells) to reduce interference. On the other hand, using a dense deployment of overlapped femtocells can result in higher bandwidth and greater mobility. Specifically, in this paper, our first contribution is a novel cooperative transmission scheme, known as Location Assisted Coding (LAC) technique that takes advantage of the receiver's location information to eliminate interference and achieve high bit rates. LAC allows multiple receivers including ones in an overlapped areas to obtain data from multiple transmitters simultaneously without interference. We provide theoretical and numerical results of LAC technique for random deployment topologies. Our second contribution is the formulation of the multi-user rate allocation problems and the corresponding algorithmic solutions.

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II. RELATED WORK

We first briefly discuss a few related work on hybrid RF-FSO communication systems then highlight the differences between our work on LAC and the popular cooperative transmission techniques Multiple Input Multiple Output (MIMO) as well as the classic results on multiuser communication theory.

Hybrid RF-FSO communication systems. There have been several studies on RF-FSO hybrid systems. The majority of these studies, however are in the context of outdoor point-to-point FSO transmission, using a powerful modulated laser beam. Due to the instability of the FSO link over long distance transmission, an RF link is often used as a back up link [5], [6], [7]. Usually, a low capacity RF link is used when the primary FSO link fails or degrades significantly due to rain, fog or other environmental conditions. [8] provided a routing framework that maximizes the fairness index, which defined as the minimal ratio of data transmitted and data required among all traffic profiles. Backup RF link is made more available when the traffic is more delay sensitive. There are also recent literature on joint optimization of simultaneous transmissions on RF and FSO channels. For example, in [9], [10],[11], and [12], the authors considered a joint coding schemes for both FSO and RF channels. In [10], the authors studied the outage probability in a FSO/RF hybrid system and presented a power allocation scheme to minimize the outage probability. There have been also work on optimizing a FSO/RF hybrid network with respect to the location of the optical transceivers [13]. A more comprehensive optimization problem is presented in [11]. Additional work on hybrid RF-FSO systems can also be found in [14], [15], [16], and [17]. Many aforementioned FSO/RF systems are designed for outdoor environments where attenuation/fading is due mainly to the weather conditions or scintillations. In contrast, our work is focused on Wi-Fi and FSO system for indoor environments where fading is due mainly to geometry of the cone beams. In this respect, WiFO is similar to many VLC systems [18], [19], [20] designed for indoor settings. Some of these VLC systems rely on diffused light and thus channel characterization, e.g., light propagation is more involved. In contrast, due to WiFO femtocell architecture, a WiFO channel is essentially flat since there is almost no fading involved. Line-of-sight signal is the main contribution to SNR.

Cooperative Transmissions. LAC is similar to Multiple-Input Multiple-Output (MIMO) techniques that have been used widely in communications systems to achieve significantly higher data rates than traditional single-input and single-output systems [21], [22], [23], [24], [25]. In both LAC and MIMO, the spatial dimension is key to increase data rate. On the other hand, due to the simplicity of On-Off Keying, or more generally, the Pulse Amplitude Modulation (PAM) used in WiFO, LAC's spatial dimension is gained through the receiver's information location. More importantly, majority of MIMO coding techniques are focused on multiple transmit and receive antennas for a single user [26], [27], [28], [29]. On the other hand, LAC's aim is to use multiple transmitters for multiple receivers simultaneously.

Multiuser information theory. From the information theory perspective, LAC is related to the well-known broadcast channel problem [30]. In this setting, single data source tries to transmit a common message to all receivers at the same time. The capacity for discrete memoryless channel is derived by Marton in [31] which generalizes the results in [30]. The achievable throughput of Gaussian broadcast channel is shown in [32] using dirty-paper coding technique. The idea behind dirty-paper coding [33] is that if the interference is known, then by adapting to the interference, the transmitter still can transmit at maximum rate despite of the interference. This result is extended to multiple receivers in [34]. There have also been several other approaches to interference management [35], [36], [37]. LAC technique is different from these approaches in several ways. First, LAC is designed for the WiFO system [1] with short distance transmissions under well-controlled environments. Second, LAC directly relies on amplitude modulation and base band transmission which are not typically used in high-rate RF transmissions as in other approaches. Third, and importantly, LAC is a high level coding technique similar to analog network coding [38],[39]. However, there is a crucial difference between analog network coding and LAC. Analog network coding techniques typically rely on the assumption that a receiver has access to side information in the form of actual information bits or packets. On the other hand, using LAC, a receiver does not need any side information. This is possible because in WiFO setting, the AP (sender) has complete knowledge about all the bits or packets wanted by all the receivers together with the receivers locations. Thus, the AP can incorporate all these information into the encoded bits, allowing a simple decoding at the receivers without side information.

III. Problem Setup

To motivate the problem, we first provide a brief background on WiFO and FSO communication. WiFO design is based on the femtocell architecture consisting of an array of triangular-lattice FSO transmitters deployed in the ceiling to provide FSO coverage for the floor area directly below. WiFO femtocells can be overlapped or non-overlapped. We note that WiFO uses inexpensive transmitters (LEDs) and receivers (silicon photodiodes (PDs)). In addition, LEDs operate around 20 mW with good SNR and well within the eye safety (850 nm). We have successfully built a WiFO prototype from off-the-shelf components. Each receiver is capable of receiving data 50-100 Mbps simultaneously over both WiFi and FSO channels.

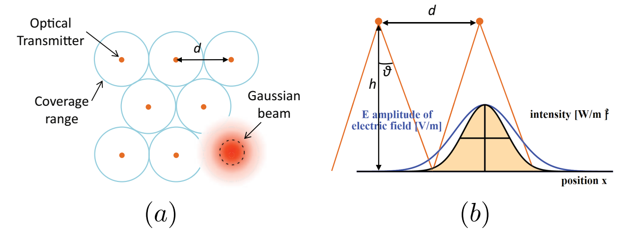

Fig. 1: (a) Configuration of the optical transmitter array; (b) coverage of optical transmitters with a divergent angle of ϑ

A demo can be seen at http://www.eecs.oregonstate.edu/∼thinhq/WiFO.html.

Fig. 1(a) shows a topology of non-overlapped triangular-lattice FSO transmitter array. The spacing between each transmitter is determined by: $d = 2h \tan \vartheta$, where h is the height of the ceiling, and ϑ is the divergent angle of the transmitter. Using $h = 5$ meters (approximate height of ceilings in typical buildings) and $\vartheta = 7.5$ degrees, the coverage area for a single FSO transmitter is approximately 1.36 meter squares. The light from the optical transmitter is a Gaussian beam with a divergent angle of ϑ as shown in Fig. 1 (b). A large ϑ will cover a larger floor area and thus reduce the total number of FSO transmitters. However, the transmit power and the minimum optical power required at the optical receivers set the upper limit of ϑ. If two transmitters are co-located, then the received signal power for a user will be doubled, and thus higher data rates can be achieved. Nevertheless, such simple deployment would increase the number of transmitters by two, without improving the mobility since there are still gaps between the circles as seen in Fig. 1(a). Although WiFi transmission can cover those gaps, the bit rates might be reduced in these areas.

One can use dense deployment of transmitter array to ensure no gaps. Using overlapped coverage will increase mobility and reduce bit error rate for a single receiver if two or more transmitters are used to send data to the single receiver. On the other hand, to avoid multi-user interference, transmitting data in overlapped areas may require TDMA or FDMA, which effectively reduces the overall capacity. We will show that this limitation is not necessary when the side information, specifically the user location, is used.

Assume that there are n FSO transmitters $T_1, T_2, \ldots T_n$, each produces a light cone that overlaps each other. We also assume that there are m receivers $R_1, R_2, \ldots R_m$, located in the coverage areas. An FSO transmitter is assumed to use On-Off Keying (OOK) modulation where high optical power represents "1" and low power represents "0" [40]. On the other hand, a receiver is assumed to be able to detect different levels of light intensities. If two transmitters send a "1" simultaneously to a receiver, the receiver would be able to detect "2" as light intensities from two transmitters add constructively. On the other hand, if one transmitter sends a "1" while the other sends a "0", the receiver would receive a "1".

Note that unlike a VLC system that relies on diffused light and thus channel characterization, e.g., light propagation is more involved, the WiFO channel is essentially flat since there is almost no fading involved, and line-of-sight signal is the main contribution to SNR. Thus, SNR depends on the received energy density which in turn, depends on the distance and angle between the transmitter and receiver. Specifically, WiFO receiver uses (adaptive) threshold decoding which is essentially an energy detector, and therefore, is much more robust than OFDM to recover the symbols (0, 1, 2, etc). Importantly, to highlight the benefits of proposed LAC, we assume that channel errors are either negligible or can be made negligible using Forward Error Correcting (FEC) codes. We note that the measurement results of our current WiFO prototype show that the bit error rate is negligible for transmission distance of less than 2 meters. When moderately strong FEC such as RS(255, 223) is applied, the resulted bit error rate is virtually zero up to 3 meters. That said, LAC as described in Section IV can be viewed as a high level coding scheme such as network coding where the received symbols (“0”, “1”, “2”, etc.) at the physical layers are assumed to be correct.

As an example, Fig. 2(a) shows a topology consisting of two FSO transmitters and two receivers. In this setting, the interference will occur at receiver R_2 if transmitter T_1 and T_2 sends independent bits to R_1 and R_2. Consequently, the resulted channel diagram for each receiver is shown in Fig. 2(b). Again, we note that there is no symbol error due to SNR. Rather, errors at each receiver is due only to interference. Furthermore, a cooperative transmission scheme is one that

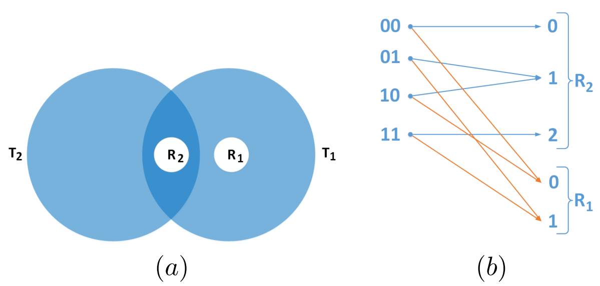

Fig. 2: (a) Topology for two FSO transmitters and two receivers; (b) Broadcast channels for two receivers.

uses both transmitters to send information to each receiver simultaneously. This cooperative transmission scheme can be viewed as a broadcast channel in which the sender broadcasts four possible symbols: “00”, “01”, “10”, and “11” with the left and right bits are transmitted by T_1 and T_2, respectively. Thus, there is a different channel associated with each receiver as shown in Fig. 2(b). There are only three possible symbols for R_2 because it is located in the overlapped coverage of two transmitters. Therefore, it cannot differentiate the transmitted patterns “01” and “10” as both transmitted patterns result in a “1” at the receiver due to additive interference. On the other hand, there are only two symbols at receiver R_1 because it is located in the light cone of a single transmitter.

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We note that the entry $\mathbf{A}_k(i,j)$ of the channel matrix denotes the probability that a transmitted symbol i to turn a symbol j

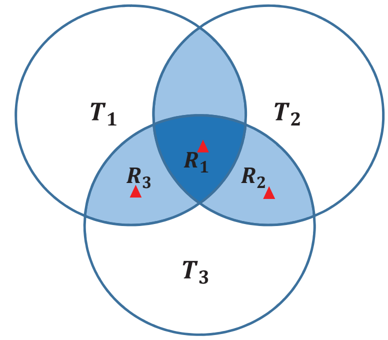

Fig. 3: Example of three cones with interference

at the receiver. Since we assume all sources of error are due to multi-user interference, $\mathbf{A}_k(i,j)$ is either 0 or 1. The same method can be used to construct the channel matrices for other topologies, and scenarios with transmission errors [41].

Our goal is to design a cooperative transmission scheme that allows the AP to send independent information to the receivers at the maximum rates. We note that our problem of characterizing the achievable region appears to be similar to the well-known broadcast channels. Specifically, when the channel is a Degraded Broadcast Channel (DBC), the capacity region has been established [30], [42], [43]. However, it can be shown that WiFO channel is not a degraded broadcast channel. Thus, the well-known results on DBC are not applicable.

IV. Location Assisted Coding

We now describe LAC encoding/decoding algorithms that allow multiple receivers to receive independent bits simultaneously, effectively eliminating interference under many settings. For simplicity, assume there are n transmitters and n receivers. Receiver R_i wants to receive bits b_i, $i = 1, 2, \dots, n$. The goal is for the transmitters $T_1, T_2, \dots, T_n$ to transmit bits $t_1, t_2, \dots t_n$ simultaneously, but yet all the receivers R_i's will be able to recover their intended bits b_i's from the received signals r_i's. By assumption, b_i, $t_i \in \{0, 1\}$. On the other hand, $r_i \in \{1, 2, \dots, n\}$ since the received signals add constructively.

Definition 1. *Let $\boldsymbol{H}$ be the matrix whose entry $\boldsymbol{H}(i,j)$ is equal to 1 if receiver i can receive signal from transmitter j and 0 otherwise. $\boldsymbol{H}$ is called a topology matrix.*

For example, the topology matrix associated with Fig. 3 is:

$$\mathbf{H}_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

We note that $\mathbf{H}$ is not a channel matrix that models the property of a physical channel. Rather, $\mathbf{H}$ is used to model the interference patterns.

Definition 2. *The system is said to achieve full rate if every receiver R_i can achieve 1 bit per transmission simultaneously.*

Note that Definition 2 is meant for On-Off Keying modulation in which, at most one bit of information can be sent by any transmitter. We have the following Proposition:

Proposition 1. *If the topology matrix* $\mathbf{H}$ *has full rank in* $\mathbf{GF}(2)$*, then it is possible for the system to achieve full rate.*

The proof for Proposition 1 is best presented via the following encoding and decoding algorithms that can achieve full rate.

A. Encoding Algorithm

Let $b_1, b_2, \ldots, b_n \in \{0,1\}$ be the bits wanted by receivers $R_1, R_2, \ldots, R_n$, and $\mathbf{H}$ is a full rank topology matrix.

Consider the following system of equations in $\mathbf{GF}(2)$:

$$\begin{cases} \mathbf{H}(1,1)t_1 \oplus \mathbf{H}(1,2)t_2 \oplus \ldots \oplus \mathbf{H}(1,n)t_n = b_1 \\ \mathbf{H}(2,1)t_1 \oplus \mathbf{H}(2,2)t_2 \oplus \ldots \oplus \mathbf{H}(2,n)t_n = b_2 \\ \ldots \\ \mathbf{H}(n,1)t_1 \oplus \mathbf{H}(n,2)t_2 \oplus \ldots \oplus \mathbf{H}(n,n)t_n = b_n \end{cases} \tag{1}$$

where $\oplus$ is addition in $\mathbf{GF}(2)$, i.e. $a \oplus b = (a+b) \mod 2$. Since $\mathbf{H}$ is a full-rank matrix in $\mathbf{GF}(2)$, we can solve the system of equations (1) above for unique $t_1, t_2, \ldots, t_n \in \{0,1\}$ in terms of $b_1, b_2, \ldots, b_n$. The solution for $t_1, t_2, \ldots, t_n$ is a linear combination of $b_1, b_2, \ldots, b_n$. We claim that if the transmitters $T_1, T_2, \ldots, T_n$ transmit the bits $t_1, t_2, \ldots, t_n$, respectively, then all the receiver $R_1, R_2, \ldots, R_n$ will be able to receive their desired bits $b_1, b_2, \ldots, b_n$, even if a receiver is in the overlapped area cover by multiple transmitters. We note that in WiFO, the AP having access to all the flows of data, transmits $t_1, t_2, \ldots t_n$ to the transmitters $T_1, T_2, \ldots, T_n$, respectively. T_i then transmits t_i. Thus, the encoding procedure involves solving a system of linear equations. One assumption is that the AP knows which regions the receivers are in, and therefore it can construct the topology matrix $\mathbf{H}$. If a receiver is associated with two given transmitters then the AP knows that the receiver is in an overlapped region of those two transmitters. When all receivers are in non-overlapped regions, the $\mathbf{H}$ is an identity matrix, and therefore full-rank. Thus, $t_i = b_i$.

B. Decoding Algorithm

A receiver R_i needs to be able to recover the bit b_i from the received signal r_i which can be represented as:

$$\begin{cases} r_1 = \mathbf{H}(1,1)t_1 + \mathbf{H}(1,2)t_2 + \ldots + \mathbf{H}(1,n)t_n \\ r_2 = \mathbf{H}(2,1)t_1 + \mathbf{H}(2,2)t_2 + \ldots + \mathbf{H}(2,n)t_n \\ \ldots \\ r_n = \mathbf{H}(n,1)t_1 + \mathbf{H}(n,2)t_2 + \ldots + \mathbf{H}(n,n)t_n \end{cases} \tag{2}$$

The receiver recovers b_i by performing

$$r_i \mod 2 = \hat{b}_i. \tag{3}$$

Note that $r_i \in \{0,1,2\}$ is a symbol obtained by thresholding a real value signal. Now we claim that $b_i = \hat{b}_i$. This can been seen by performing $\mod 2$ operations on both sides of equations (2) which results in the equations (1). Or simply, if r_i is even then R_i decodes bit b_i as "0", and "1" otherwise. As a result, each receiver can decode its bits correctly and independently in presence of interference. Furthermore, no other information regarding other users is required. Therefore, the decoding procedure is very simple.

TABLE I: Transmitted signals, received signals and recovered bits in $\mathbf{GF}(2)$ for three cones in Fig. 3

b_1	b_2	b_3	t_1	t_2	t_3	r_1	r_2	r_3	$\hat{b}_1$	$\hat{b}_2$	$\hat{b}_3$
0	**0**	**0**	0	0	0	0	0	0	**0**	**0**	**0**
0	**0**	**1**	0	1	1	2	2	1	**0**	**0**	**1**
0	**1**	**0**	1	0	1	2	1	2	**0**	**1**	**0**
0	**1**	**1**	1	1	0	2	1	1	**0**	**1**	**1**
1	**0**	**0**	1	1	1	3	2	2	**1**	**0**	**0**
1	**0**	**1**	1	0	0	1	0	1	**1**	**0**	**1**
1	**1**	**0**	0	1	0	1	1	0	**1**	**1**	**0**
1	**1**	**1**	0	0	1	1	1	1	**1**	**1**	**1**

Example 1. Consider the overlapped regions as shown in Fig. 3. The topology matrix for this case is:

$$\mathbf{H}_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

This matrix is also full-rank, therefore using LAC, one can transmit data at full rate. Specifically, we solve the following system of equations for t_1, t_2, t_3 in $\mathbf{GF}(2)$.

$$\begin{cases} t_1 \oplus t_2 \oplus t_3 = b_1 \\ t_2 \oplus t_3 = b_2 \\ t_1 \oplus t_3 = b_3 \end{cases} \tag{4}$$

or

$$\begin{cases} t_1 = b_1 \oplus b_2 \\ t_2 = b_1 \oplus b_3 \\ t_3 = b_1 \oplus b_2 \oplus b_3 \end{cases} \tag{5}$$

Now, if the three transmitters transmit bits as shown in (5), then at the receivers, the received signals are:

$$\begin{cases} r_1 = t_1 + t_2 + t_3 \\ r_2 = t_2 + t_3 \\ r_3 = t_1 + t_3 \end{cases} \tag{6}$$

The received signals and the recovered bits using (3) for all cases are shown in Table I. We can see that the recovered bits are exactly the intended bits.

C. Coding Scheme for $\mathbf{GF}(q)$

The coding scheme in previous section uses $\mathbf{GF}(2)$. It assumes that the transmitters can only transmit "0" and "1" using OOK modulation. If the transmitters can transmit with q levels from "0" to "$q-1$" where q is a prime number, then the coding scheme can be extended to $\mathbf{GF}(q)$. Specifically,

1) At the transmitter, we still use the system of equations (1) except that the addition is now performed over $\mathbf{GF}(q)$, i.e. $a \oplus b = (a+b) \mod q$. Unique $t_1, t_2, \ldots, t_n$ can be obtained if the matrix $\mathbf{H}$ is full-rank in $\mathbf{GF}(q)$.
2) At the receiver, we still have the system of equations (2) whose coefficients are in $\mathbf{GF}(q)$. Then the transmitted bits can be recovered by computing:

$$\hat{b}_i = r_i \mod q. \tag{7}$$

The advantages of using multiple levels q are twofold. First, it increases the capacity with fewer number of transmitters. Second, it is easier to have full-rank topology matrix. A matrix that is not full-rank in $\mathbf{GF}(2)$ might be full-rank in $\mathbf{GF}(q)$ with $q > 2$, leading to the potential of transmitting data at full rate.

D. Asymptotic Performance of LAC

We now consider a topology generated uniformly at random that consists of n receivers and n transmitters (cones). The probability of receiver i is belonged to transmitter j is p for all i and j. Thus, the topology matrix $\mathbf{H}$ is an $n \times n$ matrix in $\mathbf{GF}(2)$ such that:

$$\mathbf{H}_{ij} = \begin{cases} 1 \text{ with probability } p \\ 0 \text{ with probability } 1-p. \end{cases}$$

Let q be the size of the finite field, and $\gamma = 1 - 1/q$, and denote

$$h = \sum_{k=1}^{n} \binom{n}{k} \gamma^k (1-\gamma)^{n-k} [1 + (q-1)(1-p/\gamma)^k]^n.$$

We have the following proposition regarding the achievable rate.

Proposition 2. *The achievable rate R_{LAC} using LAC, defined as the average number of bits can be received per time slot, can be approximated as :*

$$R_{LAC} \approx n - \log_q (h+1) \tag{8}$$

for sufficiently large n. For $p = \frac{1}{2}$, and if there is a sufficiently large number of transmitters and receivers in a small area, the probability P_{LAC} for achieving full rate, i.e. $R_{LAC} = n$, approaches a constant. Specifically,

$$P_{LAC} = 0.289, \tag{9}$$

and the average achievable rate is:

$$R_{LAC} = \frac{1}{2^{n^2}} \sum_{k=1}^{n} k \prod_{i=0}^{k-1} \frac{(2^n - 2^i)^2}{2^k - 2^i}. \tag{10}$$

Proof: Please see the Appendix. ■

To quantify the interference gain of LAC, we compare LAC with BC. BC scheme is designed to avoid interference, thus a transmission to a receiver is taken place only if it is located in a non-overlapped area. The number of pairs of receiver and transmitter that satisfies this condition is the maximum number of bits that can be transmitted at a time. We have the following proposition for the BC performance, assuming $p = 1/2$.

Proposition 3. *The probability that the BC scheme is able to transmit a full rate (n) is:*

$$P_{BC} = \frac{n!}{2^{n^2}}, \tag{11}$$

and the average rate R for the BC scheme is:

$$R_{BC} = \frac{1}{2^{n^2}} \sum_{k=1}^{n} k! \binom{n}{k}^2 2^{(n-k)^2}. \tag{12}$$

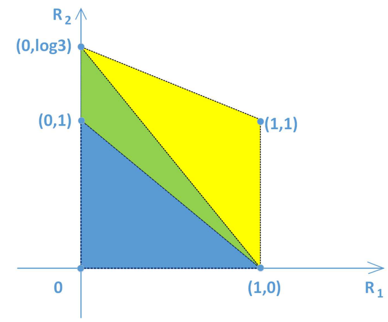

Fig. 4: Achievable rate region for R_1 and R_2.

Proof: Please see the Appendix. ■

We note that Propositions 2 and 3 show that the probability of transmitting at the full rate is approaching a constant (0.289) for LAC while this probability is exponentially decreased to zero for BC when n is large. Clearly, LAC is more efficient than the BC scheme.

E. Enlarging Achievable Rate Region Using LAC

Fig. 4 shows the three achievable rate regions for R_1 and R_2 in the scenario shown scenario in Fig. 2. Each blue, green, and yellow region depicts possible rate region using different transmission schemes. Each point (x, y) denotes the achievable rate, i.e., bits per transmission for R_1 and R_2, respectively. The blue region is achievable by simply using TDMA. Specifically, (1,0) is achievable by sending bits to R_1 exclusively and zero bits to R_2. Similarly, (0,1) is achievable by sending all the bits to R_2 and zero bits to R_1. Therefore, using TDMA and varying the fraction of time we use the strategy (0,1) and the remaining time we use the strategy (1,0), the blue achievable region can be achieved.

Such a scheme can be further enlarged by noting that the point $(0, \log 3)$ can be achieved. Indeed, $\log 3$ bits per transmission is achievable for R_2 if two transmitters are used to transmit bits to R_2. Specifically, there are three distinct symbols at R_2: 0, 1, 2. Therefore,the maximum achievable rate is $\log 3$. Now using TDMA between the strategies $(0, \log 3)$ and $(1, 0)$, the green region is achievable.

Finally and importantly, the point (1,1) in Fig. 4 is achievable using LAC since the topology has rank 2, resulting in each receiver can receive 1 bit per time unit. Consequently, the achievable region is enlarged by an additional amount shown in yellow, using a time sharing scheme that alternates among three different strategies: $(0, \log 3), (1, 0)$, and $(1, 1)$.

V. LAC-based Scheduling Algorithms

In previous Section, we describe LAC for some ideal $n \times n$ topology matrices with equal number of transmitters and receivers. In practice, the numbers of transmitters and receivers are different. Therefore, in this section, we develop three scheduling schemes with different aims for handling arbitrary topologies. All these schemes use LAC in Section IV as the primitive. The first scheme ($k - bit$ Algorithm) is a simple algorithm that produces the maximum transmission rate while ensuring that every receiver is guaranteed to receive its bit at a certain minimum rate. The second scheme (*Max Rate First*) is a solution to a convex optimization problem that produces a maximum transmission rate while minimizing the difference between a receiver's rate and its target proportional rate. The third scheme (*Proportional Rate First*) is also a solution to a convex optimization problem that ensures every receiver achieving exactly its proportional rate at the expense of reducing the overall transmission rate.

A. $k-bit$ Algorithm

Assuming that the topology matrix $\mathbf{H}$ is of size $m \times n$, and has rank k ($k \leq \min(m, n)$). As a result, only k independent single channels are used at any point in time [23]. Our goal is to derive a scheduling scheme to achieve k/n of the full rate R. This is also the theoretical maximum rate.

We begin with the following proposition.

Proposition 4. *Let $\boldsymbol{H}$ be an $m \times n$ matrix with rank k in $\mathbf{GF}(q)$. $\boldsymbol{H}$ has no row with all zero entries. Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ be k linearly independent row vectors and $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{m-k}$ be the other $m-k$ row vectors in the matrix. For each $\boldsymbol{v}_i$, there always exists a row vector $\boldsymbol{u}_j$ such that if the two vectors are swapped, $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{j-1}, \boldsymbol{v}_i, \boldsymbol{u}_{j+1}, \ldots, \boldsymbol{u}_k$ are still linearly independent.*

Proof: Please see Appendix. ■

Using Proposition 4, the following scheduling scheme can achieve the rate of $\frac{k}{n}R$ as follows.

Algorithm 1 produces a sequence of $m-k+1$ pairs.

Algorithm 1: k-bit Algorithm

1) Find a set $\mathcal{U}$ of k linearly independent rows of $\mathbf{H}$: $U = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$. Put the other $m-k$ rows to a set $\mathcal{V}$.
2) From the set of linearly independent rows $\mathcal{X} = \mathcal{U}$, create a $k \times n$ matrix $\tilde{\mathbf{H}}$. $\tilde{\mathbf{H}}$ has rank k. Therefore, we can pick k columns from $\tilde{\mathbf{H}}$ to create $k \times k$ matrix $\mathbf{H}'$ that has rank k.
3) Deploy the proposed coding schemes (Section IV-C) for k transmitters and k receivers corresponding to the full-rank matrix $\mathbf{H}'$.
4) Take a row $\mathbf{v}_i$ out from $\mathcal{V}$, search through the set $\mathcal{U}$ and find a row $\mathbf{u}_j \in \mathcal{U}$ such that if we replace $\mathbf{u}_j$ by $\mathbf{v}_i$ in the set $\mathcal{U}$, we obtain a set of linearly independent rows $\mathcal{U}'$. This is a direct consequence of Proposition 4.
5) Go to step 2) with $\mathcal{X} = \mathcal{U}'$ until $\mathcal{V}$ is empty.

Each pair includes a set of k transmitters and a set of k receivers with the corresponding full-rank $k \times k$ topology matrix $\mathbf{H}'$ from Step 3. They are the sets of active transmitters and receivers allowing k receivers to decode its signal correctly in a time slot. By periodically using the pairs of active transmitter set and receiver set in the sequence , we can achieve rate of $\frac{k}{n}R$ and allow m receivers share the bandwidth. After the algorithm terminates, each receiver appears in at least one pair of transmitter and receiver sets and therefore has a chance to receive and decode its signal at a certain minimum rate. Nevertheless, this scheduling scheme does not guarantee the throughput fairness among all users in the system.

B. Proportional Rate Allocation Scheduling

We have demonstrated that $k-bit$ Algorithm performs very well in term of bandwidth efficiency since it always operates at full rate. However, it is not clear how LAC can be used in the scenarios where multiple receivers request different transmission rates. We now formulate the Time Minimization (TM) problem that minimizes the time required to send different number of bits to different receivers. This problem turns out to be NP-hard. We then propose two variants of the Proportional Rate Allocation (PRA) problem, as approximate solutions to the TM problem, but which can be solved efficiently using convex optimization.

We use the following assumptions and notations for the TM formulation.

- There are m receivers $R_1, R_2, \ldots, R_m$. Each receiver requires a different number of bits, i.e., receiver R_i needs b_i bits.
- Let k be the rank of the topology matrix $\mathbf{H}$. Therefore, in each round, the transmitters can collectively transmit no more than k information bits to the receivers.
- Let N be the number of rounds required for the transmitters to send the requested bits to all the receivers. The goal is to find the a scheduling scheme that minimizes the number of rounds N.

Let the set $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ be the set of all rows of matrix $\mathbf{H}$. Let $\mathcal{D} = \{\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_d\}$ where $|\mathcal{D}| = d$ be the set that contains all distinct non-empty subsets $\mathcal{V}_i \subset \mathcal{U}$ such that all vectors in $\mathcal{V}_i$ are linearly independent. Since $rank(\mathbf{H}) = k$ we have $|\mathcal{V}_i| \leq k$ and $0 < d \leq \sum_{i=1}^{i=k} \binom{m}{k}$.

We can construct any LAC-based scheduling scheme C as follows. At each round, we choose a subset $\mathcal{V}_i$ for any $1 \leq i \leq d$ where $|\mathcal{V}_i| = l$ then l receivers corresponding to l independent vectors in $\mathcal{V}_i$ will be served using LAC. The process repeats until all receivers receive their desired number of bits (some receivers can receive more bits than their desired number of bits).

Let $\mathbf{A} \in [0,1]^{m \times d}$ be the matrix which represents the set $\mathcal{D}$

$$\mathbf{A}_{ij} = \begin{cases} 1 \text{ if } \mathcal{V}_j \text{ includes receiver } R_i \\ 0 \text{ otherwise} \end{cases}$$

Define $\mathbf{x} = [x_1, x_2, \ldots, x_d]^T$ where $x_i \in \mathbb{Z}^+$ denotes the number of rounds that we choose subset $\mathcal{V}_i$. The total number of rounds:

$$N = \sum_{i=1}^{m} x_i$$

Therefore, the number of bits that receiver R_i receives which requires to be no less than b_i can be written as the constraint:

$$\sum_{j=1}^{d} x_j \mathbf{A}_{ij} \geq b_i \ \forall i \tag{13}$$

$$\leftrightarrow \quad \mathbf{Ax} \succeq \mathbf{b} \tag{14}$$

where $\mathbf{b} = [b_1, b_2, \dots, b_m]^T$ is the vector representing the desired number of bits for each receiver, and $\succeq$ represents element-wise comparison.

This time minimization problem can be formulated an Integer Linear Program:

$$\textbf{Problem P1:} \quad \text{Minimize} \quad \sum_i x_i$$
$$\text{Subject to} \quad \begin{cases} \mathbf{x} \succeq \mathbf{0} \\ \mathbf{Ax} \succeq \mathbf{b} \end{cases} \tag{15}$$

with variable $\mathbf{x} \in {\mathbb{Z}^+}^{d\times 1}$ and given $\mathbf{A} \in [0,1]^{m\times d}, \mathbf{b} \in {\mathbb{Z}^+}^{m\times 1}$

We illustrates this problem with the following example.

Example 1. *We have* $m = 4, n = 3$ *and suppose* $(b_1, b_2, b_3, b_4) = (2, 2, 1, 1)$. *Assume that*

$$\boldsymbol{H} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Then $rank(\boldsymbol{H}) = 3$.

- *There are* $d = 12$ *feasible subsets in* $\mathcal{D}$: $\{(R_1); (R_2); (R_3); (R_4); (R_1, R_2); (R_1, R_3); (R_1; R_4)$ $(R_2, R_3); (R_2, R_4), (R_3, R_4), (R_1, R_2, R_3), (R_1, R_2, R_4)\}$.
- *The optimal scheme* C^* *would need* 2 *rounds:* $(R_1, R_2, R_3), (R_1, R_2, R_4)$.
- *Compare to some other scheme* C *would need* 3 *rounds:* $(R_1, R_2), (R_1, R_2), (R_3, R_4)$.

We note that (15) is the generalized form of the Covering Integer program [44] in which $b_i = 1 \ \forall i$. Furthermore, the Covering Integer program is shown to be equivalent to Set Cover problem which has been shown to be NP-hard [45]. Thus, the Time Minimization problem is NP-hard, and many heuristic algorithms can be used to solve this problem. Therefore, we will now focus on the PRA problems below.

Recall that if $rank(\mathbf{H}) = k$ then by using the $k - bit$ Algorithm, we can serve k receivers in each round. Certainly, we would prefer to transmit k bits per time slot in any round. That said, there are many scheduling schemes that can achieve the maximum rate, but we prefer a scheme that can also achieve some given target rate of each receivers. To do so, we will use a randomized approach to design our scheduling schemes.

To illustrate our approach, suppose there are 3 receivers R_1, R_2, R_3 and their associated topology matrix $\mathbf{H}$, with $rank(\mathbf{H}) = 2$ such that the systems can serve both R_1 and R_2 or both R_2 and R_3 in a round. A scheme C can be implemented as follows. In each round with probability of 0.5, we choose the subset $\mathcal{V}_1 = (R_1, R_2)$ to serve and with remaining probability of 0.5, we choose the subset $\mathcal{V}_2 = (R_2, R_3)$ to serve. By applying scheme C with a probabilistic policy $\mathbf{x} = [0.5, 0.5]$, the resulted rate distribution $\mathbf{b}$ of over three receivers R_1, R_2, R_3 is $[0.5, 1, 0.5]$ respectively or can be normalized as $[\frac{1}{4}, \frac{1}{2}, \frac{1}{4}]$.

Due to the weak law of large numbers, it is guaranteed that the average rate distribution achieved by a randomized schedule $\mathbf{x}$ will converge to its true target rate in probability, i.e, if the policy $\mathbf{x}$ is used repeatedly in a large number of rounds, the average rate distribution $\bar{\mathbf{b}}$ would be within ϵ close to the resulted rate distribution $\mathbf{b}$:

$$\lim_{n\to\infty} P(|\bar{\mathbf{b}} - \mathbf{b}| \geq \epsilon) = 0.$$

Suppose the number of bits that three receivers R_1, R_2, R_3 requires are $[500, 1000, 500]$ respectively then the desired rate distribution $\mathbf{b}$ can also be normalized as $[\frac{1}{4}, \frac{1}{2}, \frac{1}{4}]$. Hence, this desired distribution can be achieved by applying the above scheme C.

Max Rate First. We now consider the following proportional rate allocation problem. The notations are similar to the time minimization problem.

- The set $\mathcal{D}$ only includes subset $\mathcal{V}_i$ such that $|\mathcal{V}_i| = k$. Also $0 < d \leq \binom{m}{k}$
- Matrix $\mathbf{A}$ is defined similarly to the time minimization problem.

$$\mathbf{A}_{ij} = \begin{cases} 1 \text{ if the set } \mathcal{V}_i \text{ includes receiver } R_j \\ 0 \text{ otherwise} \end{cases}$$

- Let $\mathbf{x} = [x_1, x_2, \dots, x_d]^T$ where x_i be the probability that $\mathcal{V}_i$ is chosen at each round.

$$\begin{cases} \mathbf{x} \geq 0 \\ \sum_i x_i = 1. \end{cases}$$

- Hence, the resulted rate distribution $\mathbf{r}(\mathbf{x}) = \frac{1}{k}\mathbf{Ax}$.

Our goal is to find a randomized scheme that can operate with a maximum rate while achieving as close as possible to a given target rate allocation. The problem can be described as follows. Let $\mathbf{b} = [b_1, b_2, \dots, b_m]^T$ be the desired rate distribution over all m receivers. Also, $\mathbf{b}$ is normalized such that $\sum_{i=1}^{m} b_i = 1$. The goal of this problem is to find a scheme $\mathbf{x}$ such that the target rate distribution $\mathbf{r}(\mathbf{x}) = \mathbf{b}$, or as close as possible to $\mathbf{b}$ where the distance from the obtained rate allocation distribution $\mathbf{r}(\mathbf{x})$ to the target distribution $\mathbf{b}$ is defined by using vector norm $||\mathbf{r}(\mathbf{x}) - \mathbf{b}|| = ||\frac{1}{k}\mathbf{Ax} - \mathbf{b}||$. Thus, the problem can be formulated in convex optimization form as:

$$\textbf{Problem P2:} \quad \text{Minimize} \quad ||\tfrac{1}{k}\mathbf{Ax} - \mathbf{b}||$$
$$\text{Subject to} \quad \begin{cases} \mathbf{x} \succeq \mathbf{0} \\ \mathbf{1}^T\mathbf{x} = 1 \end{cases} \tag{16}$$

P2 is a non-negative least square problem (NNLS) which can be solved efficiently by the active set method [46].

Proportional Rate First. When WiFO operates at full-rate, it is possible that the desired rate allocation cannot be reached. However, if a precise target rate allocation is required, it is possible to achieve this objective at the cost of reduced overall transmission rate. In particular, we can formulate this problem as follows. Denote $\mathcal{D}^{(k)} \subseteq \mathcal{D}$ as the set which only includes $\mathcal{V}_i$ such that $|\mathcal{V}_i| = k$. Note that when $\mathcal{D}^{(k)}$ is used, the transmission rate is k. Let $\mathbf{A}^{(k)}$ be the matrix representing the set $\mathcal{D}^{(k)}$ (similar to previous $\mathbf{A}$ and $\mathcal{D}$) and $\mathbf{A}^{(k)} \in [0,1]^{m \times d^{(k)}}$ where $d^{(k)} = |\mathcal{D}^{(k)}|$. A policy can be represented by vector $\mathbf{x} = [\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(k)}]$ where $\mathbf{x}^{(i)}$ is a $d^{(i)}$-vector corresponding to the probability that $\mathbf{A}^{(i)}$ is chosen. Next, the average rate can be computed as:

$$R = k \sum \mathbf{x}^{(k)} + (k-1) \sum \mathbf{x}^{(k-1)} + \cdots + \sum \mathbf{x}^{(1)} = \sum_{i=1}^{k} \mathbf{i}^T \mathbf{x}^{(i)},$$

where each $\mathbf{i}^T$ is a $d^{(i)}$-vector with all entries of value i for $i - 1, \ldots, k$. Also the rate allocation distribution is:

$$\frac{1}{k}\mathbf{A}^{(k)}\mathbf{x}^{(k)} + \frac{1}{k-1}\mathbf{A}^{(k-1)}\mathbf{x}^{(k-1)} + \cdots + \mathbf{A}^{(1)}\mathbf{x}^{(1)} = \sum_{i=1}^{k} \frac{1}{i}\mathbf{A}^{(i)}\mathbf{x}^{(i)} = \mathbf{b}.$$

Now, the problem can be formulated as:

$$\textbf{Problem P3:} \quad \text{Maximize} \quad \sum_{i=1}^{k} \mathbf{i}^T \mathbf{x}^{(i)}$$

$$\text{Subject to} \quad \begin{cases} \mathbf{x} \succeq \mathbf{0} \\ \mathbf{1}^T \mathbf{x} = 1 \\ \sum_{i=1}^{k} \frac{1}{i} \mathbf{A}^{(i)} \mathbf{x}^{(i)} = \mathbf{b}. \end{cases} \quad (17)$$

P3 can be efficiently solved via convex optimization algorithms [47].

VI. Simulation Results

A. Asymtoptotic Performance

To quantify the interference gain of LAC over BC as described in Section IV-D, Fig. 5 shows the empirical probability of being able to send bits at full rate for LAC and BC schemes as a function of number of the receivers n. As seen, this probability for LAC is always larger than that of BC scheme. Furthermore, it decreases and approaches 0.289, verifying the correctness of Proposition 2. On the other hand, the same probability decreases to zero quickly for the BC scheme, making the BC scheme extremely inefficient for a densely populated area as shown by Proposition 3.

Also, Fig. 6 shows that the average rate of the LAC scheme is much larger than that of BC. Furthermore, the rate of LAC shows an roughly linear relation to the number of transmitters while the rate of BC decreases as the number of transmitter increases.

B. Proportional Rate Allocation

We now provide performance of *Max Rate First* scheme. Fig. 7 plots the average optimal values of the objective function vs. rank of $\mathbf{H}$'s. In this simulation, the number of transmitters is equal to the number of receivers ($m = n = 10$). The topology matrices $\mathbf{H}$ are generated uniformly at random, and the their ranks are noted. However, for any $\mathbf{H}$, we require

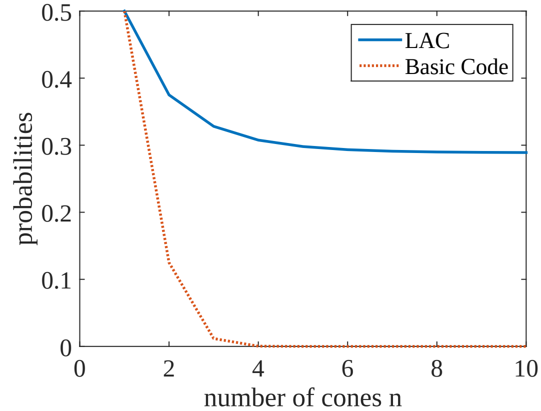

Fig. 5: Full rate transmission probabilities versus different

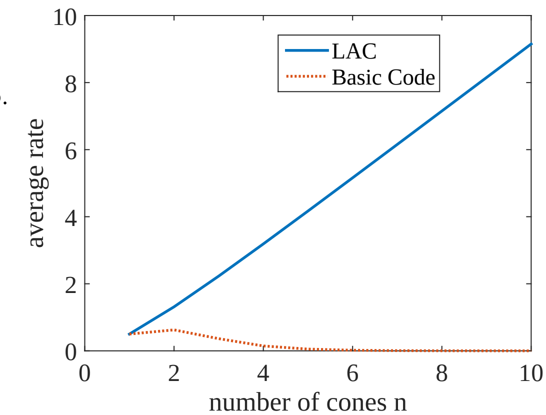

Fig. 6: Average rate versus different number of cones

that any receiver is covered by at least one transmitter. The desired rate allocation is a constant vector:

$$\mathbf{b} = \begin{bmatrix} 0.19 & 0.21 & 0.18 & 0.02 & 0.1 & 0.05 & 0.07 & 0.08 & 0.01 & 0.09 \end{bmatrix}^T.$$

A small distance implies that the solution of the corresponding randomized policy approximates the desired rate allocation well. As seen in Fig. 7, when the rank of H increases, the system can transmit at full rate which exactly equals to the rank. However, all receivers need to participate in transmission in every time slot. Thus, the policy is not sufficiently flexible to achieve the proportional target rate allocation. For example, when the matrix $\mathbf{H}$ is full rank, the only possible solution is $[0.1, 0.1, \ldots, 0.1]$ ($n = 10$) which can be far way from the target rate allocation. On the other hand, when $\mathbf{H}$ has low rank, at each time slot, there are several options of choosing which receivers to serve. Hence, the system would have more flexibility to allocate the rate as desired. For *Proportional Rate First* scheme, Fig. 8 shows the overall rate of the system vs. the rank of $\mathbf{H}$. The simulation parameters are identical to that of Fig. 7. The proportional rate is now guaranteed to be the exact target rate allocation. On the other hand, the solution might not achieve full rate. In fact, as the rank increases, hence the full rate increases, the gap between the overall resulted rate and the full rate increases.

We note that since the rank of a matrix cannot exceed its dimensions, therefore $k \in (0, \min(m, n))$ where m and n denote the numbers of transmitters and receivers. When there are many transmitters and receivers, one can expect that k

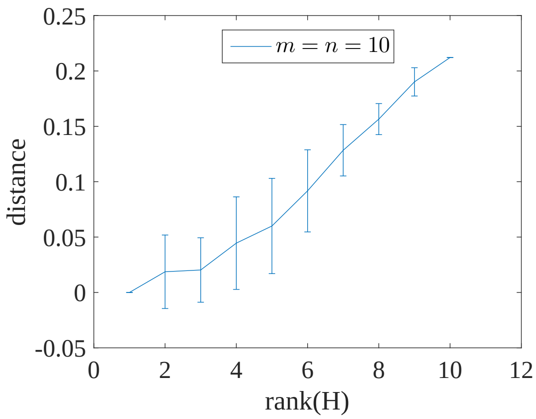

Fig. 7: Average optimal value versus k (rank of topology

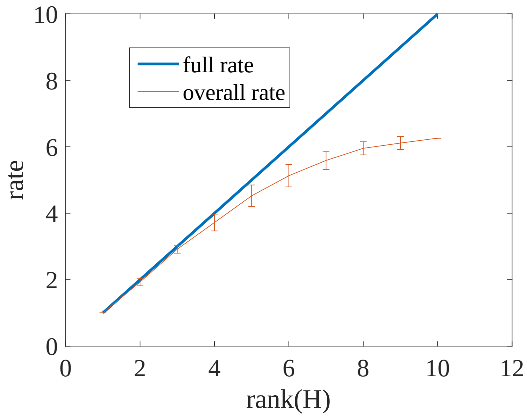

Fig. 8: Average rate versus k (rank of topology matrix $\mathbf{H}$)

will likely to be large. With higher values of k, the total bit rate of the system will increase. This is intuitive since there are more opportunities to transmit information independently from either larger number of independent transmitters or larger number of receivers

VII. Conclusions

In this paper, we briefly introduce WiFO, a hybrid high-speed WiFi-FSO network for Gbps wireless local area network (WLAN) femtocells with seamless mobility. We introduce a novel location assisted coding (LAC) technique, based on which, the number of novel rate allocation algorithms are proposed to increase throughput and reduce interference for multiple users in a dense array of overlapped femtocells. Both theoretical analysis and numerical results show orders of magnitude increase in throughput using LAC over existing schemes for various random topologies.

Appendix

Proof of Proposition 2

Proof: The average number of bits that the systems can transmit at a time (transmission rate) using LAC is computed as follows.

$$\mathbf{E}[R_{LAC}] = \mathbf{E}[rank(\mathbf{H})] = \sum_{k=1}^{n} \mathbf{P}(rank(\mathbf{H}) = k)k.$$

From [48], the expected number of linear dependencies of the rows of $\mathbf{H}$ in $\mathbf{GF}(q)$ is:

$$\mathbf{E}[l(\mathbf{H})] = \sum_{k=1}^{n} \binom{n}{k} \gamma^k (1-\gamma)^{n-k} [1 + (q-1)(1-p/\gamma)^k]^n.$$

where $\gamma = 1 - 1/q$ with $q = 2$ in this case. Since $l(\mathbf{H}) = q^{n-rank(\mathbf{H})} - 1$, the approximation of expected rank of $\mathbf{H}$, i.e., the average transmission rate can be computed as:

$$\mathbf{E}[rank(\mathbf{H})] \approx n - \log_q(\mathbf{E}[l(\mathbf{H})] + 1).$$

When $p = 0.5$, $\mathbf{H}$ is drawn uniformly at random from all the possible matrices whose entries consists of 0 or 1. Denote $C(k)$ as the number of $n \times n$ matrix of rank k in $\mathbf{GF}(2)$. According to [49], we have:

$$C(k) = \prod_{i=0}^{k-1} \frac{(2^n - 2^i)^2}{2^k - 2^i}. \tag{18}$$

Also, the total number of $n \times n$ matrix in $\mathbf{GF}(2)$ is 2^{n^2} then:

$$\mathbf{P}(rank(\mathbf{H}) = k) = \frac{C(k)}{2^{n^2}} = \frac{1}{2^{n^2}} \prod_{i=0}^{k-1} \frac{(2^n - 2^i)^2}{2^k - 2^i}. \tag{19}$$

Therefore, the average rate (average number of bits transmitted at a time) for LAC is shown as follows.

$$\begin{aligned} \mathbf{E}[R_{LAC}] = \mathbf{E}[rank(\mathbf{H})] &= \sum_{k=1}^{n} k \times \mathbf{P}(rank(\mathbf{H} = k)) \\ &= \frac{1}{2^{n^2}} \sum_{k=1}^{n} k \prod_{i=0}^{k-1} \frac{(2^n - 2^i)^2}{2^k - 2^i} \end{aligned}$$

Now, the probability that the matrix $\mathbf{H}$ is invertible [50] is:

$$\mathbf{P}(rank(\mathbf{H}) = n) = (1-2^{-n}) \dots (1-2^{-2})(1-2^{-1}) = \prod_{i=1}^{n} (1 - \frac{1}{2^i}). \tag{20}$$

From [50], one can show that

$$\lim_{n \to \infty} \mathbf{P}(rank(\mathbf{H}) = n) \approx 0.289. \tag{21}$$

■

Proof of Proposition 3

Proof: Using BC, k bits can be transmitted if there are k receivers located in k non-overlapped regions and in each of these k regions there is only one receiver. As a result, $\mathbf{H}$ would have k "1" entries and these k entries is the only non-zero entry in its row and its column. Denote $D(k)$ as the number of $n \times n$ matrices in $\mathbf{GF}(2)$ that have at least k "1" entries satisfying the condition. We have:

$$D(k) = k! \binom{n}{k}^2 2^{(n-k)^2}, k = 1, \dots, n, D(n+1) = 0. \tag{22}$$

Therefore, the number of matrices in $\mathbf{GF}(2)$ that have exactly k entries satisfying the condition is $D(k) - D(k+1)$. Consequently, the average rate is:

$$\mathbf{E}[R_{BC}] = \frac{1}{2^{n^2}} \sum_{k=1}^{n} (D(k) - D(k+1))k = \frac{1}{2^{n^2}} \sum_{k=1}^{n} D(k) \tag{23}$$

■

Proof of Proposition 4

Proof: Since the matrix $\mathbf{H}$ is of rank k in $\mathbf{GP}(q)$ and rows $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are linearly independent, the other $m-k$ rows $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{m-k}$ could be represented as linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$. In other words, for any row $\mathbf{v}_i$, we have:

$$\mathbf{v}_i = \sum_{q=1}^{k} c_s \mathbf{u}_s, \tag{24}$$

where $c_s \in \{0, 1, 2, \ldots, q-1\}$ and at least one of the coefficient c_s's is different from 0 since $\mathbf{H}$ contains no row with all zero entries. Denote that non-zero coefficient as $c_{s'}$. Now we just need to pick $\mathbf{u}_{s'}$ to be replaced by $\mathbf{v}_i$ and still obtain a set of linearly independent rows. We will prove this by contradiction.

Suppose that $\mathbf{u}_1$, $\mathbf{u}_2$, ..., $\mathbf{u}_{s'-1}$, $\mathbf{v}_i$, $\mathbf{u}_{s'+1}$, ..., $\mathbf{u}_k$ are not linearly independent. As a result, since $\mathbf{u}_1$, $\mathbf{u}_2$, ..., $\mathbf{u}_{s'-1}$, $\mathbf{u}_{s'+1}$, ..., $\mathbf{u}_k$ are linearly independent, $\mathbf{v}_i$ could be represented by a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, ..., $\mathbf{u}_{s'-1}$, $\mathbf{u}_{s'+1}$, ..., $\mathbf{u}_k$. In other words,

$$\mathbf{v}_i = \sum_{s \neq s'} c'_s \mathbf{u}_s. \tag{25}$$

From (24) and (25),

$$\begin{aligned} & \textstyle\sum_{s=1}^{k} c_s \mathbf{u}_s = \sum_{s \neq s'} c'_s \mathbf{u}_s, \\ \Leftrightarrow \quad & c_{s'} \mathbf{u}_{s'} = \textstyle\sum_{s \neq s'} (c_s - c'_s) \mathbf{u}_s, \end{aligned}$$

or $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are linearly dependent which leads to contradiction. ■

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