Approximate Inference 1

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Forward Sampling

- This section on approximate inference relies on samples / particles
- Full particles: complete assignment to all network variables eg. ($X_1 = x_1$, $X_2 = x_2$, ..., $X_N = x_N$)

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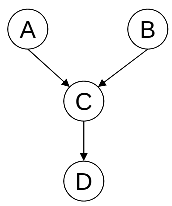
Forward Sampling

- Topological sort or order: An ordering of the nodes in the DAG where X comes before Y in the ordering if there is a directed path from X to Y in the graph.
- A topological order is equivalent to a partial order on the nodes of the graph
- There may be several topological orderings



Examples of Topological orders:

A,B,C,D

B,A,C,D

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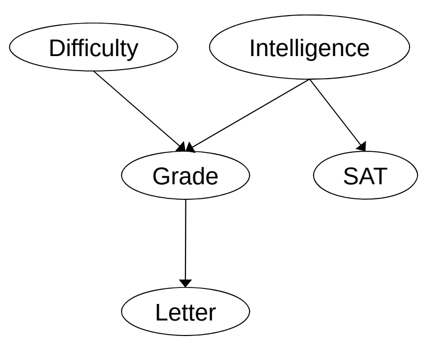
Forward Sampling

Student Example

D	P(D)
low	0.6
high	0.4

D	I	G	P(G|D,I)
low	low	C	0.3
low	low	B	0.4
low	low	A	0.3
low	high	C	0.02
low	high	B	0.08
low	high	A	0.9
high	low	C	0.7
high	low	B	0.25
high	low	A	0.05
high	high	C	0.2
high	high	B	0.3
high	high	A	0.5



I	P(I)
low	0.7
high	0.3

I	S	P(S|I)
low	low	0.95
low	high	0.05
high	low	0.2
high	high	0.8

G	L	P(L|G)
C	weak	0.99
C	strong	0.01
B	weak	0.4
B	strong	0.6
A	weak	0.1
A	strong	0.9

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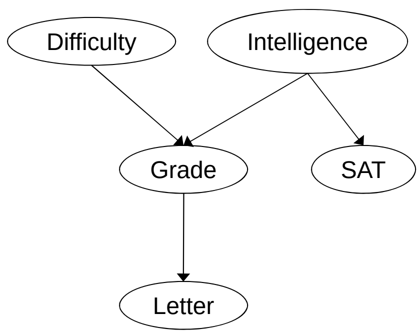
Forward Sampling



Topological ordering: D, I, G, S, L

1. Sample D from P(D) (Say you get D=high)
2. Sample I from P(I) (Say you get I=low)
3. Sample G from P(G|I=low,D=high) (Say you get G=C)
4. Sample S from P(S|I=low) (Say you get S=low)
5. Sample L from P(L|G=C) (Say you get L=weak)

You now have a sample (D=high, I=low, G=C, S=low, L=weak)

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Forward Sampling

Suppose you want to calculate $P(X_1= x_1, X_2= x_2, \ldots, X_n= x_n)$ using forward sampling on a Bayesian network. The algorithm:

1. Do a topological sort of the nodes in the Bayesian network.

2. For *j = 1* to *NUM_SAMPLES*
   For each node *i* in the ordering (starting from the top of the Bayesian network down)
   Sample the value $\hat{x}_i$ from the distribution $P(X_i \mid Parents(X_i))$
   Add $\{\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n\}$ to your collection of samples

3. Let $P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$

$$\approx \frac{\#\text{ of samples with } X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n}{NUM\_SAMPLES}$$

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Forward Sampling

- How do you sample from $P(X_i \mid Parents(X_i))$?
- Note: $P(X_i \mid Parents(X_i))$ is a multinomial distribution $P(x_i^1, ..., x_i^k \mid \theta_1, ..., \theta_k)$?

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Forward Sampling

How do you sample from a multinomial distribution $P(x_i^1, ..., x_i^k \mid \theta_1, ..., \theta_k)$?

- Generate a sample s uniformly from [0,1]
- Partition interval into k subintervals: $[0, \theta_1)$, $[\theta_1, \theta_1+\theta_2)$, ...
- More generally, the ith interval is

$$\left[\sum_{j=1}^{i-1}\theta_j, \sum_{j=1}^{i}\theta_j\right)$$

- If s is in the ith interval, the sample value is x_i.
- Use binary search to find the interval for s in time $O(\log k)$

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Forward Sampling

Suppose your list of samples looks like the following table:

D	I	G	S	L
low	low	B	low	weak
low	high	A	high	strong
low	high	A	high	weak
high	high	A	high	strong
high	low	C	low	weak

P(I=high) = 3/5 = 0.6

Note that this value becomes a lot more accurate as the number of samples heads to infinity.

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Forward Sampling

- From a set of samples ***D*** = *{ξ[1], ..., ξ[M]}*, we can estimate the expectation of any function *f* as:

$$\hat{E}_D(f) = \frac{1}{M}\sum_{m=1}^{M} f(\xi[m])$$

- To estimate *P(**y**)*

$$\hat{P}_D(\mathbf{y}) = \frac{1}{M}\sum_{m=1}^{M} \boldsymbol{I}\{\mathbf{y}[m] = \mathbf{y}\}$$

This will return 1 if the sample ***ξ[m]*** matches the values in ***y*** (eg. ***y*** is I=high) and 0 otherwise.

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Rejection Sampling

Rejection Sampling

What if we want to estimate *P(**y**|**E**=**e**)*?

- Rejection sampling: do forward sampling but throw out samples where ***E≠e***

Example:

P(I=high|L=weak) = 1/3

D	**I**	**G**	**S**	**L**
low	low	B	low	weak
low	high	A	high	strong
low	high	A	high	weak
high	high	A	high	strong
high	low	C	low	weak

Rejection Sampling

What if the evidence ***E***=***e*** is very very rare?

- For example, if $P(\mathbf{e}) = 0.001$, then for 10,000 samples, we get 10 unrejected samples
- To obtain at least M^* unrejected samples, we need to generate on average $M = M^*/P(\mathbf{e})$ samples
- If evidence is rare, we end up generating a lot of samples which wastes time

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Rejection Sampling

Bad news:

- Rare evidence is the norm!
- As # of evidence variables $k = |\mathbf{E}|$ grows, the probability of the evidence decreases exponentially with k

Need something better than rejection sampling!

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Likelihood Weighting

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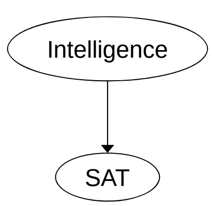
Likelihood Weighting

Intuition: Weight samples according to probability of the evidence



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high	low	0.2
high	high	0.8

Drawing I = high and S = high should be 80% of a sample

Drawing I = low and S = high should be 5% of a sample

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Likelihood Weighting

Weighted particles:

$$D = \langle \xi[1], w[1] \rangle, \ldots, \langle \xi[M], w[M] \rangle$$

Estimate:

$$\hat{P}_D(\boldsymbol{y} \mid \boldsymbol{e}) = \frac{\sum_{m=1}^{M} w[m] \boldsymbol{I}\{\boldsymbol{y}[m] = \boldsymbol{y}\}}{\sum_{m=1}^{M} w[m]}$$

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Likelihood Weighting

```
Procedure LW-Sample(
    β,        // Bayesian network over 𝒳
    Z=z       // Event in the network
)
Let X_1, …, X_n be a topological ordering of 𝒳
w ← 1
for i = 1, …, n
  u_i ← x<Pa_Xi>        // Assignment to Pa_Xi in x_1, …, x_i-1
  if X_i ∉ Z then
    Sample x_i from P(X_i | u_i)
  else
    x_i ← z<X_i> // Assignment to X_i in z
    w ← w · P(x_i | u_i) // Multiply weight by probability of desired value
  return (x_1, …, x_n), w
```

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