CS325 Assignment #5, Summer 2009, OSU

This assignment is due on Friday, 8/7/2009 at the beginning of class. You are required to type your solution.

1. For each of the following problems, prove that it is in NP (i.e. give a certificate, a verifier, and a runtime analysis showing your verifier is polynomial time).

   (a) (3 points) The Directed Hamiltonian Cycle Problem:
   Given a directed graph $G = (V, E)$, does it contain a hamiltonian cycle? Recall that a hamiltonian cycle is a cycle (i.e. it returns to the start node), that visits each node in V exactly once.

   (b) (3 points) The Traveling Salesman Problem (TSP):
   Given a directed graph $G = (V, E)$, with edge weights $w(u, v)$ for all $(u, v) \in E$, is there a path that visits each node in V exactly once, with total path weight $\leq k$?

   (c) (3 points) The (Undirected) Vertex Cover (VC) Problem:
   Given an undirected graph $G = (V, E)$, does it contain a vertex cover with $\leq k$ nodes? Recall that a vertex cover is a subset of nodes $S \subseteq V$ such that every edge has at least one end in S (i.e. edges are being covered, and the thing covering them is nodes).

2. A store is trying to analyze the behavior of its customers. They maintain a two-dimensional array A, where the rows correspond to its customers, and the columns correspond to the products it sells. The entry $A[i, j]$ specifies the quantity of the product j and that has been purchased by customer i. Let us say a subset S of the customers is diverse if no two of them have ever bought the same product.

   Now we define the Diverse Subset Problem as follows: Given an $m \times n$ array A as defined above, and a number $k \leq m$, is there a subset of at least K customers that is diverse?

   (a) (3 points) Prove Diverse Subset $\in$ NP

   (b) (5 points) The Set Packing problem is: Given a set U of n elements, a collection $S_1, ..., S_m$ of subsets of U, and a number k, does there exist a collection of at least k of these subsets such that no two of them intersect?

Assume you know that Set Packing $\in$ NP-complete, and prove that Diverse Subset $\in$ NP-hard (i.e. show that Set Packing $\leq_p$ Diverse Subset).

Together, (a) and (b) prove that Diverse Subset $\in$ NP-complete.

3. Suppose you're helping to organize a summer camp, and the following problem comes up. The camp is supposed to have at least one counselor who's skilled at each of the n sports covered by the camp. They have received job applications from m potential counselors. For each of the n sports, there is some subset of the m applications qualified in that sport. The question is: For a given number $k \leq m$, is it possible to hire at most k of the counselors and have at least one counselor qualified in each of the n sports? We call this the Efficient Recruiting Problem. You will prove that Efficient Recruiting $\in$ NP-complete.

   (a) (3 points) Prove Efficient Recruiting $\in$ NP

   (b) (5 points) The Set Cover problem is: Given a set U of n elements, a collection $S_1, ..., S_m$ of subsets of U, and a number k, does there exist a collection of at most k of these subsets whose union is equal to all of U?

   Assume you know that Set Cover $\in$ NP-complete, and prove that Efficient Recruiting $\in$ NP-hard.

4. (1 point) Write a question something we have covered in class that you would like me to review.