Space Shuttle Experiment Optimization

- Experiments $X = \{X_1, X_2, ..., X_m\}$
- Profit p_j for experiment Xj
- Instruments $I = \{I_1, I_2, ..., I_n\}$
- R_j = instruments required for X_j
- c_k = cost of taking instrument I_k
- Net revenue = profit from experiments - cost of instruments
- Goal: Find a set of experiments that maximizes net revenue

$X=\{X_1,X_2,X_3\}$ $p_1=10$ $p_2=6$ $p_3=6$

$I=\{I_1,I_2,I_3,I_4\}$ $c_1=3$ $c_2=2$ $c_3=5$ $c_4=7$

$R_1=\{I_1,I_2\}$ $R_2=\{I_1,I_3\}$ $R3=\{I_3,I_4\}$

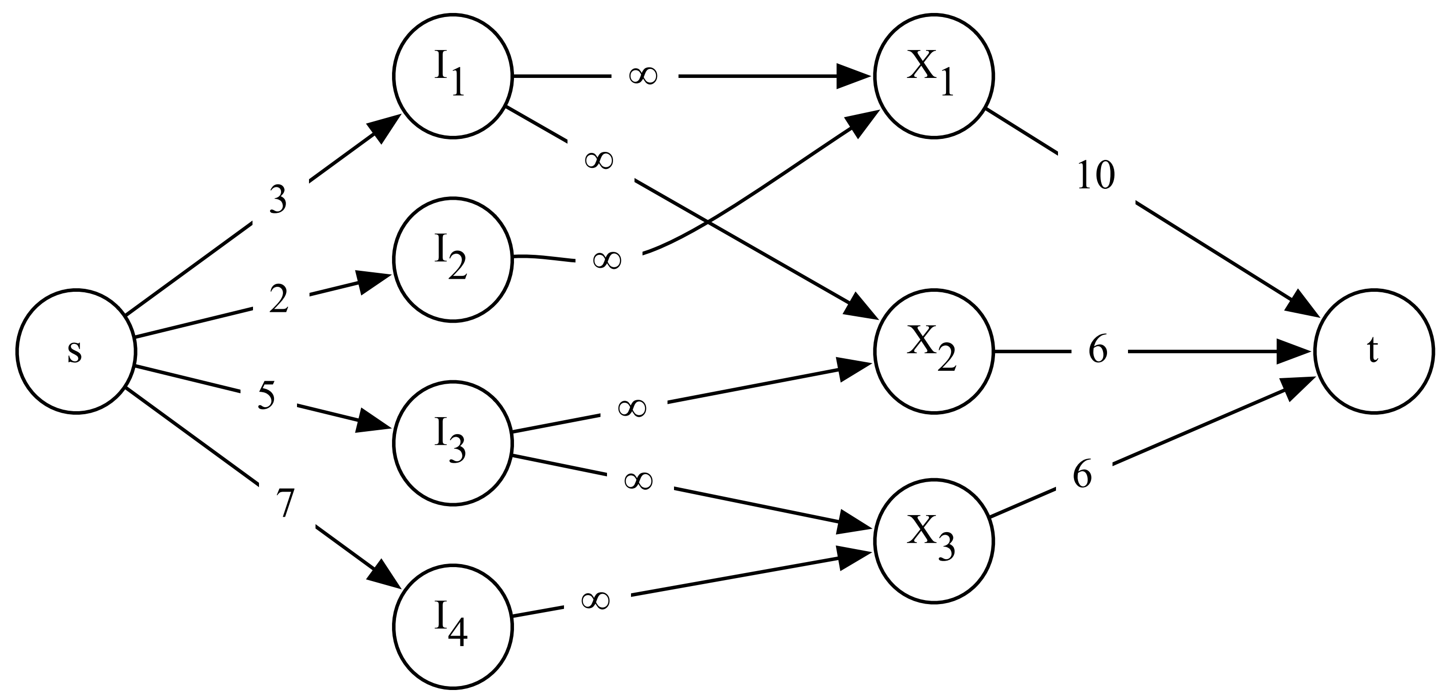
Experiments	Instruments	Profit	Cost	Net Revenue
1	1,2	10	5	5
2	1,3	6	8	-2
3	3,4	6	12	-6
1,2	1,2,3	16	10	6
1,3	1,2,3,4	16	17	-1
2,3	1,3,4	12	15	-3
1,2,3	1,2,3,4	22	17	5

There are 2^n subsets of n experiments

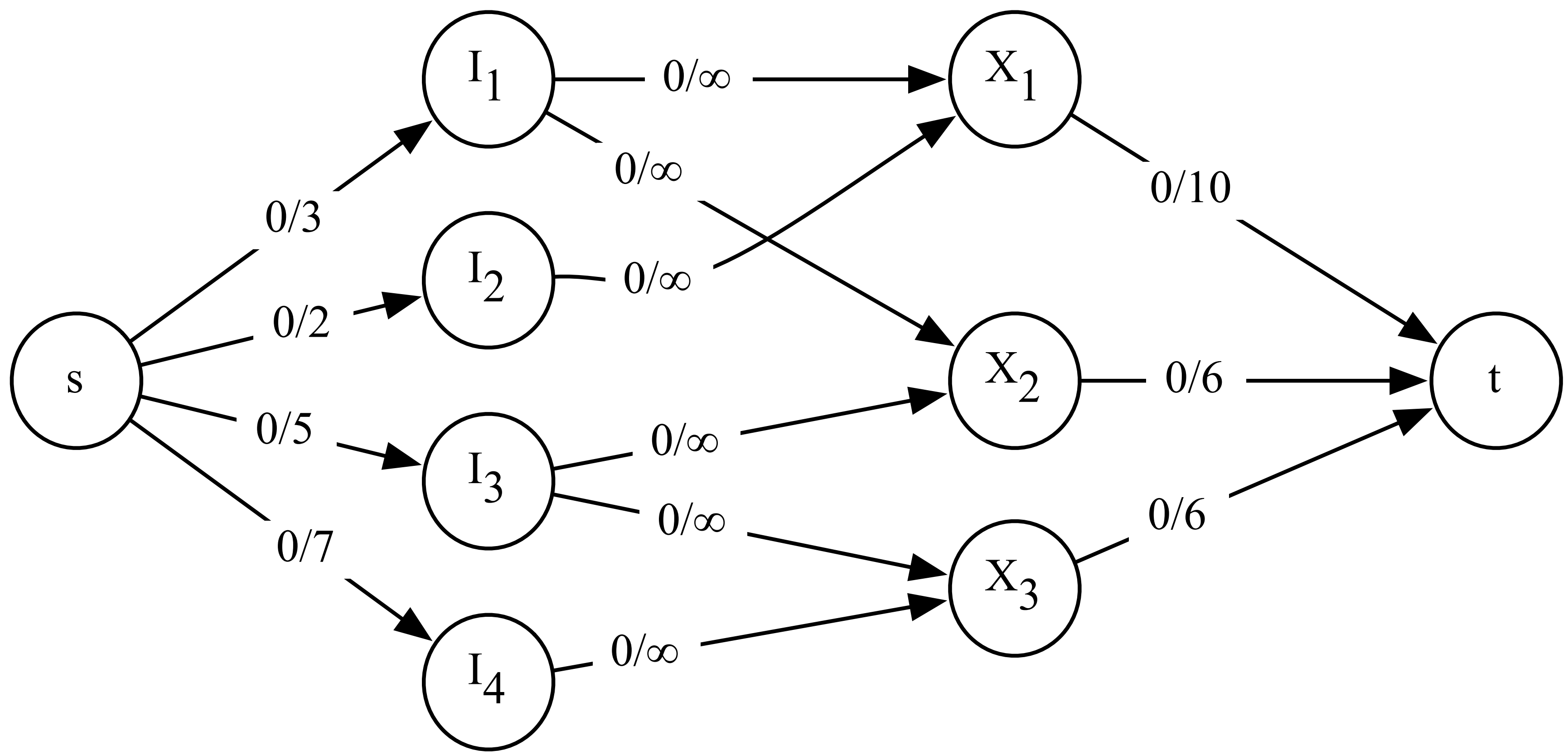
Flow Network:

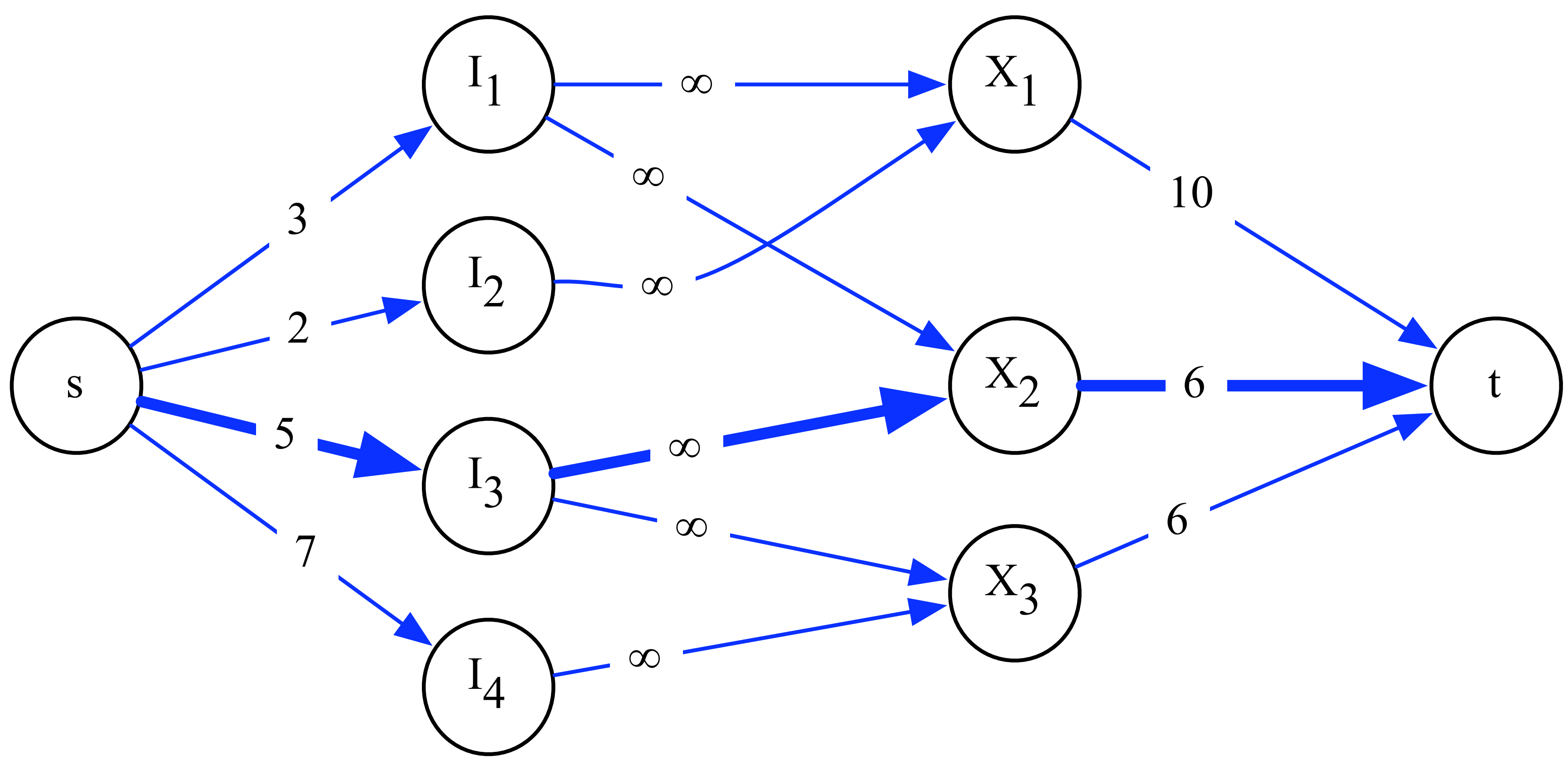
- <u>Nodes:</u> s, t, I_1, ..., I_n, X_1, ..., X_m
- <u>Edges:</u>
  - From s to each I_k w/ capacity c_k
  - From each X_j to t w/ capacity p_j
  - If $I_k \in R_j$ there is an edge from I_k to X_j w/ capacity ∞

$X=\{X_1,X_2,X_3\}$ $p_1=10$ $p_2=6$ $p_3=6$
$I=\{I_1,I_2,I_3,I_4\}$ $c_1=3$ $c_2=2$ $c_3=5$ $c_4=7$
$R_1=\{I_1,I_2\}$ $R_2=\{I_1,I_3\}$ $R3=\{I_3,I_4\}$

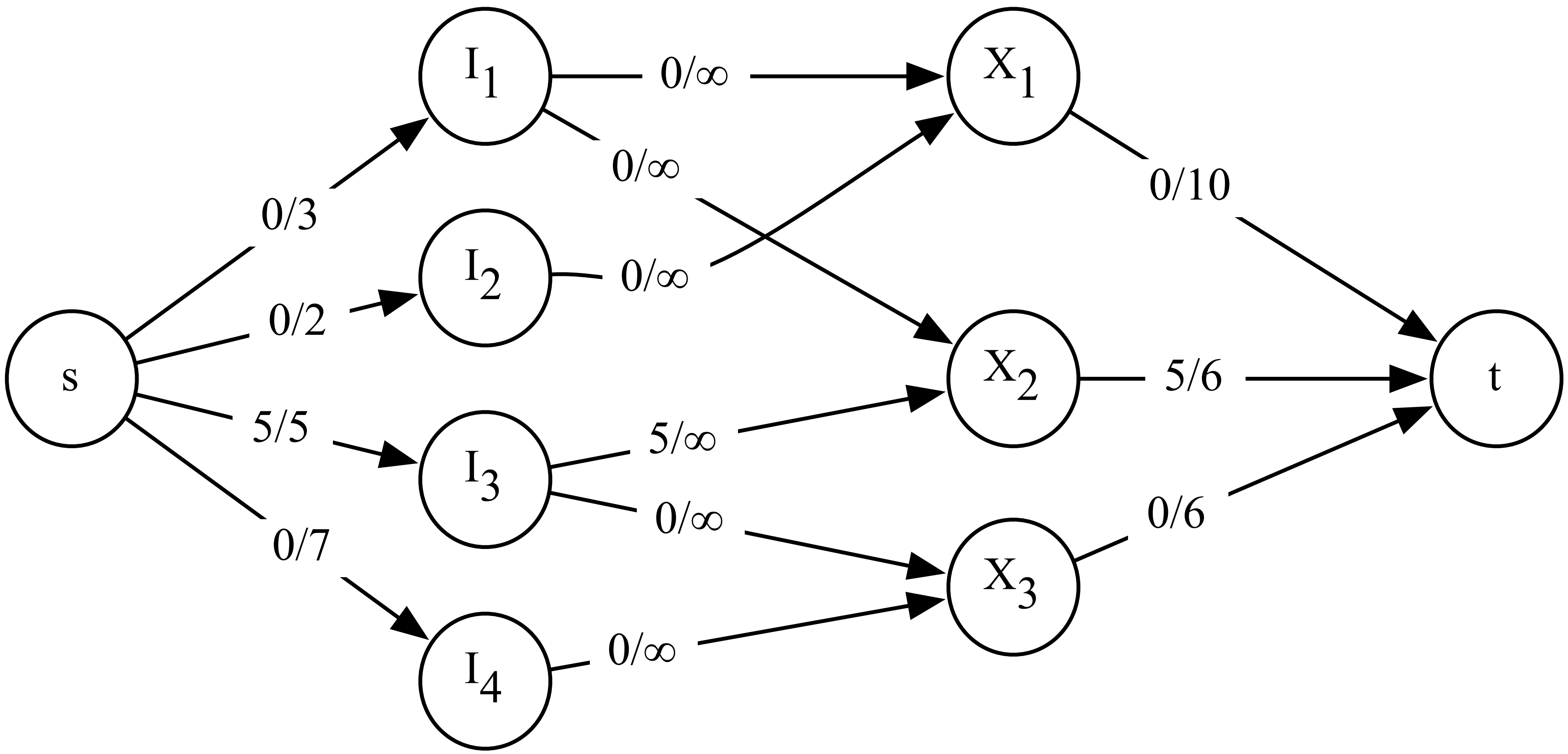


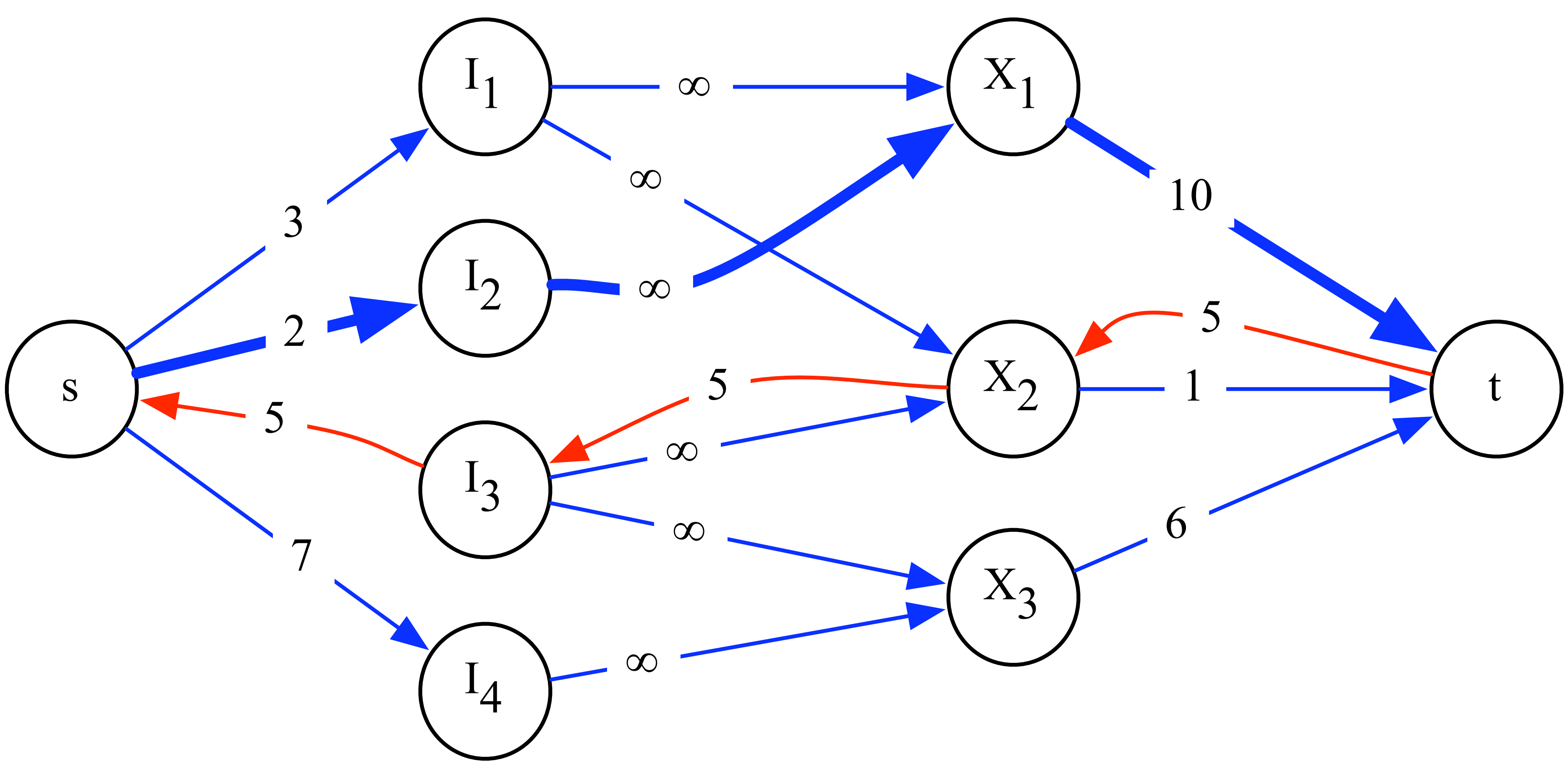
= capacity





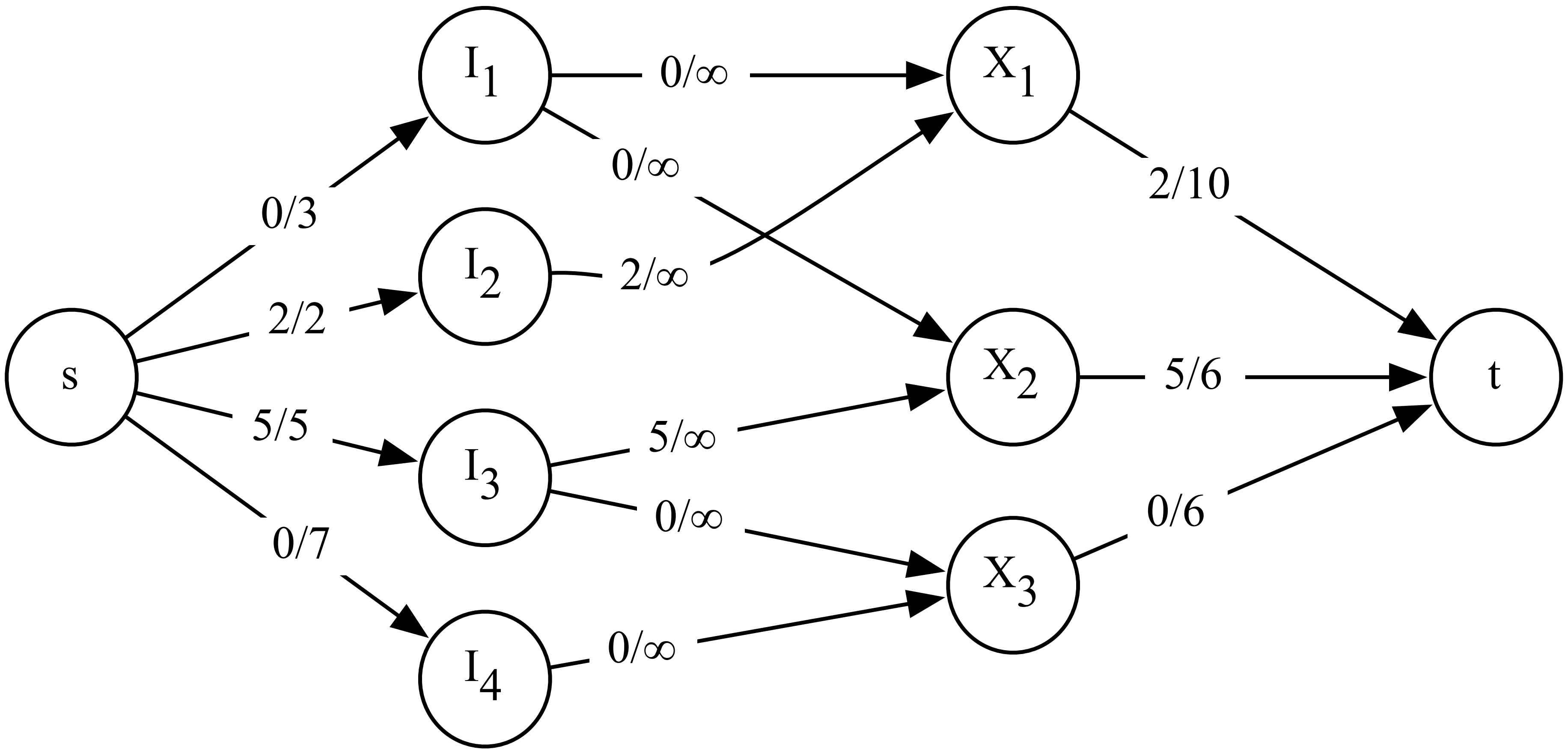
Augment s-I_1-X_2-t
bottleneck = 5

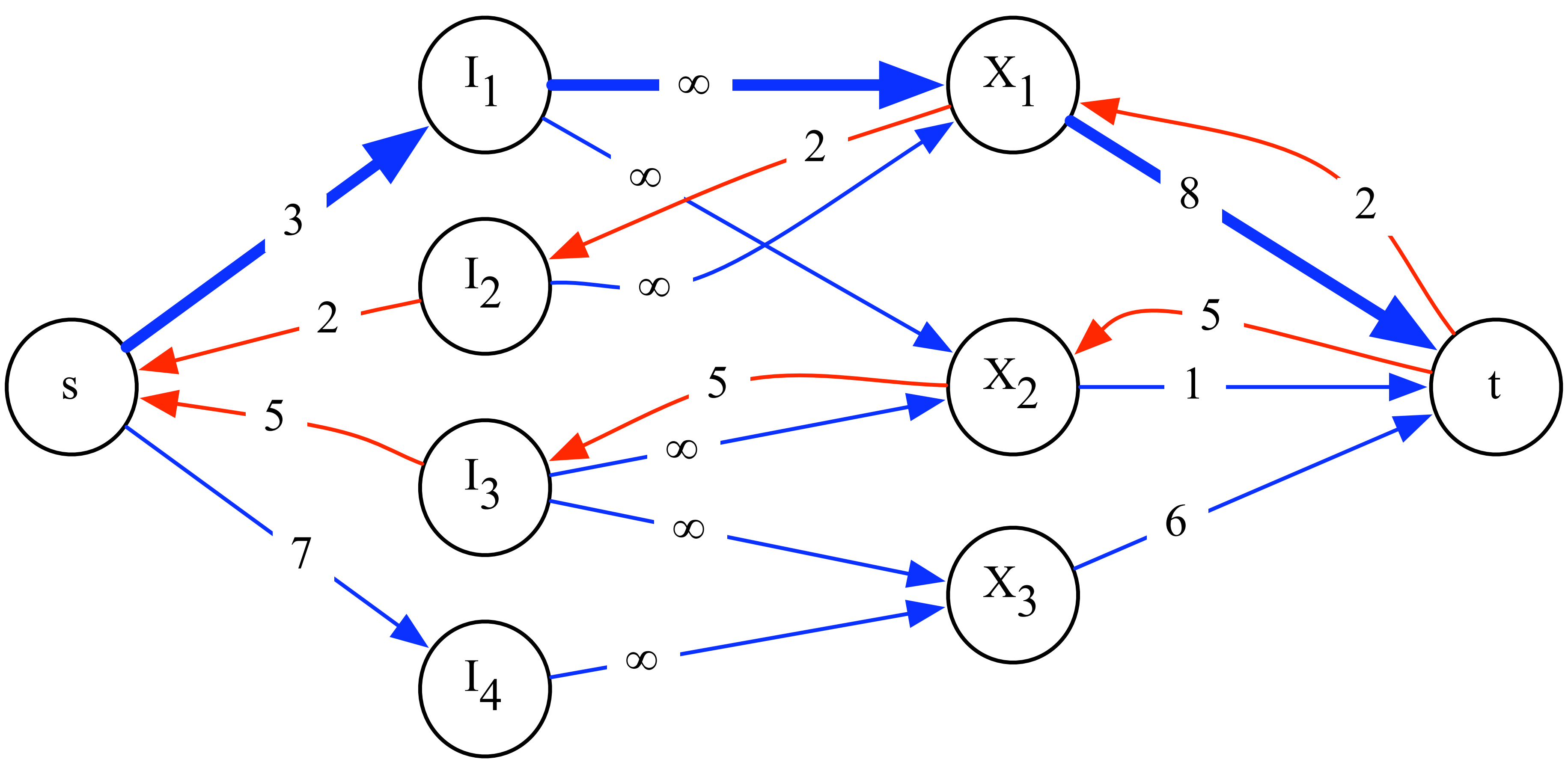




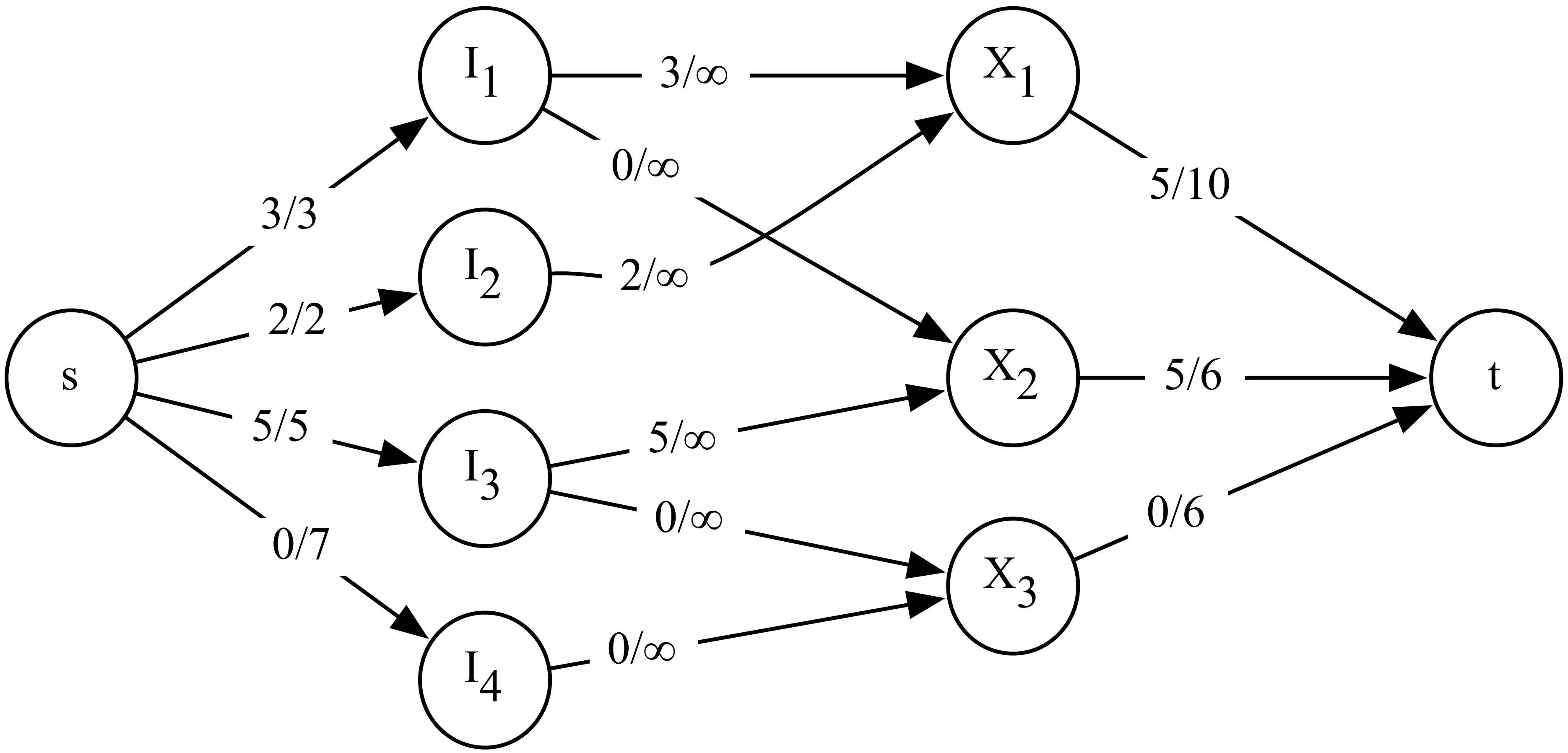
Augment s-I_2-X_1-t

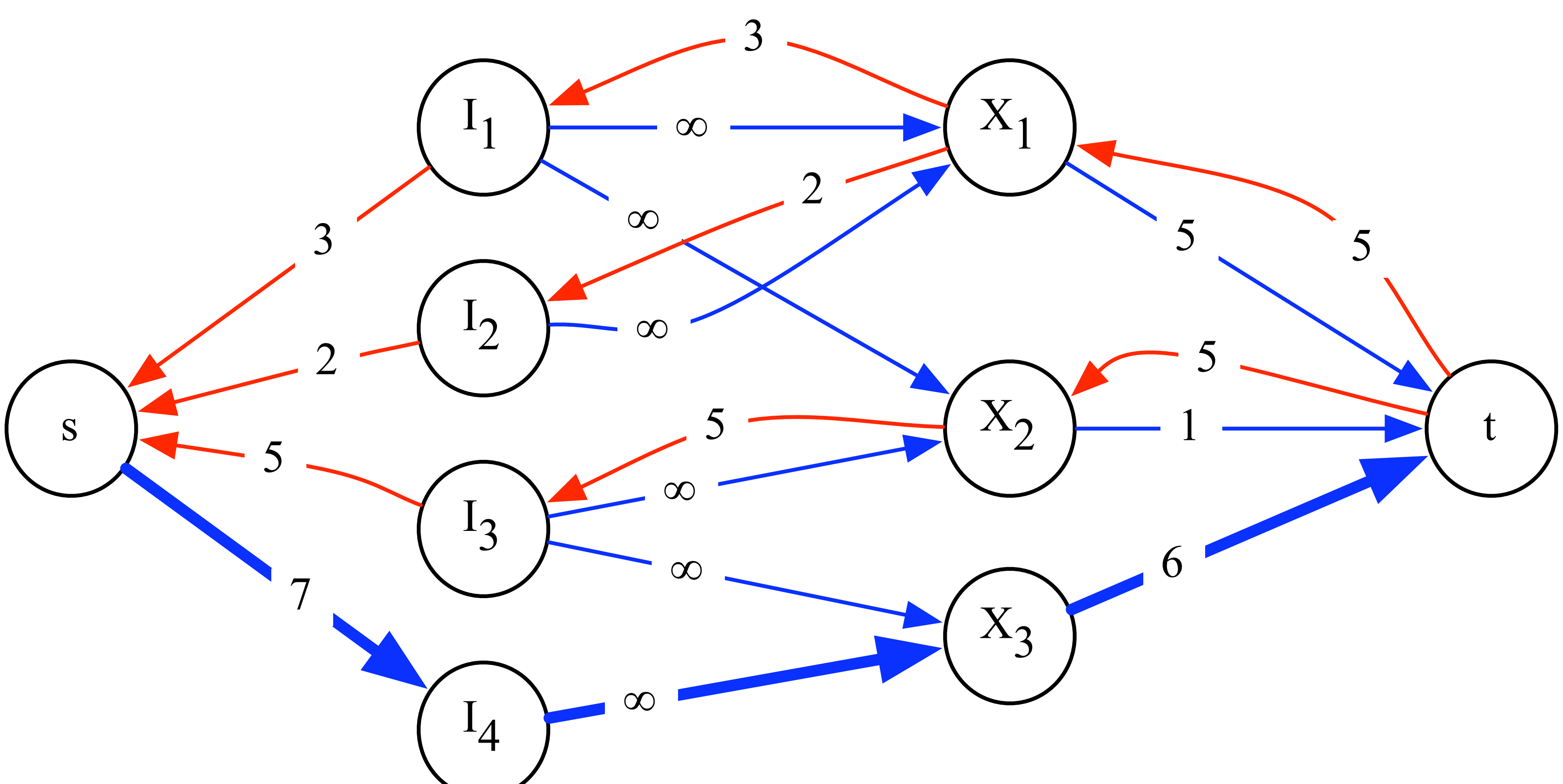
bottleneck = 2



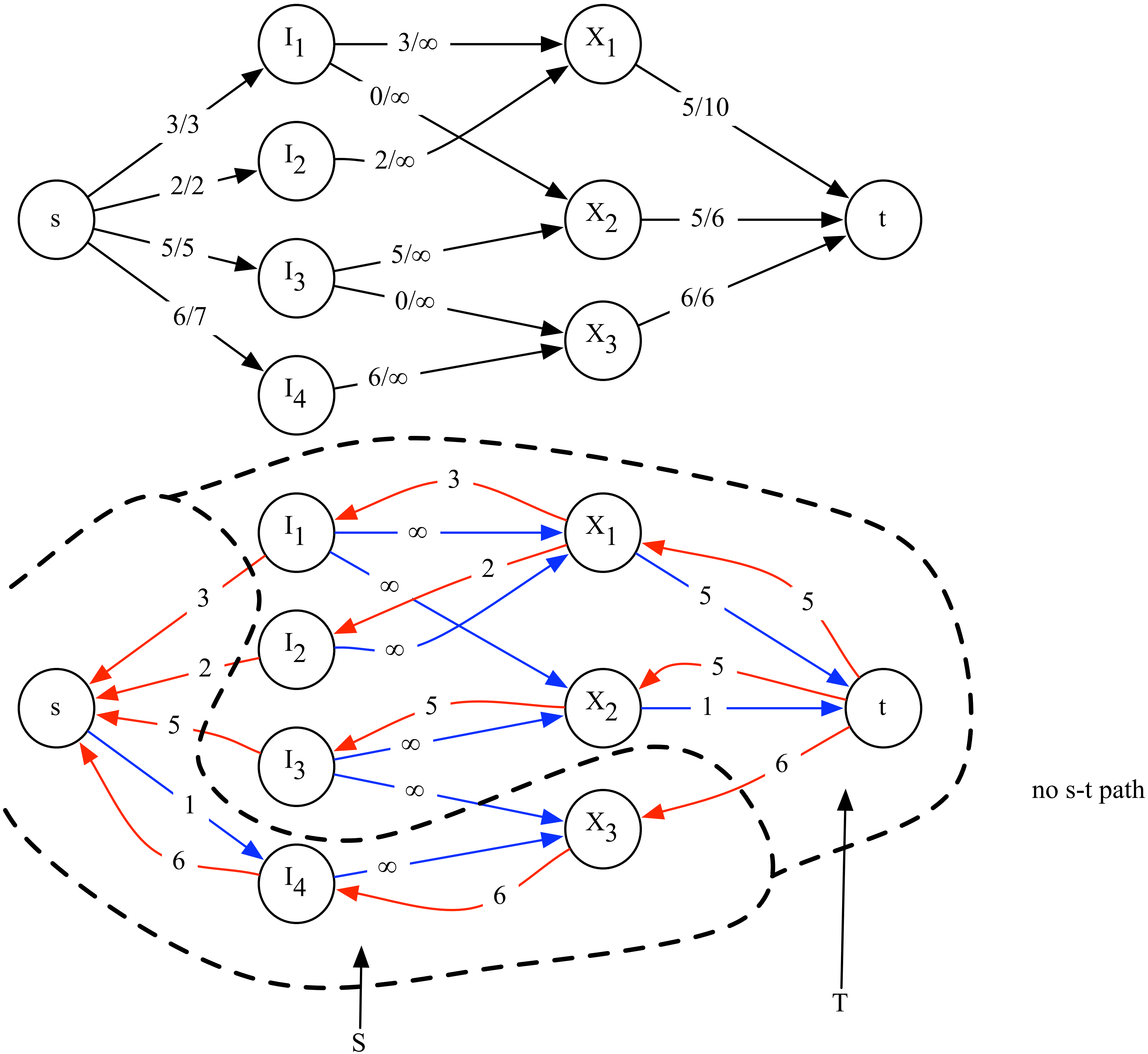


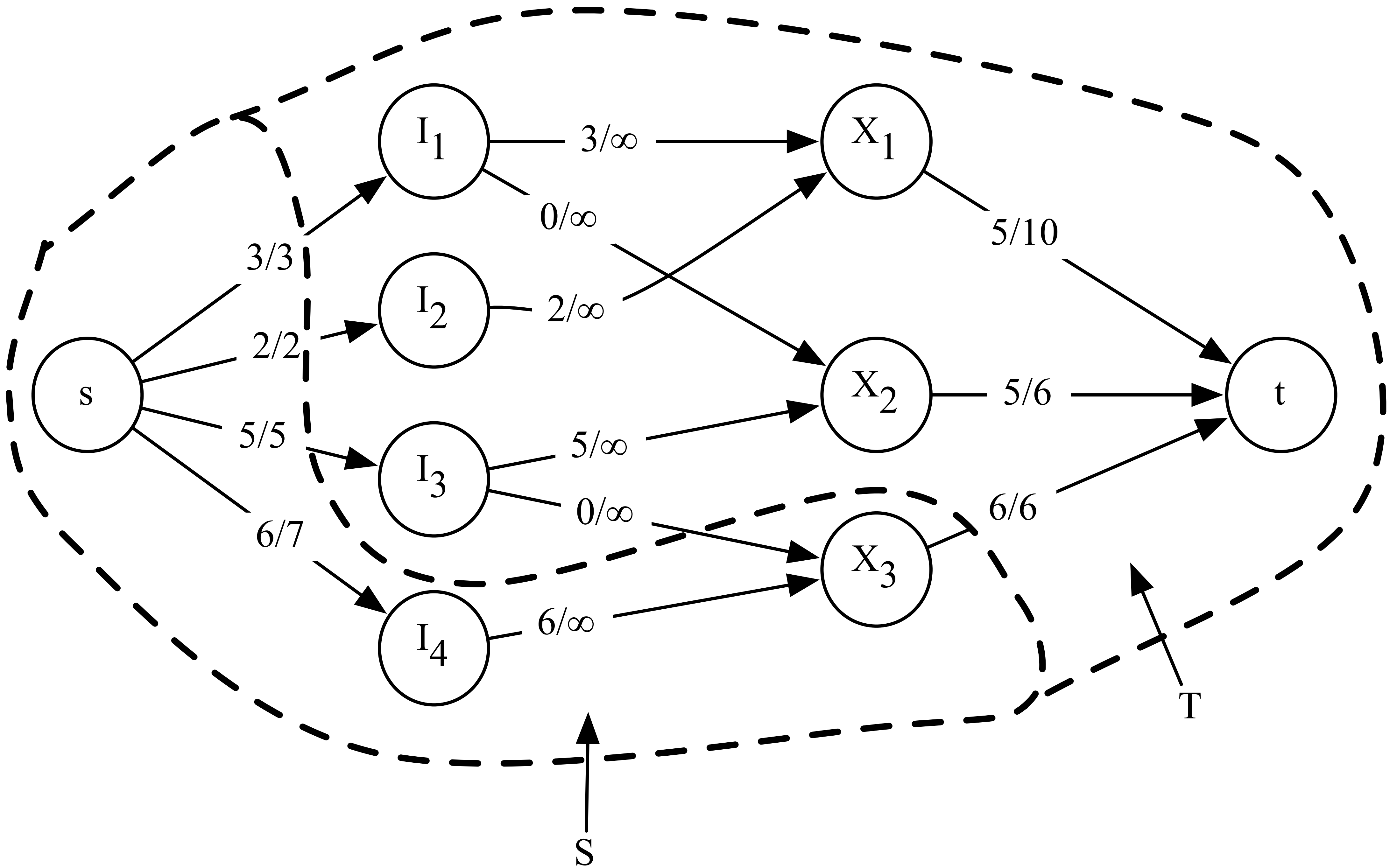
Augment s-I_1-X_1-t
bottleneck = 3





Augment s-I_4-X_3-t
bottleneck = 6


s
I_1
I_2
I_3
I_4
X_1
X_2
X_3
t
3/3
2/2
5/5
6/7
3/∞
0/∞
2/∞
5/∞
0/∞
6/∞
5/10
5/6
6/6
no s-t path
S
T



T = the set of experiments / instruments in the optimal solution

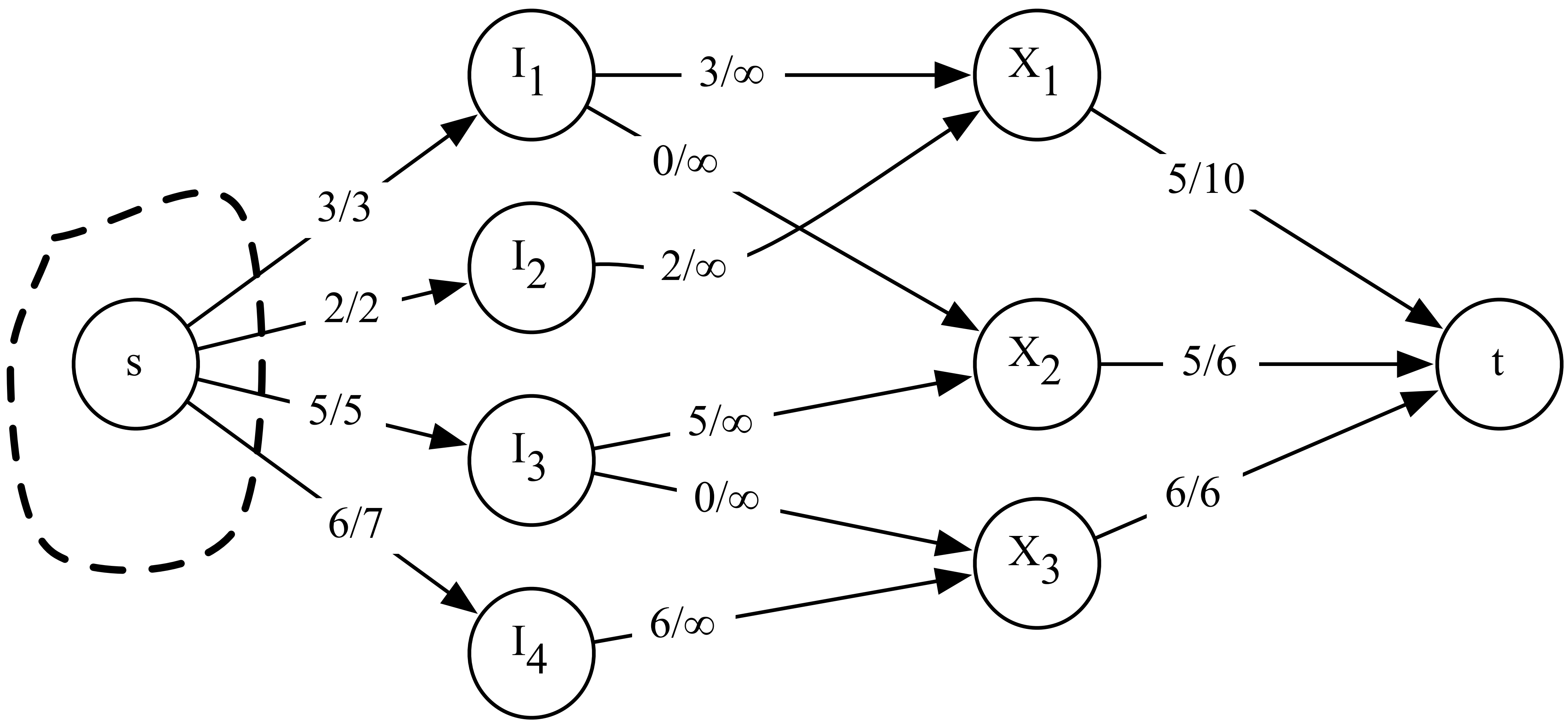
Algorithm

- Construct the flow network as described
- Find the max-flow w/Edmunds-Karp*
- Find the corresponding min-cut (S,T)
- The optimal set of experiments/ instruments is T

* Capacities are unbounded, so Ford-Fulkerson is non-polynomial.

Runtime

- $|X|=m$
- $|I|=n$
- $|V| = m + n + 2 = O(m + n)$
- $|E| = m + n + \sum|R_i| = m + n + O(mn) = O(mn)$
- Construct flow network $G = (V,E)$: $O(m + n + mn)$
- Edmunds-Karp: $O(VE^2) = O((m + n)(mn)^2)$
- BFS: $O(V + E) = O(m + n)$
- Total Runtime: $O((m + n)(mn)^2)$

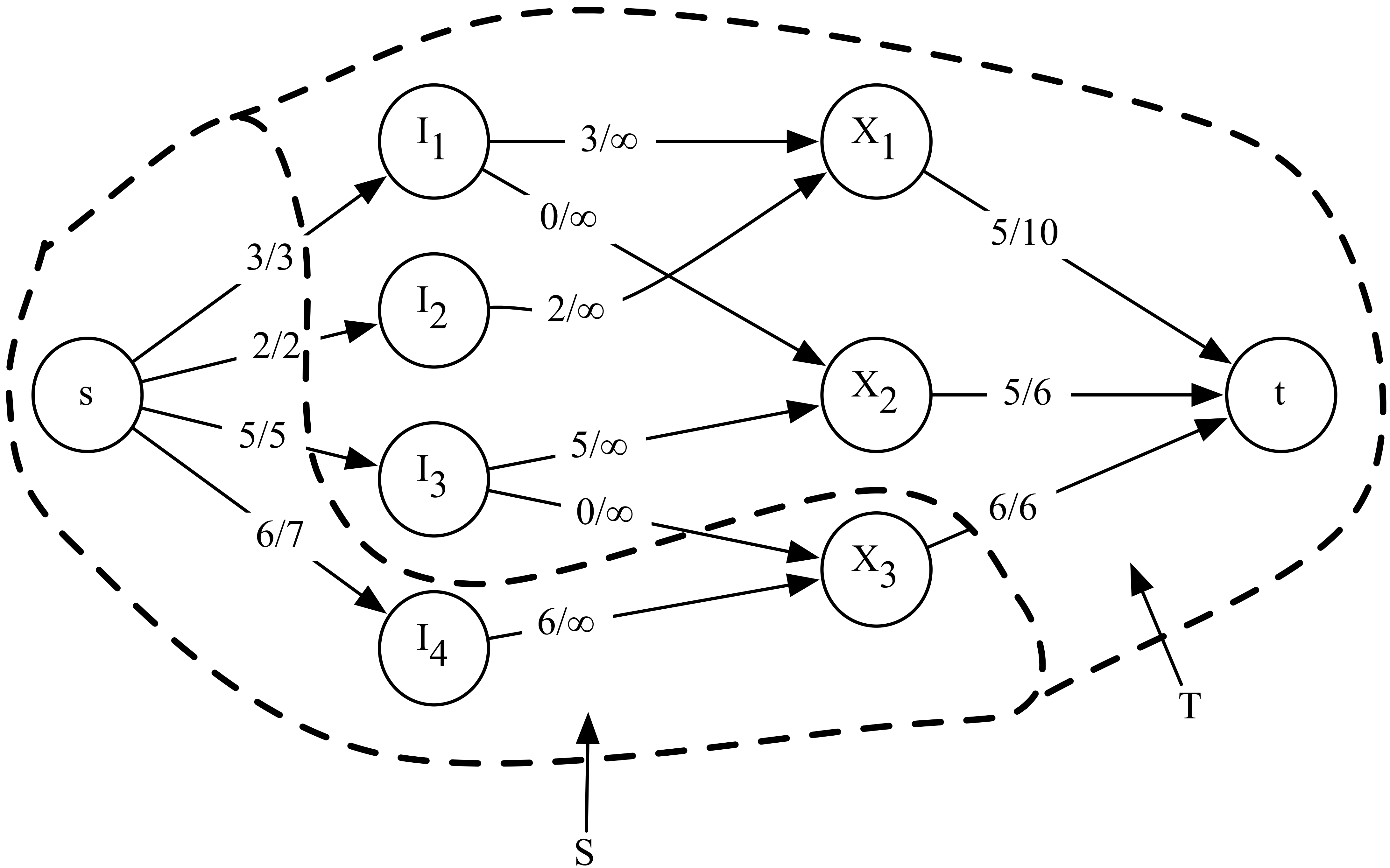


There is always a finite cut
→ the min-cut is finite

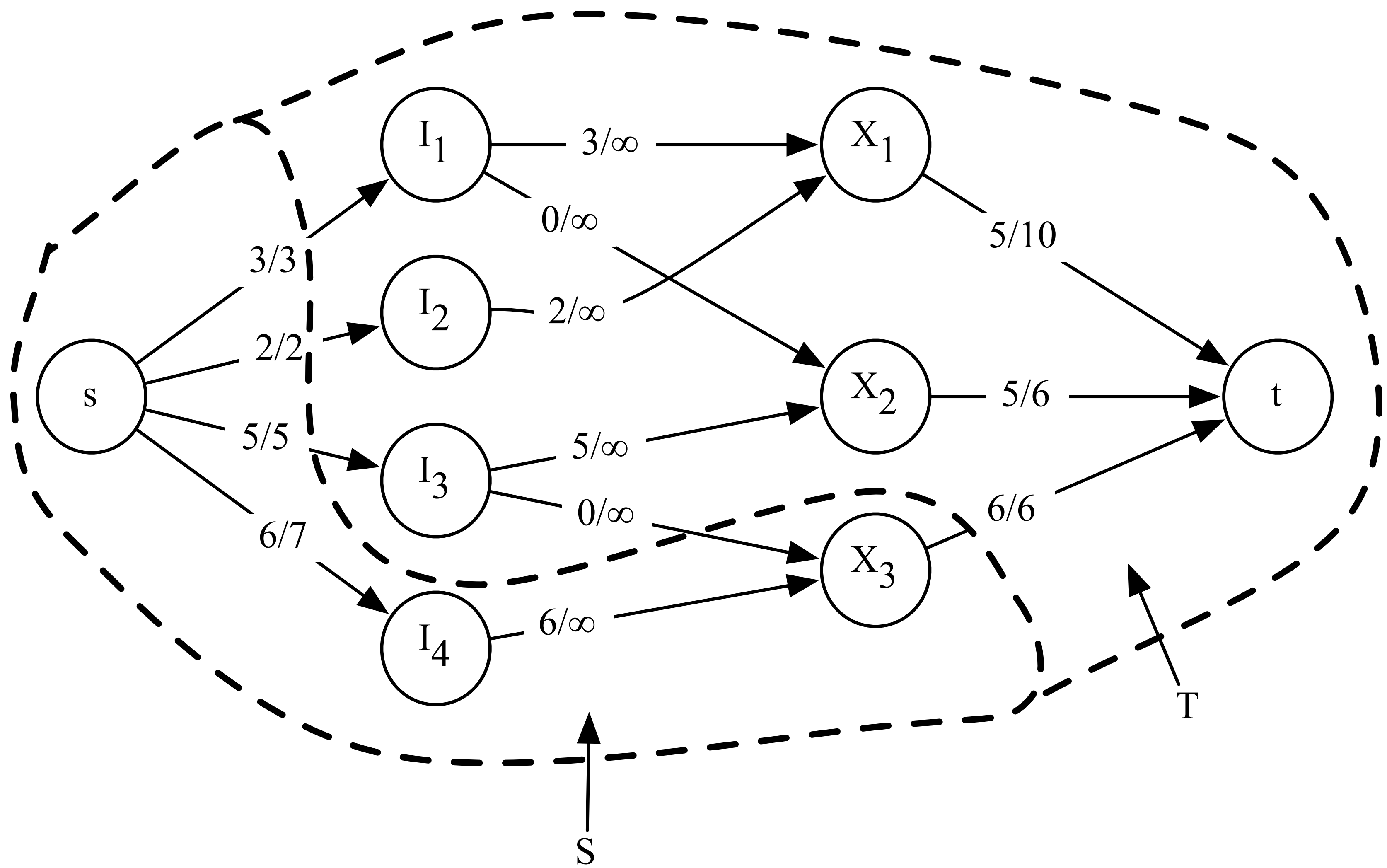
Lemma: For any finite s-t cut (S,T), if experiment X_j in T, then all of the instruments required for X_j are also in T.

Pf: A cut with finite capacity cannot have any ∞ capacity edges crossing from S to T. All ∞ capacity edges are from instruments to experiments required by them. Thus, if an experiment is in T, all of the instruments required for it must also be in T.

Every finite-cut represents a valid solution.



T = the set of experiments / instruments in the optimal solution



$$C(S,T) = 3 + 2 + 5 + 6 = 16$$

τ = sum profit for all experiments

$$= 10 + 6 + 6 = 22$$

$$\tau - C(S,T) = 6 = \text{max net revenue}$$

Lemma:

For any finite s-t cut (S,T),
net revenue = τ - C(S,T)

Notation:

I_T = instruments in T
I_S = instruments in S
X_T = experiments in T
X_S = experiments in S
τ = sum profit for all (not just chosen) experiments

Pf: Let (S,T) be an s-t cut with capacity C(S,T). For C(S,T) to be finite, only the following edges can cross from S to T:

i. Edges between s and instruments in I_T

ii. Edges between experiments in X_S and t

C(S,T) = cost of I_T + profit for X_S

τ = profit for X_T + profit for X_S

τ - C(S,T)

= (profit for X_T + profit for X_S) - (cost of I_T + profit for X_S)

= profit for X_T - cost of I_T

= net revenue

For any s-t cut (S,T), net revenue = τ - C(S,T)

τ is a constant

Edmunds-Karp finds (S,T) that minimizes C(S,T)
(by the Min-Cut Max-Flow Theorem)

So τ - C(S,T) is maximized.

So we select the set of experiments that gives the maximum net revenue.