Planarity Testing (1976; Booth, Lueker)

Glencora Borradaile, Brown University, www.cs.brown.edu/~glencora

INDEX TERMS: planarity testing, planar graph, planar embedding.
SYNONYMS: planarity testing, planar embedding.

1 PROBLEM DEFINITION

The problem is to determine whether or not the input graph G is planar. The definition pertinent to planarity-testing algorithms is: G is planar if there is an *embedding* of G into the plane (vertices of G are mapped to distinct points and edges of G are mapped to curves between their respective endpoints) such that edges do not cross. Algorithms that test the planarity of a graph can be modified to obtain such an embedding of the graph.

2 KEY RESULTS

Theorem 1. *There is an algorithm that given a graph G with n vertices, determines whether or not G is planar in $O(n)$ time.*

The first linear-time algorithm was obtained by Hopcroft and Tarjan [5] by analyzing an iterative version of a recursive algorithm suggested by Auslander and Parter [1] and corrected by Goldstein [4]. The algorithm is based on the observation that a connected graph is planar if and only if all its biconnected components are planar. The recursive algorithm works with each biconnected component in turn: find a separating cycle C and partition the edges of G not in C; define a component of the partition as consisting of edges connected by a path in G that does not use an edge of C; and, recursively consider each cyclic component of the partition. If each component of the partition is planar and the components can be combined with C to give a planar graph, then G is planar.

Another method for determining planarity was suggested by Lempel, Even, and Cederbaum [6]. The algorithm starts with embedding a single vertex and the edges adjacent to this vertex. It then considers a vertex adjacent to one of these edges. For correctness, the vertices must be considered in a particular order. This algorithm was first implemented in $O(n)$ time by Booth and Lueker [2] using an efficient implemenation of the PQ-trees data structure. Simpler implementations of this algorithm have been given by Boyer and Myrvold [3] and Shih and Hsu [8].

Tutte gave an algebraic method for giving a *straight-line embedding* of a graph that, if the input graph is 3-connected and planar, is guaranteed to generate a planar embedding. The key idea is to fix the vertices of one face of the graph to be the corners of a convex polygon and then embed every other vertex as the geometric average of its neighbors.

3 APPLICATIONS

Planarity testing has applications to computer-aided circuit design and VLSI layout by determining whether a given network can be realized in the plane.

4 OPEN PROBLEMS

None is reported.

5 EXPERIMENTAL RESULTS

None is reported.

6 DATA SETS

None is reported.

7 URL to CODE

LEDA has an efficient implementation of the Hopcroft and Tarjan planarity testing algorithm [7]: http://www.algorithmic-solutions.info/leda_guide/graph_algorithms/planar_kuratowski.html

8 CROSS REFERENCES

Entry editors, please consider the following: PQ-Trees, planar embedding, fully dynamic planarity testing, parallel planarity testing, straight-line embedding

9 RECOMMENDED READING

References

[1] L. Auslander and S. V. Parter, *On imbedding graphs in the plane*, J. Math. and Mech., 10 (1961), pp. 517–523.

[2] K. S. Booth and G. S. Lueker, *Testing for the consecutive ones property, interval graphs, and graph planarity using PQ-tree algorithms*, J. Comp. Sys. Sci., 13 (1976), pp. 335–379.

[3] J. Boyer and W. Myrvold, *Stop minding your p's and q's: a simplified o(n) planar embedding algorithm*, in SODA '99: Proceedings of the tenth annual ACM-SIAM symposium on Discrete algorithms, Philadelphia, PA, USA, 1999, Society for Industrial and Applied Mathematics, pp. 140–146.

[4] A. J. Goldstein, *An efficient and constructive algorithm for testing whether a graph can be embedded in the plane*, in Graph and Combinatorics Conf., 1963.

[5] J. Hopcroft and R. Tarjan, *Efficient planarity testing*, J. ACM, 21 (1974), pp. 549–568.

[6] A. Lempel, S. Even, and I. Cederbaum, *An algorithm for planarity testing of graphs*, in Theory of Graphs: International Symposium, Rosentiehl, P., ed., New York, 1967, Gordon and Breach, pp. 215–232.

[7] K. Mehlhorn, P. Mutzel, and S. Näher, *An implementation of the hopcroft and tarjan planarity test*, Tech. Rep. MPI-I-93-151, Saarbrücken, 1993.

[8] W.-K. Shih and W.-L. Hsu, *A new planarity test*, Theor. Comput. Sci., 223 (1999), pp. 179–191.