Optimal Rotation Angles for Quasi-Orthogonal Space-Time Codes with PSK Modulation

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***Abstract*— Full diversity of quasi-orthogonal space-time block codes with complex signals could be achieved by constellation rotation. In this letter, we derive the optimal rotation angle in the sense of maximizing coding gain for quasi-orthogonal codes with phase-shift keying (PSK) modulation using a geometry-based approach. We also prove that the rotated PSK signals with an even constellation size have higher coding gains (or diversity products) than those with an odd constellation size.**

***Index Terms*— Quasi-orthogonal space-time codes, constellation rotation, phase-shift keying (PSK).**

I. Introduction

THE transmit diversity scheme proposed in [1] is a simple and effective orthogonal space-time block code of rate one for systems with two transmit antennas. The orthogonal design was then extended to systems with an arbitrary number of transmit antennas [2]. Rate-one real orthogonal codes are available from [2], but complex orthogonal design with transmission rate one does not exist for more than two transmit antennas (see [2, Theorem 5.4.2]). Quasi-orthogonal space-time block codes that provide rate one but partial diversity were studied in [3], [4] for four transmit antennas, and in [5] for higher number of transmit antennas. To achieve full diversity for quasi-orthogonal codes, constellation rotation schemes were proposed in [6], [7]. Optimal rotation angles in the sense of maximizing coding gain for quadrature amplitude modulation were addressed in [7]. In [8], optimal rotation angles for PSK with an even constellation size were derived.

In this letter, we derive, through a geometry-based approach, the optimal rotation angles for quasi-orthogonal codes with any PSK modulation. The independent work [9] also addressed the optimal rotation angles using a completely different approach. In addition to the optimal rotation angles, we also prove that coding gain for even-sized constellations is higher than that for odd-sized constellations, which was observed but not proved in [9].

II. Optimal Constellation Rotation for PSK

We focus on the scheme given in [4] for systems with four transmit antennas for which the code matrix is expressed as

$$\boldsymbol{C} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ -x_2^* & x_1^* & -x_4^* & x_3^* \\ x_3 & x_4 & x_1 & x_2 \\ -x_4^* & x_3^* & -x_2^* & x_1^* \end{bmatrix}, \text{ and } \boldsymbol{C}^H\boldsymbol{C} = \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ b & 0 & a & 0 \\ 0 & b & 0 & a \end{bmatrix},$$

where $(\cdot)^*$ denotes complex conjugate, $(\cdot)^H$ denotes conjugate transpose, $a = \sum_{i=1}^{4} |x_i|^2$, and $b = x_1x_3^* + x_3x_1^* + x_2x_4^* + x_4x_2^*$. Analysis for other quasi-orthogonal codes (e.g., the code given in [3]) is similar.

The maximum likelihood (ML) decision metric for this code can be written as the sum of two independent terms $f_1(x_1, x_3) + f_2(x_2, x_4)$. Thus, the minimization for ML decoding can be done separately on these two terms. For the code example chosen above, let us consider (x_1, x_3) and let $x_2 = x_4 = 0$ [6] in calculating the optimal rotation angles.

Let $\mathcal{A}$ be a PSK constellation of size Q, where Q could be even ($Q = 2n$, $n > 0$ is an integer) or odd ($Q = 2n-1$, $n > 1$), and $\mathcal{B}$ be the rotated constellation of $\mathcal{A}$ expressed as $\mathcal{B} = e^{j\phi}\mathcal{A}$, where ϕ represents the rotation angle. Also let $x_1, \tilde{x}_1 \in \mathcal{A}$ and $x_3, \tilde{x}_3 \in \mathcal{B}$, where $(x_1, x_3) \neq (\tilde{x}_1, \tilde{x}_3)$. Maximizing coding gain is equivalent to maximizing $|\det[\Delta^H_{C(x_1,0,x_3,0)}\Delta_{C(x_1,0,x_3,0)}]|$, where $|\cdot|$ denotes the absolute value and $\Delta_{C(x_1,0,x_3,0)}$ is the difference matrix given as $\Delta_{C(x_1,0,x_3,0)} = \boldsymbol{C}_{(x_1,0,x_3,0)} - \boldsymbol{C}_{(\tilde{x}_1,0,\tilde{x}_3,0)}$. This is also equivalent to maximizing the minimum ζ-distance between constellations $\mathcal{A}$ and $\mathcal{B}$ expressed as [6, Eq. (23)]

$$d_{\min,\zeta}(\mathcal{A},\mathcal{B}) \triangleq \underbrace{\min}_{(x_1,x_3)\neq(\tilde{x}_1,\tilde{x}_3)} \left|(x_1-\tilde{x}_1)^2 - (x_3-\tilde{x}_3)^2\right|^{\frac{1}{2}}. \tag{1}$$

The set of values of $(x_1-\tilde{x}_1)^2$ with $x_1, \tilde{x}_1 \in \mathcal{A}$ and $(x_3-\tilde{x}_3)^2$ with $x_3, \tilde{x}_3 \in \mathcal{B}$ constitute, respectively, constellation $\mathcal{X}$ and constellation $\mathcal{Y} = e^{j2\phi}\mathcal{X}$. Note that both $\mathcal{X}$ and $\mathcal{Y}$ include the origin. The optimal rotation angle in the sense of maximizing coding gain must maximize the minimum distance between any point from $\mathcal{X}$ and any point from $\mathcal{Y}$, except the origin.

Before calculating the optimal rotation angles, let us examine the properties of $\mathcal{X}$. Examples of an even-sized constellation $\mathcal{A}$ (16PSK) and an odd-sized constellation $\mathcal{A}$ (7PSK) are shown in Fig. 1 and Fig. 2, respectively.

The corresponding new constellations $\mathcal{X}$ formed by the set of values of $(x_1 - \tilde{x}_1)^2$ with $x_1, \tilde{x}_1 \in \mathcal{A}$ are shown in Fig. 3 and Fig. 4, respectively. In the following discussion, we exclude the origin in $\mathcal{X}$ because if $x_1 - \tilde{x}_1 = 0$ or $x_3 - \tilde{x}_3 = 0$, $\left|(x_1-\tilde{x}_1)^2 - (x_3-\tilde{x}_3)^2\right|^{1/2}$ becomes a constant for any rotation angle. Let $P = Q/2$ when Q is an even number and $P = (Q-1)/2$ when Q ($Q \geq 3$) is an odd number. In the following proofs, we focus on the case when Q is an even number. The proof when Q is an odd number is similar.

Properties of $\mathcal{X}$:

1) All points in $\mathcal{X}$ can be divided into P groups, $S_1, S_2, \cdots, S_P$, based on their relative magnitudes in ascending order as $|S_1| < |S_2| < \cdots < |S_P|$.

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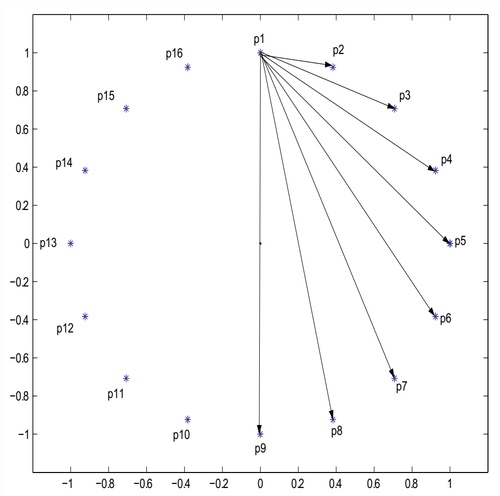

Fig. 1. An example of an even-sized constellation $\mathcal{A}$: 16PSK.

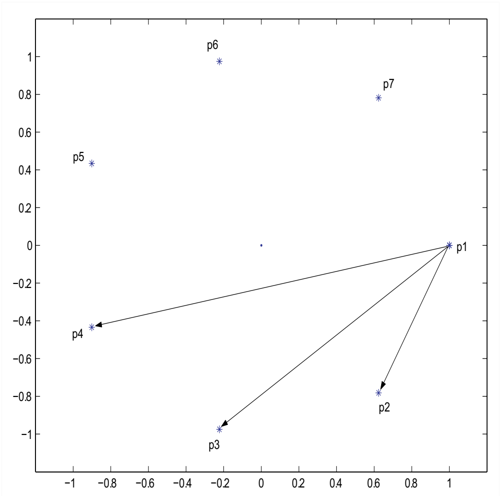

Fig. 2. An example of an odd-sized constellation $\mathcal{A}$: 7PSK.

Proof: It is easy to see from Fig. 1 that for all possible combinations of $x_1, \tilde{x}_1 \in \mathcal{A}$ and $x_1 \neq \tilde{x}_1$, $|x_1 - \tilde{x}_1|$ has $Q/2$ distinct values. Additionally, the group with the minimum magnitude is formed by two adjacent points in $\mathcal{A}$ such as $x_1 = p_1$ and $\tilde{x}_1 = p_2$. The minimum magnitude is thus obtained to be $|S_1| = |p_2 - p_1|^2 = 2 - 2\cos(2\pi/Q)$. Obviously, we have $d_{\min,\zeta}(\mathcal{A}, \mathcal{B}) \leq |S_1|^{1/2}$.

2) The distance between any two points from two different groups is greater than or equal to $|S_1|$ when Q is an even number and greater than $|S_1|$ when Q is an odd number.

Proof: Consider the worst case where two points from two adjacent groups have the same phase. The distance between these two points is $|S_i| - |S_{i-1}|$. From Fig. 1, it is easy to see that sides $\sqrt{|S_1|}, \sqrt{|S_{i-1}|}$, and $\sqrt{|S_i|}$ always constitute a triangle. Additionally, the angle opposite $\sqrt{|S_i|}$ is an obtuse angle or right angle (e.g., $\angle p_1 p_7 p_8$ is an obtuse angle and $\angle p_1 p_8 p_9$ is a right angle). Thus, we have $|S_i| - |S_{i-1}| \geq |S_1|$ with equality if and only if the triangle is a right triangle.

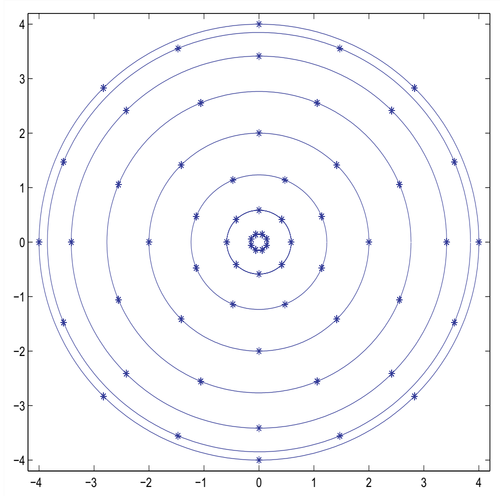

Fig. 3. Constellation $\mathcal{X}$ formed by $(x_1 - \tilde{x}_1)^2$, where $x_1, \tilde{x}_1 \in \mathcal{A}$ as shown in Fig. 1.

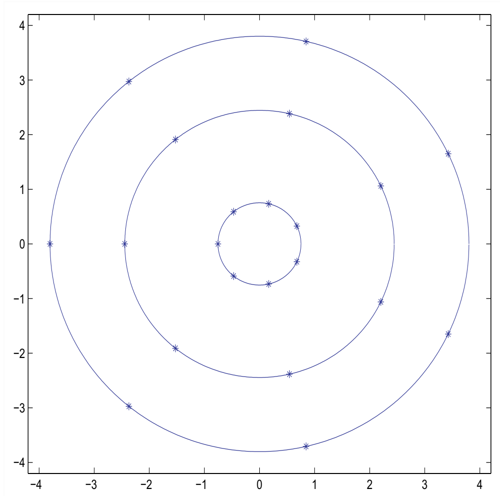

Fig. 4. Constellation $\mathcal{X}$ formed by $(x_1 - \tilde{x}_1)^2$, where $x_1, \tilde{x}_1 \in \mathcal{A}$ as shown in Fig. 2.

For the example shown in Fig. 3, the distance between the two outer circles equals the radius of the inner circle.

3) There are exactly $Q/2$ ($Q = 2n$) or Q ($Q = 2n - 1$) different points in each of the P groups, $S_1, S_2, \cdots, S_P$, and the phases of all points in any group are uniformly distributed between 0 and 2π.

Proof: We prove using group S_1 as an example. The same method applies to other groups. All points in S_1 are listed as

$$
\begin{aligned}
(p_2 - p_1)^2 &= e^{j\pi}(e^{-j2\pi/Q} - 1)^2 \\
&\vdots \\
(p_Q - p_{Q-1})^2 &= e^{-j4\pi(Q-2)/Q} \cdot e^{j\pi}(e^{-j2\pi/Q} - 1)^2 \\
(p_1 - p_Q)^2 &= e^{-j4\pi(Q-1)/Q} \cdot e^{j\pi}(e^{-j2\pi/Q} - 1)^2.
\end{aligned}
$$

It is found that the phase difference between any two adjacent points in S_1 is $4\pi/Q$ when Q is an even number, and is $2\pi/Q$ when Q is an odd number.

A. The Optimal Rotation Angles

According to properties 1-3 of constellation $\mathcal{X}$, the optimal rotation angle for quasi-orthogonal codes with PSK modulation is π/Q when Q is even and $\pi/(2Q)$ when Q is odd.

Proof: If we let $\phi = \pi/Q$, constellation $\mathcal{Y}$ is related to constellation $\mathcal{X}$ as $\mathcal{Y} = e^{j2\phi}\mathcal{X} = e^{j2\pi/Q}\mathcal{X}$. According to property 3 of $\mathcal{X}$, the minimum distance between any two points $s \in S_i$ and $\tilde{s} \in e^{j2\pi/Q}S_i$ is maximized. Additionally, from property 2 of $\mathcal{X}$, the minimum distance between $\alpha \in S_i$ and $\beta \in e^{j2\pi/Q}S_j,\ i \neq j$, is always greater than or equal to $|S_1|$. Therefore, the minimum ζ-distance is determined by $|S_1|$, and $\phi = \pi/Q$ is the optimal rotation angle that maximizes $d_{\min,\zeta}(\mathcal{A},\mathcal{B})$ when Q is even. Following a similar procedure, we can prove the conclusion for odd-sized constellations. Note that rotation with $\phi = \pi/Q$ for odd-sized constellations does not even provide full diversity ([10], Theorem 2.2). This is clear seen from Fig. 4: if $\mathcal{Y} = e^{j2\pi/Q}\mathcal{X}$, then $\mathcal{Y}$ and $\mathcal{X}$ overlap and $\left|(x_1-\tilde{x}_1)^2-(x_3-\tilde{x}_3)^2\right|^{\frac{1}{2}}$ could be zero.

The corresponding optimal minimum ζ-distance is

$$d_{\min,\zeta}(\mathcal{A},\mathcal{B}) = \begin{cases} \min\left(\left|2|S_1|^2(1-\cos(\frac{2\pi}{Q}))\right|^{\frac{1}{4}}, |S_1|^{\frac{1}{2}}\right), & Q=2n \\ \min\left(\left|2|S_1|^2(1-\cos(\frac{\pi}{Q}))\right|^{\frac{1}{4}}, |S_1|^{\frac{1}{2}}\right), & Q=2n-1 \end{cases}$$

The above expression can be simplified as

$$d_{\min,\zeta}(\mathcal{A},\mathcal{B}) = \begin{cases} 2\sin\left(\frac{\pi}{Q}\right), & Q=2,4 \\ \sqrt{8\sin^3\left(\frac{\pi}{Q}\right)}, & Q=2n\geq 6 \\ \sqrt{8\sin\left(\frac{\pi}{2Q}\right)\sin^2\left(\frac{\pi}{Q}\right)}, & Q=2n-1 \end{cases} \quad (2)$$

For the specific case of 8PSK ($Q = 8$), the optimal rotation angle based on the conclusion in this letter is $\phi = \pi/8$ and the minimum ζ-distance is 0.6696, which are the same as the results obtained via computer search in [7].

B. Relative Coding Gain between $Q=2n$ and $Q=2n-1$

Proposition 1: The cases of $Q = 2n, n \geq 3$, have larger optimal minimum ζ-distances than those of $Q = 2n-1$. Mathematically, *Proposition 1* is expressed as

$$\sin^3\left(\frac{\pi}{2n}\right) - \sin\left(\frac{\pi}{2(2n-1)}\right)\sin^2\left(\frac{\pi}{2n-1}\right) > 0. \quad (3)$$

Before proving *Proposition 1*, let us prove the following inequality:

$$x > \sin(x) > x - x^3/6, \quad \text{for} \quad x > 0. \quad (4)$$

Let $f_1(x) = x - \sin(x)$ and $f_2(x) = \sin(x) - x + x^3/6$. Obviously, $f_1(x)|_{x=0} = 0$ and $f_1'(x) \geq 0,\ \forall x > 0$, where $(\cdot)'$ denotes derivative. Thus, the left inequality of (4) follows. The derivative of $f_2(x)$ with respect to x is written as

$$\begin{aligned} f_2'(x) &= \cos(x) - 1 + \frac{x^2}{2} \\ &= 2\left[\left(\frac{x}{2}\right)^2 - \sin^2\left(\frac{x}{2}\right)\right] > 0. \end{aligned} \quad (5)$$

Thus, the right inequality of (4) follows.

Let $\xi_1 = \left[\frac{\pi}{2n} - \frac{(\pi/(2n))^3}{6}\right]^3$ and $\xi_2 = \frac{\pi}{2(2n-1)}\left(\frac{\pi}{2n-1}\right)^2$. For $n \geq 3$

$$\xi_1 - \xi_2 = (n+0.92)(n-0.36)(n-2.98) > 0.$$

Notice that from (4) we have

$$\begin{aligned} \xi_1 &< \sin^3\left(\frac{\pi}{2n}\right) \\ \xi_2 &> \sin\left(\frac{\pi}{2(2n-1)}\right)\sin^2\left(\frac{\pi}{2n-1}\right). \end{aligned}$$

Thus, *Proposition 1* expressed in (3) is proved.

III. Conclusion

We have derived, using a geometry-based method, the optimal constellation rotation angles for quasi-orthogonal space-time block codes for four-antenna systems with PSK modulation. Through constellation rotation with these rotation angles, the coding gain is maximized and full diversity of quasi-orthogonal codes is achieved. We have also proved that PSK signals with a constellation size $Q = 2n$ have larger optimal minimum ζ-distances than those with a constellation size $Q = 2n-1$ ($n \geq 3$).

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