Adaptive modulation with finite rate feedback for QR decomposition-successive interference cancellation-based multiple-in multiple-out systems

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Abstract: This article studies adaptive modulation for QR decomposition and successive interference cancellation receivers in multiple-input multiple-output systems over slow- and fast-fading channels. In slowly fading channels, adaptive modulation schemes with finite-rate feedback under the constraints of a constant total transmit power, discrete-rate and a target bit-error-rate (BER) are proposed. Specifically, the authors first develop adaptive modulation with modulation order feedback only. For power allocation and modulation order feedback, the authors establish an optimisation scheme that can be solved easily and analyse the effect of power-level quantisation on the achievable data rates. For the fast fading channel case, the authors analyse the probability distribution of the ‘R’ matrix and derive a scheme that only requires feedback of the statistical information of the channel for adaptive modulation with optimal bit loading to guarantee that the BER target is met.

1 Introduction

Common approaches to cancel inter-data-stream interferences in spatial multiplexing multiple-input multiple-output (MIMO) antenna systems [1, 2] include precoding [3, 4] and successive interference cancellation (SIC) [5, ch. 5]. Precoding requires channel state information (CSI) at the transmitter, typically through feedback. In practice, CSI must be quantised to allow feedback with limited number of bits [6]. SIC based on QR decomposition (QRD) is simple to implement and is widely used in practice.

To improve the spectrum efficiency, adaptive modulation and power allocation could be employed [7–9], which require feedback. In precoding-based adaptive modulation systems, CSI (including precoding matrix information) must be sent to the transmitter. Since perfect CSI is not possible in practice, adaptive modulation using CSI with prediction error [10, 11], mean feedback CSI [12], outdated CSI [13] and limited feedback CSI [14] has been studied. A MIMO orthogonal frequency division multiplexing system with RQ decomposition is studied in [15]. This scheme, which is similar to QRD, requires the rotation information be sent to the transmitter to update the precoding matrix. In [16], antenna selection and link adaptation for MIMO QRD-based systems with a given throughput and a constant transmit power are analysed and a recursive algorithm is developed to solve the optimisation problem. Adaptive modulation for MIMO systems with QRD and successive interference cancellation (QRD-SIC) is rarely studied in existing work.

This paper analyses adaptive modulation for QRD-SIC MIMO systems in slow- and fast-fading channels. In slowly fading channels, modulation is optimised for equal power allocation (EPA) that requires modulation-feedback only and for optimal bit loading that requires feedback of the power allocation strategy and modulation order [17, 18, p. 66] via limited number of bits. In fast fading channels, modulation is optimised for the case of optimal bit loading using the statistical information of the channel obtained via feedback. The major advantage of this system is that precoding at the transmitter and the associated feedback are not required while the achievable transmission rate is maximised. Major contributions of this paper include: (i) proving that the signal-to-noise (SNR) ratio of each data stream in an adaptive modulation QRD-SIC MIMO system is dominated by the diagonal values of the ‘R’ matrix. Thus the QRD-SIC-based MIMO system with adaptive modulation can be viewed as a parallel channel system and consequently adaptive modulation can be applied to each data stream, (ii) proposing two adaptive modulation schemes with finite-rate feedback for slowly fading channels and analysing the effect of the number of feedback bits on the achievable rate, (iii) analysing the distribution of ‘R’ matrix and proposing adaptive modulation with optimal bit loading for fast fading channels and (iv) deriving the system performance in both slow- and fast-fading channels.

2 QRD-SIC-based MIMO systems

2.1 System model

Consider a point-to-point MIMO system shown in Fig. 1, where the transmitter has $n_t$ antennas and the receiver has $n_r$
Generally in an uncoded system, the BER target is the total transmit power allocation only needs to consider the diagonal elements of $G$, whose $(i,j)$th element is denoted by $g_{i,j}$. Note that $r = \text{rank}(H) \leq \min(n_r, n_t)$. Let $D_g = \text{diag}(g_{1,1}, \ldots, g_{r,r}, \ldots, g_{n_t,n_t})$ be a diagonal matrix. The maximum mutual information (MMI) of QRD-SIC-based MIMO systems assuming no error propagation is written as [5, p. 33]

$$R_{\text{MMI}} = \max_{p_t, p_{\text{rec}}} \left\{ \log \det \left( \mathbf{I}_{n_t} + \frac{1}{N_0} D_g Q Q^H D_g^H \right) \right\}$$

$$= \max_{p_t, p_{\text{rec}}} \left\{ \prod_{i=1}^{n_t} \log \left( 1 + \frac{p_i}{N_0 g_{i,i}} \right) \right\}$$

(4)

By using the Lagrange multiplier algorithm, we have

$$p_i = \begin{cases} \left( 1 - \frac{N_0}{\eta g_{i,i}} \right)^+ & i \leq r \\ 0 & n_t \geq i > r \end{cases}$$

(5)

where $\eta$ is the Lagrange multiplier and $(a)^+ = \begin{cases} a & a \geq 0 \\ 0 & a < 0 \end{cases}$.

### 2.3 Bit-error-rate performance of quadrature amplitude modulation

$M$-ary quadrature amplitude modulation (QAM) ($M = 2^k$, $k = 1, 2, \ldots, 8$) will be assumed in this paper. A unified expression of the approximate bit-error-rate (BER) of $M$-QAM signals over an additive Gaussian noise channel is written as [12]

$$\Pr(M, \gamma_{\text{rec}}) \approx 0.2 \exp(-\psi(M) \gamma_{\text{rec}})$$

(6)

where $\gamma_{\text{rec}}$ is the received SNR and $\psi(M)$ is a constellation-specific quantity defined as

$$\psi(M) = \begin{cases} 1.5/(M - 1) & \text{for square } M-QAM \\ 6/(5M - 4) & \text{for rectangular } M-QAM \end{cases}$$

(7)

### 3 Adaptive modulation with finite-rate feedback in QRD-SIC-based MIMO systems

When adaptive modulation is employed, the BER of the $i$th data stream is guaranteed to be below the BER target $\text{BER}_t$.

**Lemma 1**: In QRD-SIC-based MIMO systems with adaptive modulation, the SNR of the $i$th data stream is dominated by $(g_{i,i}^2 p_i/N_0)$.

**Proof**: Generally in an uncoded system, the BER target is small in practice (e.g. lower than $10^{-3}$). The average SNR of the $i$th data stream is

$$E[\gamma_i] = \sum_{k=0}^{i} \left( \frac{i-1}{k} \right) (1 - \text{BER})^{i-1-k} \text{BER}_t^k \gamma_{i,k}$$

(8)

where $\gamma_i$, $k$ represents the SNR of the $i$th data stream when the $i$th data stream is interfered by $k$ data streams. It is assumed that when the number of interfering data streams is the same, the received SNR of the $i$th data stream is the same. In general cases, $n_t$ is smaller than 20 in practical systems, that is $r < 20$ and $\gamma_i$, $i(k=0, \ldots, i-1)$ are of the same order
of magnitude. Therefore
\[ E[\gamma_i] \simeq (1 - \text{BER})^{i-1} \gamma_0 \]
\[ \simeq \gamma_0 = \frac{2^2 P_i}{N_0} \tag{9} \]

Thus, the QRD-SIC-based MIMO system with adaptive modulation can be viewed as a parallel channel system. Next, we propose two finite-rate feedback adaptive modulation schemes for the slowly fading channel.

### 3.1 Adaptive modulation in slowly fading channels

In a slowly fading channel, it is reasonable to assume that the channel does not change over two adjacent transmission periods. The SNR of the \( i \)th data stream is expressed as
\[ \gamma_i = \frac{2^2 P_i}{N_0} \tag{10} \]
and the BER of \( i \)th data stream is
\[ e_i = \text{Pr}(2^k, \gamma_i) = 0.2 \exp(-\psi(2^k) \gamma_i) \tag{11} \]

#### 3.1.1 Adaptive modulation with modulation order feedback only:

In this case, power is equally allocated to each data stream at the transmitter; that is \( P_1 = P_2 = \cdots = P_r = (P_i/r) \). Considering a BER target and (11), we define the SNR threshold as
\[ \text{th}_k = -\frac{\ln(5 \text{ BER})}{\psi(2^k)}, \quad k = 1, 2, \ldots, 8 \tag{12} \]

where \( \text{th}_1 < \text{th}_2 < \cdots < \text{th}_8 \).

The order of modulation is determined by the received SNR and is expressed as
\[ M_i = \begin{cases} 0 & p_i g_{iJ}^2 < \text{th}_1 \\ 2^k & \text{th}_k \leq p_i g_{iJ}^2 < \text{th}_{k+1}, \quad k = 1, 2, \ldots, 7 \\ 256 & p_i g_{iJ}^2 \geq \text{th}_8 \end{cases} \tag{13} \]

For example, if \( \text{th}_k \leq (p_i g_{iJ}^2/N_0) < \text{th}_{k+1} \), then \( 2^k \)-QAM is chosen for the \( i \)th stream; if \( (p_i g_{iJ}^2/N_0) < \text{th}_1 \), then no data will be transmitted. For the \( i \)th data stream, it can be shown that
\[ k_i = \sum_{k=1}^{8} U \left( p_i g_{iJ}^2/N_0 - \text{th}_k \right) \tag{14} \]

where \( U(\cdot) \) is the unit step function and the modulation order is \( M_i = 2^{k_i} \). After the modulation orders are determined, the receiver sends the information back to transmitter.

#### 3.1.2 Adaptive modulation with power allocation and modulation order feedback (optimal bit loading):

Now consider the case that the transmitter no longer allocates the power equally; instead, the constraint of a constant transmit power, discrete data rate and a BER target are to be satisfied. The goal is to maximise the transmission rate. The optimisation problem (OP1) can be expressed as
\[ \max_{P_i} \left\{ \sum_{i=1}^{r} \sum_{k=1}^{8} U \left( p_i g_{iJ}^2/N_0 - \text{th}_k \right) \right\} \tag{15} \]
subject to
\[ \begin{align*}
\sum_{i=1}^{r} p_i &= P_i \\
p_i &\geq 0, \quad i = 1, 2, \ldots, r
\end{align*} \tag{16} \]

The SNR threshold corresponding to the \( k \)th modulation scheme \((2^k \text{-QAM})\) for the \( i \)th sub-channel is defined as
\[ \text{ThSNR}_{i,k} = \frac{\text{th}_k}{g_{iJ}^2}, \quad i = 1, 2, \ldots, r; \quad k = 1, 2, \ldots, 8 \tag{17} \]

To maximise power efficiency, the power allocated to each sub-channel only needs to satisfy
\[ P_i \subset \left\{ \text{ThSNR}_{i,k} = \frac{\text{th}_k}{g_{iJ}^2} N_0, \quad k = 1, \ldots, 8 \right\} \tag{18} \]

Let us define the incremental power as
\[ \Delta_{i,k}^P = \begin{cases} \text{ThSNR}_{i,k} - \text{ThSNR}_{i,k-1} & k = 1, 3, \ldots, 8 \\
\text{ThSNR}_{i,k} \leq \text{ThSNR}_{i,k+1} \end{cases} \tag{19} \]

where \( \Delta_{i,k}^P \leq \Delta_{i,k+1}^P \) for each sub-channel. If the modulation of the \( i \)th sub-channel is \( 2^k \)-QAM, then the power allocated to the \( i \)th sub-channel is \( p_i = \sum_{k=1}^{8} \Delta_{i,k}^P \). A special case is \( k_i = 0 \), which means that no data is sent through the \( i \)th sub-channel.

Consequently, OP1 is equivalent to the following optimisation problem (OP2)
\[ \max_{k_i} \left\{ R_{\text{achieve}} = \sum_{i=1}^{r} k_i \right\} \tag{20} \]
subject to
\[ \sum_{i=1}^{r} \sum_{k=1}^{8} \Delta_{i,k}^P \leq P_i \tag{21} \]
which can be solved through the following two steps [17]

1. Sort the \( 8r \) values of \( \Delta_{i,k}^P \) for \( (i = 1, \ldots, r; k = 1, \ldots, 8) \) in ascending order expressed as \( Y_1 \leq Y_2 \leq \cdots \leq Y_{8r} \).
2. Find the maximum value of \( R_{\text{achieve}} \), subject to \( \sum_{i=1}^{r} Y_i \leq P_i \).
Let $2^{k_{\text{opt}}}-\text{QAM}$ $(K_{\text{opt}} \in \{0, 1, \ldots, 8\})$ denote the optimal modulation scheme for the $i$th sub-channel. The power allocated to the $i$th sub-channel is determined to be

$$P_{i,\text{opt}} = \begin{cases} \sum_{k=0}^{k_{\text{opt}}} \Delta k & k_{\text{opt}} \geq 1 \\ 0 & k_{\text{opt}} = 0 \end{cases}$$ (22)

If $\sum_{i=1}^{r} P_{i,\text{opt}} \leq P_t$, then the remaining power is given by $\Phi_{P_i} = P_t - \sum_{i=1}^{r} P_{i,\text{opt}}$, which can be equally allocated to all subchannels; that is, the remaining power allocated to the $i$th subchannel is $P_{i,\text{opt}} + \Phi_{P_i}/r$. Ultimately, the power allocated to each sub-channel is $P_{i,\text{opt}} = P_{i,\text{opt}} + \Phi_{P_i}/r$ $(i = 1, 2, \ldots, r)$. The transmission rate achieved is

$$R_{\text{achieve}} = \sum_{i=1}^{r} \sum_{k=1}^{8} U \left( \frac{P_{i,\text{opt}} S_{i,k}^2}{N_0} - \theta_k \right)$$ (23)

If $m$ bits are used to quantify the proportion of the power allocated for each data stream to the total transmitted power, the total number of feedback bits are $(m + 3) r$, of which 3 bits are used to send the modulation order to the transmitter. In the next section, we analyse the effect of power allocation quantisation.

### 3.1.3 Effect of quantisation:

When the power allocation strategy is represented by finite number of bits, the quantisation error of the optimum power values will affect the data rate. If $m$ bits are used to represent the power level for each data stream, the resolution becomes $2^{-m}$. The proportion of power for the $i$th stream to the total power is $P_{i} = \tilde{P}_{i,\text{opt}}/P_t$. The difference between the optimal power and the quantised power level for the $i$th stream is

$$\tilde{\xi}_{P_i} = \tilde{P}_{i,\text{opt}} - P_i Q(\rho)$$ (24)

where

$$Q(\rho) = 2^{-m} \left[ \frac{x}{2^{-m} + 1} \right]$$

($\lfloor \cdot \rfloor$ is the floor function). The achieved data rate is

$$R_{\text{achieve}} = \sum_{i=1}^{r} \sum_{k=1}^{8} U \left( \frac{P_{i,\text{opt}} S_{i,k}^2}{N_0} - \theta_k \right)$$ (25)

If $P_{i,\text{opt}} - \tilde{\xi}_{P_i} > 0$, then the achieved data rate of the $i$th stream becomes

$$k_{i,\text{sub+}} = k_{\text{opt}} + \sum_{k=k_{\text{opt}}+1}^{k_{\text{opt}}} U \left( \frac{P_{i,\text{opt}} - \tilde{\xi}_{P_i}}{N_0} - \theta_k \right)$$ (26)

If $P_{i,\text{opt}} - \tilde{\xi}_{P_i} < 0$, then the achieved data rate of the $i$th stream is

$$k_{i,\text{sub-}} = k_{\text{opt}} - \sum_{k=0}^{k_{\text{opt}}-1} U \left( -P_{i,\text{opt}} + \tilde{\xi}_{P_i} \right)$$ (27)

where $\theta_k = \sum_{i=k_{\text{opt}}+1}^{k_{\text{opt}}} \theta_k$. After quantisation the data rate of the $i$th stream is

$$k_i = U \left( \frac{\Phi_{P_i}}{r} - \tilde{\xi}_{P_i} \right) k_{i,\text{sub+}} + \left( 1 - U \left( \frac{\Phi_{P_i}}{r} - \tilde{\xi}_{P_i} \right) \right) k_{i,\text{sub-}}$$ (28)

This analysis clearly shows that the achieved data rate is a function of $\tilde{\xi}_{P_i}$ and $P_i$. Since $\tilde{\xi}_{P_i} \approx \frac{P_i}{2r}$, for $P_i \leq 2r^{-3}$, the maximum transmission rate, the following relationship must be satisfied:

$$\Phi_{P_i}/r - P_i 2^{-(m+1)} \geq 0$$ (29)

Therefore

$$m \geq \left\lceil \log_2 \left( \frac{P_t \rho}{\Phi_{P_i}} \right) - 1 \right\rceil$$ (30)

where $\lceil \cdot \rceil$ is the ceiling function.

### 3.2 Adaptive modulation in fast fading channels

In this paper, fast fading refers to the case that the coherence time of the channel is much shorter than the minimum feedback delay of channel coefficients from the receiver to the transmitter. Although the channel changes rapidly and obtaining accurate CSI at the transmitter is impossible, the statistical property of the channel can still be used at the transmitter. Since $h_i \sim \mathcal{CN}(0, \nu^2)$, the probability density function of $G$ is [20, (5.51)]

$$f_G(G) = \Gamma(n_i) (\det C)^{-n_i} \exp(-\tau C^{-1} G^H G)$$

$$\cdot \prod_{i=1}^{n_i} g_i^{-2(n_i-i)+1}$$

where 

$$\Gamma(\nu) = \sum_{i=0}^{\nu} \Gamma(n_i) = \sum_{i=0}^{\nu} \Gamma(n_i)$$

and

$$\Gamma(\nu) = \int_0^{\infty} \frac{z^{\nu-1} \exp(-z)}{z} dz$$

### Lemma 2

The elements $g_{i,j} \sim \chi^2(n_i-i+1)$, and $g_{i,j} \sim \mathcal{CN}(0, \nu^2)$ ($1 \leq i < j \leq n_i$). The average BER of the $i$th stream is

$$\text{BER}_{\text{avg,i}} = \int_0^{\infty} \text{BER}(x_{i,j}^2) \, dx_{i,j}$$ (32)

Since $s_{i,j}^2 \sim \chi^2(2(n_i-i)+2)$, (33) is obtained as

$$\text{BER}_{\text{avg,i}} = 0.2 \left( \psi(2k_i^2) \frac{v_i^2 P_i}{2N_0} + 2 \right)^{-(n_i-i+1)}$$ (33)

This equation shows that the average BER is a function of $n_i$ and $v_i$. The receiver only needs to send $n_i$ and $v_i$ to the transmitter.
The SNR threshold for the $i$th data stream is

$$\text{ThSNR}_{fi}^k = \frac{(5\text{BER}_t)^{1/(n_r-i+1)} - 2}{\psi(2^n)(\nu^2/2)}$$

(34)

where $i = 1, 2, \ldots, r; k = 1, 2, \ldots, 8$.

Similar to the slowly fading channel case described in Section 3.1.2, adaptive modulation with optimal bit loading can also be applied here.

4 Simulation results

In all the simulations, the elements of $H$ are independent and identically distributed, complex, zero-mean, Gaussian random variables and a $4 \times 4$ MIMO system is used as an example. SNR is defined as $\gamma = P_t/N_0$ and the target BER is $\text{BER}_t = 10^{-3}$. Note that if the channel gain is normalised, that is, $\{|h_{i,j}|^2\} = 1$, then $\gamma$ equals the received SNR.

4.1 Slowly fading channel

In a slowly fading channel, it is assumed that $h_{i,j} \sim \mathcal{CN}(0, 1)$. Fig. 2 shows the transmission rate of different configurations. It is observed from Fig. 2 that the MMI of QRD-SIC-based scheme as described by (4) and MMI of the EPA scheme are almost the same. In the low-SNR region, the gap between the MMI of these schemes and channel capacity is $\sim 1.5$ dB; in the high-SNR region, the difference is slightly smaller. Fig. 2 also shows the achieved transmission rate of the proposed schemes. When EPA is employed and only the modulation order is adaptively controlled, the difference between the achieved transmission rate and channel capacity is $\sim 9$ dB. When power allocation and the modulation order are both sent to the transmitter, the performance improves by $\sim 2.5$ dB. For comparison, the performance of the discrete rate, variable rate and variable power scheme with perfect CSI at the transmitter studied in [7], which can be viewed as the upper bound, is also presented. The difference between the second proposed scheme and the upper bound [7] is $\sim 3$ dB, because of the imperfect CSI at the transmitter.

Fig. 3 shows the achieved data rate as a function of $m$. It is observed that in the low-SNR region (below 10 dB), the rate is nearly independent of $m$; in the high-SNR region, the performance with $m = 2$ is close to that with $m = \infty$. Fig. 4 depicts the quantisation effect at $\text{SNR} = 10$ dB. When $m > 6$, the achieved data rate nearly levels off; thus, $m = 6$ is a good choice in practice.

Simulated BER results with the maximum likelihood QAM receiver are shown in Fig. 5. The achieved BER for all cases are below the target of $10^{-3}$. Note that the BER-SNR curves do not follow the usual trend as in a normal communications system without adaptive power and modulation. For example, the curves are not smooth and at different SNR and $m$ values, the trends of these curves do not seem to follow certain typical patterns. The reason is that at different SNR values, the achieved data rates are in general different as can be seen in Fig. 3.

4.2 Fast fading channel

In fast fading channels, it is assumed $h_{i,j} \sim \mathcal{CN}(0, \nu^2)$. Fig. 6 shows the maximum achieved data rate of the proposed algorithm for different values of $\nu^2$. As $\nu^2$ increases, that is, as the channel gain increases, the maximum data rate...
becomes larger. The simulated BER performance is shown in Fig. 7, which shows that for all cases, the proposed algorithm effectively maintains the BER target of $10^{-3}$. Further improvement of BER performance can be achieved, if the remaining power $F_p$ is greater than zero.

Similar to the slowly fading case, the BER-SNR curves in Fig. 7 do not follow the usually trends as in a normal communications system without adaptive power and modulation either. The same reason applies here. For example, with $\nu^2 = 1$, the BER at SNR = 20 dB is worse than that at SNR = 15 dB, because the achieved data rate at SNR = 20 dB is higher than that at SNR = 15 dB and correspondingly the modulation order at SNR = 20 dB is higher. There are cases that BER is lower at a higher SNR. For example, with $\nu^2 = 1$, the BER at SNR = 25 dB is lower than that at SNR = 20 dB. Although the modulation order at SNR = 25 dB is higher, there is more remaining power at SNR = 25 dB and the gain in BER because of further allocation of the remaining power at SNR = 25 dB is higher.

5 Conclusion

We have proposed adaptive modulation schemes with finite rate feedback for QRD-SIC-based MIMO systems in slow and fast fading channels. The system is optimised under the constraints of a constant total transmit power, discrete rate and a BER target. In slowly fading channels, two cases are studied in detail: EPA with modulation order feedback only and power allocation and modulation order both feedback. The effect of quantisation power level to allow finite-rate feedback on the achievable rate is also analysed. Simulation results show that the achieved transmission rate of the scheme with modulation order and power allocation feedback (limited rate) is close to the MMI rate. For fast fading channels, since obtaining CSI for real-time feedback is difficult, we have analysed the probability distribution of the $R$ matrix and derived a scheme that requires only the statistical information of the channel. Simulation results show that the achieved rate of the proposed scheme with channel statistical information only in a fast fading channel is close to that of the scheme with modulation order and power allocation feedback in a slowly fading channel.

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7 References

Appendix

8.1 Appendix 1. Proof of Lemma 2

The density function of $G$ is

$$f_G(G) = \frac{2^n \pi^{-n(n-1)/2}}{\prod_{i=1}^{n} \Gamma(n_r - i + 1)} \left( \det C \right)^{-n_r} \cdot \exp\left(-\text{tr}(C^{-1}G^HG)\right) \prod_{i=1}^{n} g_{ij}^{2(n_r-i)+1} \sum_{i \leq j} A \left( g_{ij} \right)$$

where $A$ stands for exterior product. This equation shows that the elements $g_{ij}$ are independent and $g_{ij} \sim \mathcal{CN}(0, \nu^2) (1 \leq i < j \leq n)$. The probability density function of $g_{ij}^2$ is

$$f_{g_{ij}^2}(g_{ij}^2) = \frac{1}{\Gamma (n_r - i + 1)} \exp\left(-\frac{g_{ij}^2}{\nu^2}\right) \left( \frac{2(n_r-i)}{\nu^2} \right) \frac{(g_{ij}^2)^{2(n_r-i)-1}}{\Gamma(2(n_r-i)+1)}$$

By letting $s_{ij}^2 = (2/\nu^2)g_{ij}^2$, we have the probability density function of $s_{ij}^2$ as

$$f_{s_{ij}^2}(s_{ij}^2) = \frac{\exp\left(-\frac{s_{ij}^2}{(2/\nu^2)}\right)}{2^{2(n_r-i)+2}\Gamma((1/2)(2n_r - 2i + 3))}$$

Therefore $s_{ij}^2 \sim \chi^2_{2(n_r-i)+2}$, which means $2s_{ij}^2/\nu^2 \sim \chi^2_{2(n_r-i)+2}$, where $\chi^2_n$ represents the chi-square distribution with $n$ degrees of freedom.