For *Notes and records*

The interest of G.H. Hardy, F.R.S. in the philosophy and the history of mathematics

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SUMMARY

Hardy's research career shows an orthodox concern with certain areas of mathematics, where he contributed with great distinction. But throughout his career there is also evidence of regular concern with the philosophy and the history of mathematics. The influence of Bertrand Russell is quite marked.

INTRODUCTION

G.H. Hardy (1877-1947) is well remembered as a major figure in British mathematics in the first half of the 20th century, especially with his collaboration from around 1910 with J.E. Littlewood, F.R.S. (1885-1977). From them, sometimes individually, there appeared a steady stream of major papers on many topics in real- and complex-variable analysis and in analytical number theory. Hardy also produced a highly influential textbook initially entitled *A course of pure mathematics* though largely concerned with real-variable analysis, in editions from the first in 1908 to the ninth in 1944 (a posthumous tenth appeared in 1956); monographs on number theory, inequalities, and infinite series; and three volumes for the Cambridge Tracts series on aspects of analysis. (Several of these books involved co-authors.) He and Littlewood also trained a notable number of graduate students at Cambridge (between 1920 and 1931 for Hardy at Oxford), several of whom became major figures in their own generations.[1] Some were involved in the preparation of his collected papers (including those written with Littlewood), which appeared as seven hefty quarto volumes between 1966 and 1979 under the auspices of the London Mathematical Society, where he had been a major figure.[2]

All this production reveals Hardy as a superb craftsman, typical for the branches of mathematics in which he specialised in proving theorems or related results (such as improved inequalities) and concerned about rigour without troubling with the niceties of proof *theory* : he is also above the average in his care to cite others and provide bibliographies in his books. One aid here will have been his own extensive library; after his death part of it was left to Littlewood and the rest to the London Mathematical Society, where it now constitutes part of the holdings of the libraries of the Society (books) and of University College London (journals).[3]

But another side peeps out, especially in the second half of the final volume of the edition: the author of some papers on set theory; the reviewer of books of a philosophical or historical character as well as those in his usual areas; a few general articles on mathematics; and an excellent obituarist. There is also the book by which his name is best known to the public in general: a personal 'apology' for being a mathematician, which appeared in 1940.[4] These items are facets of a minor but active aspect of his mathematical career, which is the subject of concern here: his interest in the philosophy and the history of mathematics, with implications for mathematical education. It started in the early 1900s and manifested itself fairly regularly to the end of his life; the account below largely follows chronology.

THE INFLUENCE OF BERTRAND RUSSELL: SET THEORY AND MATHEMATICAL LOGIC

Hardy studied at Trinity College from 1897 to 1899, performing well in the Tripos competition (fourth Wrangler). In 1900 he became a junior fellow at the college and college lecturer six years later; at that time reforms in the Tripos system were being effected, which he doubtless supported. During this period his senior colleagues included A.N. Whitehead (1861-1947) and Bertrand Russell (1872-1970) (both F.R.S.), and he fell strongly and durably under the influence of the latter: the points of contact involved not only mathematics but also pacifism and socialism, and seems to be unique among the relationships of each man.[5] All three men were also members of the Apostles, a self-perpetuating discussion society of undergraduates.

Their relationship began to develop soon after Hardy's appointment as a Fellow in 1900. That August Whitehead and Russell learnt from the Italian mathematician and logician Giuseppe Peano (1858-1932) the discipline of 'mathematical logic' (Peano's phrase in this sense), which was based upon the propositional and predicate calculi deploying propositional functions (for example, 'x is a man' with 'x' a variable) and quantification ('for all...' and 'there is a …') over both x and propositional functions. Soon they were converting his practice of expressing mathematical theories in this logic into Russell's 'logicist' philosophy of mathematics (to use the modern name), in which those theories were held to be derivable *solely* from logical resources, not only means of reasoning but also the pertaining objects. As already with Peano, the fulfilment of this claim engaged Whitehead and Russell heavily in the set theory of Georg Cantor (1845-1918), which had recently become widely used after a somewhat troublesome period of creation and reception. They studied both the topology of sets of points, with its profound consequences for the techniques and rigour of mathematical analysis and related topics; and the theory of infinitely large cardinal and ordinal numbers, and the associated general theory of sets, with its potential for founding mathematics in some generality.[6] The link with mathematical logic lay in the natural association between a set (of men, say) and a propositional function (such as 'man'), although Russell was soon to find a nasty counter-example, as we shall see below.

The young Hardy may have attended Russell's lecture course on mathematical logic held at Trinity College in the winter of 1901-1902, and certainly heard Whitehead's course a year later.[7] Another auditor of both courses was an undergraduate, Philip Jourdain (1879-1919); he went on to work in set theory himself with much less distinction than his colleagues but gained considerable celebrity as a historian of these and other branches of mathematics. He corresponded at length with Russell and Hardy, and also with Cantor.[8]

The first major account of Russell's logicist philosophy was given in a book on 'the principles of mathematics', which was published by Cambridge University Press in 1903 after a difficult gestation.[9] Hardy reviewed it acutely for the *Times literary supplement* in an anonymous piece.[10] Feeling that Russell 'seems to have proved his point' about logicism, he found the book 'a good deal more difficult than was absolutely necessary' by being 'much too short' given that the theories were unfamiliar to most readers. He noted Russell's unfortunate way of lumping inference and logical consequence under the notion of implication, with surprising results such as 'every false proposition implies every other proposition, true or false'. He also noted a blunder in Russell's treatment of causality in thinking that from a finite number of states of a system its future behaviour could be predicted; for if a particle were 'projected from the ground, and take the second time to be that at which it reaches the ground again. How can we tell that it has not been at rest?'.

As well as using the techniques of set theory in his more usual kinds of paper, between 1904 and 1910 Hardy wrote five papers explicitly on aspects of it.[11] During these years a major exchange broke out among mathematicians and some philosophers on the status and uses of an axiom called the 'axiom of choice', which asserted that the members chosen one each and independently from an infinitude of sets could be allowed to form a set on a par with its parent sets. The legitimacy of this assumption was doubted by many, as was the need to add it to the assumptions underlying proofs of theorems in set theory and related areas of mathematics; there were hopes that it could be proved from more respectable assumptions.[12]

Both Russell and Hardy were among the many mathematicians who had intuitively used it or an equivalent assumption; indeed, Russell had discovered its axiomatic status in the summer of 1904 though the publicity had come that autumn from its independent discovery by the German mathematician Ernst Zermelo. Hardy told Russell that 'I should (in default of proof) be disposed to assume it and hope for the best', and played only a minor role in the discussion.[13] But he was working in most of the areas of mathematics in which its presence was then being detected, and the debates must have sharpened his sensitivity to the subtleties of proof in mathematics. His most overt allusion to it, and to other difficulties attending set theory, came in a paper of 1907 on Cantor's theory of infinitely large numbers, where he also criticised a rather vague theory due to his senior Cambridge colleague E.W. Hobson (1856-1933) for solving the paradoxes of set theory.[14]

These paradoxes were another case of independent findings by Russell and Zermelo, though in this case Russell came second: the discovery of the bad news that the set of all sets that are not members of themselves is a member of itself if and only if it is not. Found by Russell in 1901 (two years after Zermelo, who did not publish it) and reported in *The principles*, it was a major disaster for the fulfilment of Russell's logicist aims, and caused much difficulty in completing that book; means of 'solving' it and other paradoxes that he soon collected were a major concern. Although he lived then near Oxford for much of the time, he kept in good touch with Hardy. For example, in an important letter of 1905 he showed how he had discovered his paradox from modifying a proof-method introduced in set theory by Cantor.[15] Again, for about 18 months from late in 1905 he developed a 'substitutional theory' as his solution of the paradoxes, and in December he wrote for Hardy's benefit an extensive summary of the theory as it then stood.[16] Its main novelty was the abandonment of propositional functions; however, during 1907 he found it inadequate in various ways, and formed instead a 'theory of logical types', with these functions back in centre stage, with which to found a hopefully paradox-free logicism.

By October 1909 Whitehead and Russell had worked out this solution in symbolic detail, and Cambridge University Press put out three hefty volumes of *Principia mathematica* in 1910, 1912 and 1913.[17] Again Hardy reviewed anonymously the first volume for the *Times literary supplement*.[18] He made a point of lamenting mathematicians' lack of interest in such studies, and of its various features he stressed that 'mathematics, one may say, is the science of propositional functions' as used within mathematical logic, and 'The theory of "incomplete symbols" is one of the authors' triumphs'. This latter feature had been introduced by Russell in 1905 into philosophy in general, as a means of coping with grammatical phrases such as 'the present King of France' which made sense and yet had no referent. Russell's main motivation was mathematical, as it gave him both the means of defining mathematical functions in terms of propositional ones (a feature rather misrepresented in Hardy's remark above) and of obeying the stipulation that they should be single-valued. Hardy was equally aware of the need for single-valuedness, and he referred to Russell's theory several times here and there in later books and papers. He was more reserved on the credentials of the theory of types; however, in wondering if there could be infinitude of them he had not read the book carefully enough, for their finite number was explicitly mentioned; indeed, it constituted a serious limitation to logicism.[19]

WORK SHORTLY BEFORE AND DURING THE GREAT WAR

Two years before this review the *Times literary supplement* had Hardy handle T.L. Heath's edition of Euclid's *Elements.* Taking a much more guarded view than did Heath of its mathematical calibre while praising the editor's endeavours, he did not question the then prevalent habit of converting many of Euclid’s theorems into algebraic form.[20] In the next year came a new venue, the *Cambridge review,* where he castigated a book on the supposed ‘mysticism of mathematical objects such as points at infinity and complex numbers, a study which Whitehead a Jourdain were also to maul;[21] he also had to respond to the author’s self-defence, not a daunting task.[22]

Between 1914 and 1919 Hardy offered a free course each summer term on ‘Elements of mathematics (for non-mathematicians)’; seemingly the only one in his career.[23] The philosopher G.E. Moore (1871-1958) was wont to sit in on colleagues’ courses and take extensive notes, and one year (internal evidence suggests 1915 or 1916) he heard this one. His notes show that it was largely guided by the content of *Principia mathematica* : much set theory including variables, finite and transfinite cardinal and ordinal arithmetic, mathematical induction, the axiom of choice, continuity, and some elementary geometry.[24]

Hardy also had to contend at this time with the dismissal of Russell from his fellowship at Trinity College because of pacifist resistance; during the next war he wrote a short book on the affair,[25] which helped to reinstate Russell there and secure a fellowship from 1944 to 1949. In 1917 Hardy read to the Cambridge Moral Sciences Club an essay on Russell’s philosophy of religion, one atheist praising another one against the advocacy of Christianity by the Cambridge neo-Hegelian philosopher J.M.C. MacTaggart (1866-1925). ‘I should regard it as highly probable that, in another 100 years, belief in anything fairly recognizable as Christianity will be practically extinct, except perhaps among savage races’.[26] Not a profound piece, its existence is very surprising.

During these years Hardy was also in touch with the young Norbert Wiener (1894-1964), after the 18-year-old arrived in Cambridge in 1913 already armed with a Harvard doctorate in which he compared the logic of relations in *Principia mathematica* with that practised in the algebraic logic of Ernst Schröder. The conclusions drawn countered many of Russell’s philosophical aims; a robust discussion ensued, with the sad consequence that Wiener never published any of this study of a topic which indeed remains little examined even now.[27] Hardy was not involved in their exchanges, but he did handle some of Wiener’s related writings. In particular, in 1914 he communicated to the Cambridge Philosophical Society Wiener’s definition of the ordered pair which has become famous.[28] In 1919 he acted similarly for the London Mathematical Society over Wiener’s adaptation of the theory of measurement given in *Principia mathematica* after Russell had warmly refereed it.[29]

In 1918 Hardy sent a paper of his own to the Cambridge Philosophical Society which is unique in his output, mathematical in content but with a strong historical component. One of the principal refinements to rigour in mathematical analysis which had emerged in the late 19th century was the realisation that infinite series of functions could converge in various modes. The insight was made by Karl Weierstrass (1815-1897) and developed by him and various followers, especially from the 1890s onwards: the adjectives ‘uniform’, ‘nonuniform' and 'quasi-uniform' had became attached to some of these modes, and the distinctions between them studied. However, as usual there had been some partial anticipations, including by the Cambridge mathematician G.G. Stokes (1819-1903); and in his paper Hardy compared this work with the successors, defining all the modes, tracking down the original literature, and showing that the subtleties attending the distinctions had not always been noticed.[30]

INTER-WAR ACTIVITIES

Soon after the end of the Great War Hardy became much concerned with reconciliation; for example, he corresponded extensively with the senior Swedish mathematician (and Germanophile) Gösta Mittag-Leffler (1846-1927).[31] Some years later he began to develop contacts in the USA, including the Princeton mathematician Oswald Veblen (1880-1960), which were to involve him during the 1930s in the problem of finding appointments for German refugee mathematicians.[32]

In 1922 Hardy sought Russell's advice again over a long paper on 'Notational relativity' by the Harvard logician Henry Sheffer (1882-1964) submitted to (I think) the Cambridge Philosophical Society; despite Russell's enthusiasm no decision was reached, and this interesting piece remains unpublished.[33] During that year he also published in the new edition of the *Encyclopaedia Britannica* three summaries with bibliographies of his three specialities in mathematics: number theory, infinite series, and the theories of functions.[34]

From 1920 to 1931 Hardy held the Savilian Professorship in Geometry in Oxford University. He began his inaugural lecture by considering appropriate topics for such an occasion; they included the 'important applications of mathematical logic to philosophy', but felt unable to do its justice. So he tried to exhibit the intrinsic interest number theory. This lecture was a main inspiration to his later book-length *Apology*.[35]

Although Hardy did not continue with his introductory mathematics course there, he became more involved with mathematical education. Already a distinguished reviewer in the *Mathematical gazette* of the Mathematical Association,[36] he served as its President in 1925 and 1926. The first of his Addresses posed the question 'What is geometry?', an improbable topic for him perhaps inspired by the title of his chair; dismissive of philosophers on the question, he hardly surpassed them in making rather obvious contrasts between the analytic and projective branches.[37] But he was silent over Whitehead, who had not only written at some length on geometry but also on its educational place, including a decade earlier as President of the Association;[38] the contacts between these two Cambridge mathematicians with philosophical concerns were much less than might be expected.

In the following year, 1926, President Hardy tackled a topic closer to his speciality, in arguing against the Cambridge Mathematical Tripos.[39] He will have supported the elimination in 1906 of the notorious order of "merit" of the Wranglers, and the consequent modernisation of the syllabus for mathematical analysis; his textbook, which had appeared in

1925 in its fourth edition after revisions in 1914 and 1921 of its 1908 début, was a standard source. His view now from Oxford was that the entire scheme should be ‘abolished’, where ‘I mean “abolished”, and not “reformed”’. Reviewing its history, he recalled his own experience of it in the late 1890s, and faulting the syllabus rather than the teachers operating within it. For example, ‘I do remember Mr. Bertrand Russell telling me that he studied electricity for three years, and at the end of them he had never heard of Maxwell’s equations’; but this is a very surprising memory given the undoubted strength of mathematical physics in Cambridge at that time, and curious to be invoked here by a very pure mathematician.[40] By implication the Oxford Mathematics Greats was also to go; he repeated his attack in shorter order in 1930 in the *Oxford magazine,* where he also called for the university to create an institute for mathematics,[41] a move that was effected with all convenient speed in 1953.

Hardy’s criticisms of the Tripos lay largely on grounds of excessive training for examinations, and led to a desire for a different manner of examining. He did not attack the lack of heuristics in the Tripos syllabus, and the reforms that he was to initiate were little better from this point of view. Indeed, they were to lead to its own narrowness, in that his (and Littlewood’s) own favoured branches of mathematics were to be promulgated in British universities to the quite severe exclusion of others.[42]

Hardy’s concern with philosophical questions (as he understood them) increased to the extent that by 1927 he seems to have promised to contribute a volume of ‘Mathematics for philosophers’ to a series edited by the Cambridge philosopher C.K. Ogden (1889-1957).[43] In the end nothing was produced; but part of the intended line of thought may be detected in the Rouse Ball Lecture on ‘Mathematical proof’ delivered at Cambridge in 1928 and published in the philosophical journal *Mind* in the following year.[44] Admitting that ‘mathematical logic also is a subject for professionals’, of which he was not one, he wished to enter into ‘the doubtful ground disputed by mathematics, logic and philosophy’.[45] This phrase nicely captured the current situation in the foundation of mathematics. Logicism was competing with an approach favoured by the German mathematician David Hilbert (1862-1943) and his followers, in which a mathematical theory was axiomatised along with the attendant mathematical logic (in fact more or less the calculi as laid down in *Principia mathematica*) and their consistency, completeness and independence studied formally in ‘metamathematics’. At that time an acrimonious discord had broken out between Hilbert and the ‘intuitionism’ of the Dutchman L.E.J. Brouwer (1881-1966), in which the logical law of excluded middle was abandoned along with attendant mathematical proof methods such as by contradiction; mathematics was to be literally constructed, though by means involving some incoherent philosophy of Brouwer’s creation which assigned a principal status to time.[46]

None of these philosophical positions had great impact on mathematics in general — for example, Hardy’s own. In his article he was most in favour of logicism, although he criticised type theory in certain respects. He characterised the other two positions as ‘finitists’ and disliked both, even the more intelligible version of intuitionism then being expounded by

Hermann Weyl (1885-1955). However, he did not properly distinguish the *meta* mathematical finitism of ‘the logic of Hilbert and his school’ from the mathematical finitism of Brouwer and Weyl. He also used the word ‘formalism’ to describe Hilbert’s position, which in fact had been introduced recently as a criticism by Brouwer and was *never* used by Hilbert himself. Hardy had discussed these issues, and also the epistemological status of truth, with Russell’s main followers of that time, Ludwig Wittgenstein (1889-1951) and Frank Ramsey (1903-1930), and he also reported on their views.[47]

Like other mathematicians of that time interested in foundational issues, especially those for whom proof was an important aspect of their practice, Hardy appears to have become most attracted to metamathematics among the three positions; early in 1929, during a sabbatical year in the USA, he lectured twice on ‘Hilbert’s logic’.[48] It would have been fascinating to see him expand his lectures and the recent article into the book for Ogden, especially after Kurt Gödel (1906-1978) was to show in 1931 that neither logicism nor metamathematics could be executed in the manner in which their proponents envisioned it.[49] However, Hardy’s normal interests, especially the collaboration with Littlewood, dominated, which must have been fortified by his return to a chair at Cambridge in 1931 upon the retirement of Hobson. His later writings of a historical character were some general articles on mathematical analysis or number theory, and fine obituaries of figures in those areas, including the remarkable work in number theory of the Indian mathematician Srinavasa Ramanujan, F.R.S. (1887-1920), the amazing auto-didact whom he had brought to professional development and recognition in the 1910s.[50]

APOLOGISING AND OTHER WORK IN THE 1940S

Hardy’s health and mathematical powers seem to have declined from the mid 1930s; at all events the production of papers fell to around four a year, though he published the sixth and seventh editions of his textbook in 1933 and 1938, when he also published with his former student E.M. Wright a major book on number theory.[51]

In 1940 Hardy produced the short book mentioned earlier which became his best-known general writing: *A mathematician’s apology,* a rather ironic personal statement about mathematics, stimulated as a extension of his inaugural lecture on number theory in 1920.[52] Early on in the book he surprisingly defended the philosopher F.H. Bradley (1846-1924), whose neo-Hegelianism was in some decline after confrontation by other positions, especially the ‘logical positivism’ of Russell.[53] Later on, for once his historical knowledge was faulty; within a few lines of stating that ‘I do not know an instance of a major mathematical advance initiated by a man past fifty’ he mentioned the political career of P.S. Laplace (1749-1827), who produced major results right to the end.[54] Another counter-example is Hilbert, whose prosecution of metamathematics in his sixties during the 1920s had gained Hardy’s active interest at the end of the decade![55]

Hardy's defence of mathematics was rather trivial, based upon personal factors such as ambition as incentives for research. For more objective factors he appealed to generality of theorems and depth of their consequences; number theory provided most of the examples, but set theory was also mentioned. However, a major feature of the book, for which it is indeed often cited, was his advocacy of the uselessness of mathematics, meaning its irrelevance to practical needs, which could include evil ones. But his snobbish ignorance of applications, reinforced by his pacifism, was especially evident in a section which was based upon a lecture on 'Mathematics in wartime' recently inflicted upon Cambridge undergraduates. He contrasted 'the real mathematics of the real mathematicians' with 'the "trivial" mathematics' of applications, which was dismissed as 'repulsively ugly and intolerably dull; even Littlewood could not make ballistics respectable, and if he could not who can?'.[56] It beggars belief that anyone could publish such insensitive remarks in Britain in *1940* of all years, when ballistics was essential to saving as well as to destroying life.

In 1942 Hardy wrote the short history noted earlier of Russell's dismissal from Trinity College during the Great War.[57] Despite his important role in the reinstallation of Russell in the college in 1944, they do not seem then to have enjoyed any special relationship.

Hardy produced the eighth and ninth editions of his textbook in 1941 and 1944. His last major project was a monograph on the theory of summing infinite series. It includes some of his most valuable historical research, especially on methods used by Euler in the 18th century and the views held by some English analysts in the early decades of the 19th century. It appeared posthumously, after completion by various former students led by L.S. Bosanquet.[58]

Hardy's last review was for *Mathematical gazette*, appearing after a pause there of 26 years. The eminent French mathematician Jacques Hadamard had published an essay on 'the psychology of invention in the mathematical field', recording both his own experiences and that of some colleagues over this little-studied aspect of mathematics.[59] In a lengthy review Hardy followed Hadamard in noting the (many) possible combinations of ideas, and shared his bewilderment concerning the means by which fruitful ones are chosen for theorising. 'I think almost entirely in words and formulae, written if possible, or visualised on paper' and requiring a comfortable environment for effective operation.[60] He ended with a repetition of his preferred divorce of mathematics from applications. Strangely, neither he nor Hadamard mentioned Ramanujan, who surely excites questions about mathematical creation.

CONCLUDING REMARK

Hardy died in the following year, 1947. Among his contemporary mathematicians, an obvious comparison for our purpose is with the Austrian mathematician Hans Hahn (1879-1934). Also a specialist in mathematical analysis (in his case the theory of functions and the calculus of variations), he was strongly interested in philosophy in general and of mathematics and mathematical logic in particular. But he worked in a much stronger philosophical environment than did Hardy, especially after the formation of the ‘Vienna Circle’ in the mid 1920s.[61]

By contrast, Hardy seems never to have received sufficient stimulus, even from Russell, to deepen and formalise his philosophical sense,[62] or to expand his natural interest in history into some substantial account. His concern with mathematical education, doubtless sparked off by the Cambridge reform in the mid 1900s and the writing and revision of his textbook, was also serious but rather intermittent. We owe much important mathematics to him, but might regret that the measure of insightful history and philosophy was modest, especially as he lived through a time when logic, set theory and mathematics became entwined in such novel but profound and perplexing ways.[63] As he correctly wrote himself in 1910, ‘the philosophy of mathematics has suffered a complete revolution during the last fifty years, and there is still much that has not been made finally clear’,[64] and by the 1930s this judgement held still greater force.

ACKNOWLEDGEMENT

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[1] The main source on Hardy’s place in mathematics remains the sequence of essays that several of his former students contributed to the *Journal of the London Mathematical Society,* **25** (1950). Some still living took part in meetings in Oxford and Australia in 1997 to commemorate the 50th anniversary of his death; most of them gave just personal reminiscences, but a much broader context of mathematics training in Hardy’s time is provided by R.A. Rankin in ‘G.H. Hardy as I knew him’, *Gazette of the Australian Mathematical Society,* **25,** 73-81 (1998).

While some manuscripts are held at Trinity College Cambridge and New College Oxford, there is no collective Hardy *Nachlass.* He was notorious for returning letters to their senders with his replies; but as he was a quite prolific correspondent, there are some good collections (notes 8, 13, 24 and 33 below use or mention examples).

[2] G.H. Hardy, *Collected papers,* 7 vols. (Oxford, Clarendon Press, 1966-1979); hereafter cited as ‘*CP* ’. It begins with the obituary written by E.C. Titchmarsh from *Obituary notices of the Royal Society,* **6**, 447-458 (1949); a rather longer version appeared in the sequence cited in the previous footnote. Our main concern is with the last volume, where the responsibilities of editorship fell largely to Rankin.

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[3] The story must be more complicated than this, since I saw some volumes on sale in a second-hand bookshop in Cambridge a few years ago; perhaps they had belonged to the Littlewood portion.

[4] G.H. Hardy, *A mathematician's apology* (Cambridge, Cambridge University Press, 1940); various reprints, including one with a foreword by C.P. Snow (1967).

[5] For details see I. Grattan-Guinness, 'Russell and G.H. Hardy: a study of their relationship', *Russell*, new ser., **11**, 165-179 (1991-1992). The total absence of Hardy from Russell's *Autobiography* (1967-1969) is one of its many deficiencies (compare note 9 below, for example).

[6] On these developments, see I. Grattan-Guinness, *The search for mathematical roots, 1870-1940. Logics, set theories and the foundations of mathematics from Cantor through Russell to Gödel* (Princeton, Princeton University Press, 2000), chs. 2-5 and the extensive further literature cited.

[7] Hardy himself so testified in his 'A theorem concerning the infinite cardinal numbers', *Quarterly journal of pure and applied mathematics*, **35**, 87-94 (1903); also in *CP*, vol. 7, 427-434 (p. 434).

[8] Jourdain Papers (two notebooks containing many letters and draft replies), Institut Mittag-Leffler, Djursholm, Sweden. The letters with Hardy mostly deal with technical details of set theory; those with Russell are broader and much more numerous, and formed the main excuse for I. Grattan-Guinness, *Dear Russell - dear Jourdain. A commentary on Russell's logic, based on his correspondence with Philip Jourdain* (London, Duckworth; New York, Columbia University Press; 1977).

[9] B. Russell, *The principles of mathematics* (Cambridge, Cambridge University Press, 1903; repr. London, Allen and Unwin, 1937). His autobiographical account of its writing is *very* inaccurate; see I. Grattan-Guinness, 'How did Russell write *The principles of mathematics* (1903)?', *Russell*, new ser. **16**, 101-127 (1996-1997).

[10] G.H. Hardy, Review of Russell, *The principles*, *Times literary supplement*, 263 (1903); also in *CP*, vol. 7, 851-854.

[11] These papers are collectively gathered in *CP*, vol. 7, 421-451. They deal with the exponentiation of transfinite cardinal numbers (*op. cit.* note 7), Cantor's continuum hypothesis on the cardinality of the set of real numbers (note 14 below), the cardinality of closed sets, and re-ordering infinite convergent series of terms.

[12] This history is very well covered. See G. H. Moore, *Zermelo's axiom of choice* (New York, Springer, 1982), chs. 1 and 2 *passim* ; F.A. Medvedeff, *Rannyaya istoriya aksiomi vybora* (Moscow, Nauka, 1982), esp. ch. 5; and J. Cassinet and M. Guillemot, 'L'axiome du choix dans les mathématiques de Cauchy (1821) à Gödel (1940)', 2 vols. (University of Toulouse double *docteur d'état des sciences*, 1983).

[13] Undated letter (I suspect written in 1905) to Russell (Russell Archives, McMaster University).

[14] G.H. Hardy, ‘The continuum and the second number-class’, *Proceedings of the London Mathematical Society*, **(2)4**, 10-17 (1906-1907); also in *CP*, vol. 7, 438-445.

[15] See I. Grattan-Guinness, ‘How Bertrand Russell discovered his paradox’, *Historia mathematica*, **5**, 127-137 (1978).

[16] This manuscript will appear in due course in B. Russell, *Collected papers*, vol. 5. On his struggles at this time see Grattan-Guinness *op. cit.* note 6, 360-364.

[17] A.N. Whitehead and B. Russell, *Principia mathematica*, 3 vols. (Cambridge, Cambridge University Press, 1910-1913); on its preparation, publication and reception see Grattan-Guinness *op. cit.* note 6, esp. chs. 7-8. Whitehead was supposed to write on his own a fourth volume on aspects of geometry, but he abandoned it during the Great War and after his death in 1947 all his manuscripts were destroyed following his instructions.

[18] G.H. Hardy, Review of *Principia mathematica*, vol. 1 (1910), *Times literary supplement*, 321-322 (1911); also in *CP*, vol. 7, 859-862 (quote from p. 861).

[19] The drawback is that Cantor’s numbers ↵$_{\omega}$ and ω_{ω}, and thereby all larger ones, cannot be defined logistically.

[20] G.H. Hardy, Review of T.L. Heath, *The thirteen books of Euclid’s Elements* (1909), *Times literary supplement*, 420-421 (1909); also in *CP*, vol. 7, 855-858.

[21] G.H. Hardy, Review of H. Berkeley, *Mysticism in modern mathematics* (1910), *Cambridge review*, **31**, lit. supp., xiii-xiv (1909-1910); also in *CP*, vol. 7, 864-866. Whitehead’s review, a little-known piece, appeared as ‘The philosophy of mathematics’, *Science progress*, **5,** 234-239 (1910-1911); Jourdain’s were in *Mind*, new ser. **20**, 88-97 (1911), and in *Mathematical gazette*, **5,** 364-366 (1909-1911).

[22] See the correspondence between Hardy and Berkeley in *Cambridge review*, **31**, 466-467, 495-496 (1909-1910).

[23] *Cambridge University reporter*, (1913-14), 90; (1914-15), 90; (1915-16), 82; (1916-17), 81; (1917-18), 93; (1918-19), 80.

[24] G.E. Moore Papers, Cambridge University Library, file 10/6/1-4. File 8H/5 contains some letters from Hardy.

[25] G.H. Hardy, *Bertrand Russell and Trinity: a college controversy of the last war* (Cambridge, Cambridge University Press, 1942); repr. 1970 with an introduction by C.D. Broad, and also text only (New York, Arno Press, 1977). For a fine modern appraisal, see P. Delany, ‘Russell’s dismissal from Trinity: a study in high table politics’, *Russell*, new ser. **6**, 39-61 (1986). He draws upon Russell’s and Hardy’s letters of the time among other sources, and contrasts Hardy’s position with the vacillations of Whitehead (‘To the Master and Fellows of Trinity College, Cambridge’, *Ibidem*, 62-70).

[26] G.H. Hardy, 'Mr. Russell as a religious teacher', *Cambridge magazine*, **6**, 624-626, 650-653 (1917); also in *Russell*, new ser. **1**, 119-135 (1981-1982). On the context see J. Pitt, 'Russell and the Cambridge Moral Sciences Club', *Ibidem*, 103-118.

[27] As a partial substitute see I. Grattan-Guinness, 'Wiener on the logics of Russell and Schröder. An account of his doctoral thesis, and of his subsequent discussion of it with Russell', *Annals of science*, **32**, 103-132 (1975).

[28] N. Wiener, 'A simplification of the logic of relations', *Proceedings of the Cambridge Philosophical Society*, 17, 387-390 (1914); also in his *Collected works*, vol. 1 (Cambridge, Mass., MIT Press, 1976), 29-32. His title reflects his motivation, which was to reduce a certain assumption made in Russell's theory of types. Hardy also handled for the Society two more logical papers by Wiener at that time, which re-appear in Wiener's *Works*, 34-57.

[29] N. Wiener, 'A new theory of measurement: a study in the logic of mathematics', *Proceedings of the London Mathematical Society*, **(2)19**, 181-205 (1921); also in *Works* (*ibidem*), 58-82.

[30] G.H. Hardy, 'Sir George Stokes and the concept of uniform convergence', *Proceedings of the Cambridge Philosophical Society*, **19**, 148-156 (1918); also in *CP*, vol. 7, 505-513.

[31] See J.W. Dauben, 'Mathematics and World War I: the international diplomacy of G.H. Hardy and Gösta Mittag-Leffler as reflected in their personal correspondence', *Historia mathematica*, **7**, 261-288 (1980).

[32] There are substantial files of letters from Hardy in the Veblen Papers, Washington (D.C.), Library of Congress.

[33] H.M. Sheffer, 'The general theory of notational relativity' (1921, manuscript): copies in Sheffer Papers, Cambridge (Massachusetts), Harvard University, Houghton Library; and in Russell Archives (which also has Russell's report). On this famously non-publishing logician, see M. Scanlan, 'The known and unknown H.M. Sheffer', *Transactions of the C.S. Peirce Society*, **36**, 193-224 (2000).

[34] *Encyclopaedia Britannica*, 13th ed., vol. 31 (1922), 876-878; not in his *Papers*, nor in the bibliography there. A companion piece on mathematical logic by Russell's recent student Jean Nicod (1893-1930) precedes on pp. 874-876.

[35] G.H. Hardy, *Some famous problems of the theory of numbers and in particular Waring's problem* (Oxford, Oxford University Press, 1920); also in *CP*, vol. 1, 647-679 (pp. 649-650). For some information on mathematics at Oxford at the time, see J. Fauvel, R. Flood and R. Wilson (eds.), *Oxford figures* (Oxford, Oxford University Press, 2000), ch. 14.

[36] Hardy's reviews there, from 1903 to 1920 and one in 1946 (see note 60 below), are gathered together in *CP*, vol. 7, 808-838.

[37] G.H. Hardy, 'What is geometry?', *Mathematical gazette*, **12**, 309-316 (1925); also in *CP*, vol. 7, 519-526.

[38] On the context of the Mathematical Association, see M. Price, *Mathematics for the multi-tude? A history of the Mathematical Association* (Leicester, Mathematical Association, 1994), esp. ch. 5. On Whitehead's educational interests, see my *op. cit.* note 6, 412-413.
[39] G.H. Hardy, 'The case against the mathematical Tripos', *Mathematical gazette*, **13**, 61-71 (1926); also in **32**, 134-145 (1948); also in *CP*, vol. 7, 527-538.
[40] *Ibidem*, 531. On Russell's powers of recollection , compare note 9 and text.
[41] G.H. Hardy, 'Mathematics', *Oxford magazine*, **48,** 819-821 (1930); also in *CP*, vol. 7, 607-609. The fifth edition of his textbook had appeared in 1928.
[42] Some broad issues of British university mathematics in the 20th century, including Hardy's role, are raised here; not well studied in general, they are beyond the scope of this article.
[43] The publisher of the series was Kegan Paul, Trench, Trubner. I have seen Hardy's book listed at the front of Russell's *An analysis of matter* (1927); doubtless it was listed also in other volumes of the time. No letters on it survive in the Ogden Papers, McMaster University (information from Carl Spadoni).
[44] G.H. Hardy, 'Mathematical proof', *Mind*, new ser. **38**, 1-25 (1929); also in *CP*, vol. 7, 581-606; and in P. Ewald (ed.), *From Kant to Hilbert. A source book in the foundations of mathematics*, 2 vols., (New York and Oxford, Clarendon Press, 1996), 1243-1263. Hardy's request to editor Moore about publishing there is in the file cited in note 24.
[45] *Ibidem* (*CP*), 581.
[46] For sources on these developments, see P. Mancosu (ed.), *From Brouwer to Hilbert* (New York and Oxford, Oxford University Press, 1998).
[47] Hardy *op. cit.* note 44, 590-591, 603-605. On this context see Grattan-Guinness *op. cit.* note 6, 436-438.
[48] *Bulletin of the American Mathematical Society*, **35**, 280 (11 January, Lehigh University), 425 (23 March, University of Iowa) (1929).
[49] On the reception of Gödel's theorems, see Grattan-Guinness *op. cit.* note 6, ch. 9 *passim* ; and P. Mancosu, 'Between Vienna and Berlin: the immediate reception of Gödel's incompleteness theorems', *History and philosophy of logic*, **20**, 33-45 (1999).
[50] The main obituaries are gathered together in *CP*, vol. 7, 701-801; in addition to Ramanujan, Hilbert, Mittag-Leffler and Hobson, he treated C. Jordan, J.W.L. Glaisher, T.J.l'A. Bromwich, R.E.A.C. Paley, E. Landau, W.H. Young, H. Lebesgue and J.R. Wilton. His obituary of Ramanujan reappeared at the head of the edition of Ramanujan's papers which he co-edited in 1927; he also wrote an appraisal in 1937 (pp. 612-630).

In 1930 Hardy warmly reviewed for *Nature* the source book in mathematics recently published by the historian D. E. Smith (p. 844); however, he lamented the absence of Cantor's writings from the chosen texts.

[51] G.H. Hardy and E.M. Wright, *An introduction to the theory of numbers* (Oxford, Clarendon Press, 1938).

[52] See Hardy's text in *Apology op. cit.* note 4, sects. 6 and 19. However, his historical judgement of the contributions to number theory of John Farey (sect. 8) is corrigible.

[53] Hardy *ibidem,* sect. 2. See I. Grattan-Guinness, 'Russell's logicism versus Oxbridge logics, 1890-1925. A contribution to the real history of twentieth-century English philosophy', *Russell*, new ser., **5**, 101-131 (1985-1986).

[54] Hardy *ibidem,* sect. 4. On Laplace last major results, see C.C. Gillipsie with R. Fox and I. Grattan-Guinness, *Pierre Simon Laplace. A life in exact science* (Princeton, Princeton University Press, 1998), esp. ch. 28.

[55] Further candidates include Euler, Lagrange, Gauss, Weierstrass, Kronecker, … .

[56] G.H. Hardy, 'Mathematics in wartime', *Eureka*, **1**, no. 3, 5-8 (1940); also in *CP*, vol. 7, 631-643 (p. 631). The article is the basis of *Apology,* sect. 28.

[57] G.H. Hardy *op. cit.* note 25.

[58] G.H. Hardy, *Divergent series* (Oxford, Clarendon Press, 1949); see esp. chs. 2 and 1.

[59] J. Hadamard, *The psychology of invention in the mathematical field* (Princeton, Princeton University Press, 1945).

[60] G.H. Hardy, Review of Hadamard *ibidem*, *Mathematical gazette*, **30**, 111-115 (1946); also in *CP*, vol. 7, 834-838. The quotation was motivated by Hadamard's p. 84.

[61] Hahn's philosophical writings are gathered together in his *Gesammelte Abhandlungen,* vol. 3, (Vienna and New York, Springer, 1997).

[62] An intriguing comparison lies with the Hungarian mathematician Georg Pólya (1887-1985) who from the 1950s was famously to stress intuition in mathematics and its relationship to rigour. As a young man he studied under Hardy in the mid 1920s, and co-authored with him and Littlewood the book *Inequalities* (Cambridge, Cambridge University Press, 1934). He was mentioned in Hadamard *op. cit.* note 59, pp. 84-85.

[63] For a comparison of these two mathematicians and also Weyl and Wiener, see I. Grattan-Guinness, 'Mathematics and symbolic logics: some notes on an uneasy relationship', *History and philosophy of logic,* **20**, 159-167 (1999, publ. 2000).

[64] Hardy *op. cit.* note 21, 864.