Lecture 8: More on Simply-Typed Lambda Calculus:
substitution lemma, preservation, erasure and type inference

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Typing rules

\[ \Gamma \vdash \text{true} : \text{Bool} \]  \hspace{1cm} (T-\text{TRUE})

\[ \Gamma \vdash \text{false} : \text{Bool} \]  \hspace{1cm} (T-\text{FALSE})

\[ \Gamma \vdash t_1 : \text{Bool} \quad \Gamma \vdash t_2 : T \quad \Gamma \vdash t_3 : T \]
\[ \Gamma \vdash \text{if } t_1 \text{ then } t_2 \text{ else } t_3 : T \]  \hspace{1cm} (T-IF)

\[ \Gamma, x : T_1 \vdash t_2 : T_2 \]
\[ \Gamma \vdash \lambda x : T_1 . t_2 : T_1 \rightarrow T_2 \]  \hspace{1cm} (T-ABS)

\[ x : T \in \Gamma \]
\[ \Gamma \vdash x : T \]  \hspace{1cm} (T-VAR)

\[ \Gamma \vdash t_1 : T_{11} \rightarrow T_{12} \quad \Gamma \vdash t_2 : T_{11} \]
\[ \Gamma \vdash t_1 \ t_2 : T_{12} \]  \hspace{1cm} (T-APP)
Typing derivations

**Exercise 9.2.2:** Show (by drawing derivation trees) that the following terms have the indicated types:

1. $f : \text{Bool} \rightarrow \text{Bool} \vdash f (\text{if false then true else false}) : \text{Bool}$

2. $f : \text{Bool} \rightarrow \text{Bool} \vdash \lambda x : \text{Bool}. f (\text{if } x \text{ then false else } x) : \text{Bool} \rightarrow \text{Bool}$
The two typing relations

Question: What is the relation between these two statements?

1. $t : T$
2. $\vdash t : T$

First answer: These two relations are completely different things.

We are dealing with several different small programming languages, each with its own typing relation (between terms in that language and types in that language).

For the simple language of numbers and booleans, typing is a binary relation between terms and types ($t : T$).

For $\vdash$, typing is a ternary relation between contexts, terms, and types ($\vdash t : T$).

(When the context is empty — because the term has no free variables — we often write $t : T$ to mean $\vdash t : T$.)
The two typing relations

Question: What is the relation between these two statements?

1. \( t : T \)
2. \( \Gamma \vdash t : T \)

First answer: These two relations are completely different things.

- We are dealing with several different small programming languages, *each with its own typing relation* (between terms in that language and types in that language).
- For the simple language of numbers and booleans, typing is a *binary* relation between terms and types (\( t : T \)).
- For \( \lambda \rightarrow \), typing is a *ternary* relation between contexts, terms, and types (\( \Gamma \vdash t : T \)).

(When the context is empty — because the term has no free variables — we often write \( \vdash t : T \) to mean \( \emptyset \vdash t : T \).)
Conservative extension

Second answer: The typing relation for $\lambda \rightarrow$ conservatively extends the one for the simple language of numbers and booleans.

- Write “language 1” for the language of numbers and booleans and “language 2” for the simply typed lambda-calculus with base types $\text{Nat}$ and $\text{Bool}$.
- The terms of language 2 include all the terms of language 1; similarly typing rules.
- Write $t :_1 T$ for the typing relation of language 1.
- Write $\Gamma \vdash t :_2 T$ for the typing relation of language 2.
- Theorem: Language 2 conservatively extends language 1: If $t$ is a term of language 1 (involving only booleans, conditions, numbers, and numeric operators) and $T$ is a type of language 1 (either $\text{Bool}$ or $\text{Nat}$), then $t :_1 T$ iff $\emptyset \vdash t :_2 T$. 
Preservation (and Weaking, Permutation, Substitution)
Review: Proving progress

Let’s quickly review the steps in the proof of the progress theorem:
- inversion lemma for typing relation
- canonical forms lemma
- progress theorem
Inversion

Lemma:

1. If $\Gamma \vdash \text{true} : R$, then $R = \text{Bool}$.
2. If $\Gamma \vdash \text{false} : R$, then $R = \text{Bool}$.
3. If $\Gamma \vdash \text{if } t_1 \text{ then } t_2 \text{ else } t_3 : R$, then $\Gamma \vdash t_1 : \text{Bool}$ and $\Gamma \vdash t_2, t_3 : R$.
4. If $\Gamma \vdash x : R$, then
Inversion

Lemma:

1. If $\Gamma \vdash \text{true} : R$, then $R = \text{Bool}$.
2. If $\Gamma \vdash \text{false} : R$, then $R = \text{Bool}$.
3. If $\Gamma \vdash \text{if } t_1 \text{ then } t_2 \text{ else } t_3 : R$, then $\Gamma \vdash t_1 : \text{Bool}$ and $\Gamma \vdash t_2, t_3 : R$.
4. If $\Gamma \vdash x : R$, then $x : R \in \Gamma$.
5. If $\Gamma \vdash \lambda x : T_1 . t_2 : R$, then
Lemma:

1. If $\Gamma \vdash \text{true} : R$, then $R = \text{Bool}$.
2. If $\Gamma \vdash \text{false} : R$, then $R = \text{Bool}$.
3. If $\Gamma \vdash \text{if } t_1 \text{ then } t_2 \text{ else } t_3 : R$, then $\Gamma \vdash t_1 : \text{Bool}$ and $\Gamma \vdash t_2, t_3 : R$.
4. If $\Gamma \vdash x : R$, then $x : R \in \Gamma$.
5. If $\Gamma \vdash \lambda x : T_1 . t_2 : R$, then $R = T_1 \rightarrow R_2$ for some $R_2$ with $\Gamma, x : T_1 \vdash t_2 : R_2$.
6. If $\Gamma \vdash t_1 \ t_2 : R$, then
Lemma:

1. If $\Gamma \vdash \text{true} : R$, then $R = \text{Bool}$.
2. If $\Gamma \vdash \text{false} : R$, then $R = \text{Bool}$.
3. If $\Gamma \vdash \text{if } t_1 \text{ then } t_2 \text{ else } t_3 : R$, then $\Gamma \vdash t_1 : \text{Bool}$ and $\Gamma \vdash t_2, t_3 : R$.
4. If $\Gamma \vdash x : R$, then $x : R \in \Gamma$.
5. If $\Gamma \vdash \lambda x : T_1 . t_2 : R$, then $R = T_1 \rightarrow R_2$ for some $R_2$ with $\Gamma, x : T_1 \vdash t_2 : R_2$.
6. If $\Gamma \vdash t_1 \ t_2 : R$, then there is some type $T_{11}$ such that $\Gamma \vdash t_1 : T_{11} \rightarrow R$ and $\Gamma \vdash t_2 : T_{11}$.
Canonical Forms

Lemma:
Canonical Forms

**Lemma:**

1. If \( v \) is a value of type \( \text{Bool} \), then \( v \) is either \text{true} or \text{false}.
2. If \( v \) is a value of type \( T_1 \rightarrow T_2 \), then \( v \) has the form \( \lambda x : T_1 . t_2 \).
Progress

**Theorem:** Suppose $t$ is a closed, well-typed term (that is, $\vdash t : T$ for some $T$). Then either $t$ is a value or else there is some $t'$ with $t \rightarrow t'$. 
Preservation

Theorem: If $\Gamma \vdash t : T$ and $t \rightarrow t'$, then $\Gamma \vdash t' : T$.

Steps of proof:
- Weakening
- Permutation
- Substitution preserves types
- Reduction preserves types (i.e., preservation)
Weakening and Permutation

Weakening tells us that we can *add assumptions* to the context without losing any true typing statements.

*Lemma:* If $\Gamma \vdash t : T$ and $x \notin \text{dom}(\Gamma)$, then $\Gamma, x:S \vdash t : T$. 
Weakening and Permutation

Weakening tells us that we can add assumptions to the context without losing any true typing statements.

*Lemma:* If $\Gamma \vdash t : T$ and $x \notin \text{dom}(\Gamma)$, then $\Gamma, x:S \vdash t : T$.

Permutation tells us that the order of assumptions in (the list) $\Gamma$ does not matter.

*Lemma:* If $\Gamma \vdash t : T$ and $\Delta$ is a permutation of $\Gamma$, then $\Delta \vdash t : T$. 
Weakening and Permutation

Weakening tells us that we can add assumptions to the context without losing any true typing statements.

*Lemma:* If $\Gamma \vdash t : T$ and $x \notin \text{dom}(\Gamma)$, then $\Gamma, x : S \vdash t : T$.

Moreover, the latter derivation has the same depth as the former.

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*Lemma:* If $\Gamma \vdash t : T$ and $\Delta$ is a permutation of $\Gamma$, then $\Delta \vdash t : T$.

Moreover, the latter derivation has the same depth as the former.
Preservation

*Theorem:* If $\Gamma \vdash t : T$ and $t \rightarrow t'$, then $\Gamma \vdash t' : T$.

*Proof:* By induction
Preservation

*Theorem:* If \( \Gamma \vdash t : T \) and \( t \rightarrow t' \), then \( \Gamma \vdash t' : T \).

*Proof:* By induction on typing derivations.

Which case is the hard one??
Preservation

**Theorem:** If $\Gamma \vdash t : T$ and $t \longrightarrow t'$, then $\Gamma \vdash t' : T$.

**Proof:** By induction on typing derivations.

**Case T-App:** Given

- $t = t_1 t_2$
- $\Gamma \vdash t_1 : T_{11} \to T_{12}$
- $\Gamma \vdash t_2 : T_{11}$
- $T = T_{12}$

Show $\Gamma \vdash t' : T_{12}$
Preservation

*Theorem:* If $\Gamma \vdash t : T$ and $t \rightarrow t'$, then $\Gamma \vdash t' : T$.

*Proof:* By induction on typing derivations.

Case $T\text{-App}$: Given $t = t_1 t_2$

\begin{align*}
\Gamma &\vdash t_1 : T_{11} \rightarrow T_{12} \\
\Gamma &\vdash t_2 : T_{11} \\
T &\equiv T_{12}
\end{align*}

Show $\Gamma \vdash t' : T_{12}$

By the inversion lemma for evaluation, there are three subcases...
Preservation

**Theorem:** If $\Gamma \vdash t : T$ and $t \rightarrow t'$, then $\Gamma \vdash t' : T$.

**Proof:** By induction on typing derivations.

Case T-App: Given $t = t_1 \ t_2$

$\Gamma \vdash t_1 : T_{11} \rightarrow T_{12}$

$\Gamma \vdash t_2 : T_{11}$

$T = T_{12}$

Show $\Gamma \vdash t' : T_{12}$

By the inversion lemma for evaluation, there are three subcases...

Subcase: $t_1 = \lambda x : T_{11}. \ t_{12}$

$t_2$ a value $v_2$

$t' = [x \mapsto v_2]t_{12}$
Preservation

**Theorem:** If $\Gamma \vdash t : T$ and $t \rightarrow t'$, then $\Gamma \vdash t' : T$.

**Proof:** By induction on typing derivations.

**Case T-App:** Given $t = t_1 \ t_2$

- $\Gamma \vdash t_1 : T_{11} \rightarrow T_{12}$
- $\Gamma \vdash t_2 : T_{11}$
- $T = T_{12}$

Show $\Gamma \vdash t' : T_{12}$

By the inversion lemma for evaluation, there are three subcases...

**Subcase:** $t_1 = \lambda x : T_{11}. \ t_{12}$

- $t_2$ a value $v_2$
- $t' = [x \mapsto v_2]t_{12}$

Uh oh.
Preservation

**Theorem:** If $\Gamma \vdash t : T$ and $t \rightarrow t'$, then $\Gamma \vdash t' : T$.

**Proof:** By induction on typing derivations.

**Case T-App:** Given 

$t = t_1 \ t_2$

$\Gamma \vdash t_1 : T_{11} \rightarrow T_{12}$

$\Gamma \vdash t_2 : T_{11}$

$T = T_{12}$

Show $\Gamma \vdash t' : T_{12}$

By the inversion lemma for evaluation, there are three subcases...

**Subcase:** $t_1 = \lambda x : T_{11}. \ t_{12}$

$t_2$ a value $v_2$

$t' = [x \mapsto v_2]t_{12}$

Uh oh. What do we need to know to make this case go through??
The “Substitution Lemma”

Lemma: If $\Gamma, x:S \vdash t : T$ and $\Gamma \vdash s : S$, then $\Gamma \vdash [x \rightarrow s]t : T$.

I.e., “Types are preserved under substitution.”
The “Substitution Lemma”

**Lemma:** If $\Gamma, x:S \vdash t : T$ and $\Gamma \vdash s : S$, then $\Gamma \vdash [x \leftarrow s]t : T$.

**Proof:** By induction on the *depth* of a derivation of $\Gamma, x:S \vdash t : T$. Proceed by cases on the final typing rule used in the derivation.
The “Substitution Lemma”

Lemma: If $\Gamma, x:S \vdash t : T$ and $\Gamma \vdash s : S$, then $\Gamma \vdash [x \mapsto s]t : T$.

Proof: By induction on the depth of a derivation of $\Gamma, x:S \vdash t : T$. Proceed by cases on the final typing rule used in the derivation.
The “Substitution Lemma”

**Lemma:** If $\Gamma, x:S \vdash t : T$ and $\Gamma \vdash s : S$, then $\Gamma \vdash [x \mapsto s]t : T$.

**Proof:** By induction on the *depth* of a derivation of $\Gamma, x:S \vdash t : T$. Proceed by cases on the final typing rule used in the derivation.

**Case** $T$-**App:**

$t = t_1 \cdot t_2$

$\Gamma, x:S \vdash t_1 : T_2 \to T_1$

$\Gamma, x:S \vdash t_2 : T_2$

$T = T_1$

By the induction hypothesis, $\Gamma \vdash [x \mapsto s]t_1 : T_2 \to T_1$ and $\Gamma \vdash [x \mapsto s]t_2 : T_2$. By $T$-**App**, $\Gamma \vdash [x \mapsto s]t_1 \cdot [x \mapsto s]t_2 : T$, i.e., $\Gamma \vdash [x \mapsto s](t_1 \cdot t_2) : T$. 


The “Substitution Lemma”

**Lemma:** If $\Gamma, x:S \vdash t : T$ and $\Gamma \vdash s : S$, then $\Gamma \vdash [x \mapsto s]t : T$.

**Proof:** By induction on the depth of a derivation of $\Gamma, x:S \vdash t : T$. Proceed by cases on the final typing rule used in the derivation.

**Case T-VAR:** $t = z$

with $z:T \in (\Gamma, x:S)$

There are two sub-cases to consider, depending on whether $z$ is $x$ or another variable. If $z = x$, then $[x \mapsto s]z = s$. The required result is then $\Gamma \vdash s : S$, which is among the assumptions of the lemma. Otherwise, $[x \mapsto s]z = z$, and the desired result is immediate.
The “Substitution Lemma”

Lemma: If $\Gamma, x:S \vdash t : T$ and $\Gamma \vdash s : S$, then $\Gamma \vdash [x \leftrightarrow s]t : T$.

Proof: By induction on the depth of a derivation of $\Gamma, x:S \vdash t : T$. Proceed by cases on the final typing rule used in the derivation.

Case T-Abs: \[ t = \lambda y:T_2.t_1 \quad T = T_2 \to T_1 \]
\[ \Gamma, x:S, y:T_2 \vdash t_1 : T_1 \]

By our conventions on choice of bound variable names, we may assume $x \not= y$ and $y \not\in FV(s)$. Using permutation on the given subderivation, we obtain $\Gamma, y:T_2, x:S \vdash t_1 : T_1$. Using weakening on the other given derivation ($\Gamma \vdash s : S$), we obtain $\Gamma, y:T_2 \vdash s : S$. Now, by the induction hypothesis, $\Gamma, y:T_2 \vdash [x \leftrightarrow s]t_1 : T_1$. By T-Abs, $\Gamma \vdash \lambda y:T_2. \ [x \leftrightarrow s]t_1 : T_2 \to T_1$, i.e. (by the definition of substitution), $\Gamma \vdash [x \leftrightarrow s]\lambda y:T_2. \ t_1 : T_2 \to T_1$. 
Erasure and Typability
Erasure

We can transform terms in $\lambda \rightarrow$ to terms of the untyped lambda-calculus simply by erasing type annotations on lambda-abstractions.

\[
\begin{align*}
erase(x) &= x \\
erase(\lambda x : T_1. \ t_2) &= \lambda x. \ erase(t_2) \\
erase(t_1 \ t_2) &= \ erase(t_1) \ erase(t_2)
\end{align*}
\]
Typability

Conversely, an untyped $\lambda$-term $m$ is said to be typable if there is some term $t$ in the simply typed lambda-calculus, some type $T$, and some context $\Gamma$ such that $\text{erase}(t) = m$ and $\Gamma \vdash t : T$.

This process is called type reconstruction or type inference.
Typability

Conversely, an untyped $\lambda$-term $m$ is said to be *typable* if there is some term $t$ in the simply typed lambda-calculus, some type $T$, and some context $\Gamma$ such that $\text{erase}(t) = m$ and $\Gamma \vdash t : T$.

This process is called *type reconstruction* or *type inference*.

Example: Is the term

$$\lambda x. \ x \ x$$

typable?
More About Bound Variables
Substitution

Our definition of evaluation is based on the “substitution” of values for free variables within terms.

\[(\lambda x. t_{12}) v_2 \rightarrow [x \mapsto v_2]t_{12} \quad (E\text{-AppAbs})\]

But what is substitution, exactly? How do we define it?
Substitution

For example, what does

\[(\lambda x. \ x \ (\lambda y. \ x \ y)) \ (\lambda x. \ x \ y \ x)\]

reduce to?

Note that this example is not a “complete program” — the whole term is not closed. We are mostly interested in the reduction behavior of closed terms, but reduction of open terms is also important in some contexts:

- program optimization
- alternative reduction strategies such as “full beta-reduction”
Formalizing Substitution

Consider the following definition of substitution:

\[
[x \mapsto s]x = s \\
[x \mapsto s]y = y \quad \text{if } x \neq y \\
[x \mapsto s](\lambda y.t_1) = \lambda y. ([x \mapsto s]t_1) \\
[x \mapsto s](t_1 \ t_2) = ([x \mapsto s]t_1)([x \mapsto s]t_2)
\]

What is wrong with this definition?
Formalizing Substitution

Consider the following definition of substitution:

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[x \mapsto s]x = s \\
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[x \mapsto s](\lambda y. t_1) = \lambda y. ([x \mapsto s]t_1) \\
[x \mapsto s](t_1 \ t_2) = ([x \mapsto s]t_1)([x \mapsto s]t_2)
\]

What is wrong with this definition?

It substitutes for free and \textit{bound} variables!

\[
[x \mapsto y](\lambda x. \ x) = \lambda x. y
\]

This is not what we want!
Substitution, take two

\[ [x \mapsto s]x = s \]
\[ [x \mapsto s]y = y \quad \text{if } x \neq y \]
\[ [x \mapsto s](\lambda y. t_1) = \lambda y. ([x \mapsto s]t_1) \quad \text{if } x \neq y \]
\[ [x \mapsto s](\lambda x. t_1) = \lambda x. t_1 \]
\[ [x \mapsto s](t_1 \ t_2) = ([x \mapsto s]t_1)([x \mapsto s]t_2) \]

What is wrong with this definition?
Substitution, take two

\[
[x \mapsto s]x = s \\
[x \mapsto s]y = y \quad \text{if } x \not= y \\
[x \mapsto s](\lambda y. t_1) = \lambda y. ([x \mapsto s]t_1) \quad \text{if } x \not= y \\
[x \mapsto s](\lambda x. t_1) = \lambda x. t_1 \\
[x \mapsto s](t_1 \ t_2) = ([x \mapsto s]t_1)([x \mapsto s]t_2)
\]

What is wrong with this definition?

It suffers from \textit{variable capture}!

\[
[x \mapsto y](\lambda y. x) = \lambda x. x
\]

This is also not what we want.
Substitution, take three

\[[x \mapsto s]x = s\]
\[[x \mapsto s]y = y\] if \(x \neq y\)
\[[x \mapsto s](\lambda y. t_1) = \lambda y. ([x \mapsto s]t_1)\] if \(x \neq y, y \not\in FV(s)\)
\[[x \mapsto s](\lambda x. t_1) = \lambda x. t_1\]
\[[x \mapsto s](t_1 \ t_2) = ([x \mapsto s]t_1)([x \mapsto s]t_2)\]

What is wrong with this definition?
Substitution, take three

\[ [x \mapsto s]x = s \]
\[ [x \mapsto s]y = y \quad \text{if } x \neq y \]
\[ [x \mapsto s](\lambda y. t_1) = \lambda y. ([x \mapsto s]t_1) \quad \text{if } x \neq y, y \notin \text{FV}(s) \]
\[ [x \mapsto s](\lambda x. t_1) = \lambda x. t_1 \]
\[ [x \mapsto s](t_1 \ t_2) = ([x \mapsto s]t_1)([x \mapsto s]t_2) \]

What is wrong with this definition?

Now substitution is a \textit{partial function}!

E.g., \([x \mapsto y](\lambda y. x)\) is undefined.

But we want an result for every substitution.
Bound variable names shouldn’t matter

It’s annoying that the “spelling” of bound variable names is causing trouble with our definition of substitution.

Intuition tells us that there shouldn’t be a difference between the functions $\lambda x.x$ and $\lambda y.y$. Both of these functions do exactly the same thing.

Because they differ only in the names of their bound variables, we’d like to think that these are the same function.

We call such terms *alpha-equivalent*. 
Alpha-equivalence classes

In fact, we can create equivalence classes of terms that differ only in the names of bound variables.

When working with the lambda calculus, it is convenient to think about these equivalence classes, instead of raw terms.

For example, when we write \( \lambda x \cdot x \) we mean not just this term, but the class of terms that includes \( \lambda y \cdot y \) and \( \lambda z \cdot z \).

We can now freely choose a different representative from a term’s alpha-equivalence class, whenever we need to, to avoid getting stuck.
Substitution, for alpha-equivalence classes

Now consider substitution as an operation over *alpha-equivalence classes* of terms.

\[
\begin{align*}
[x \mapsto s]x &= s \\
[x \mapsto s]y &= y & \text{if } x \neq y \\
[x \mapsto s](\lambda y . t_1) &= \lambda y . ([x \mapsto s]t_1) & \text{if } x \neq y, y \not\in \text{FV}(s) \\
[x \mapsto s](\lambda x . t_1) &= \lambda x . t_1 \\
[x \mapsto s](t_1 \ t_2) &= ([x \mapsto s]t_1)([x \mapsto s]t_2)
\end{align*}
\]

Examples:

1. \([x \mapsto y](\lambda y . x)\) must give the same result as \([x \mapsto y](\lambda z . x)\).
   We know the latter is \(\lambda z . y\), so that is what we will use for the former.

2. \([x \mapsto y](\lambda x . z)\) must give the same result as \([x \mapsto y](\lambda w . z)\).
   We know the latter is \(\lambda w . z\) so that is what we use for the former.
Review

So what does

\[(\lambda x. \ x \ (\lambda y. \ x \ y)) \ (\lambda x. \ x \ y \ x)\]

reduce to?