General Analysis of Longitudinal Dispersion in Nonuniform Flow

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Abstract. An analytical method is proposed by which the effects of flow nonuniformity and variable dispersion coefficients can be evaluated for problems involving longitudinal dispersion in porous media. A boundary layer approximation is used to develop general solutions of the one-dimensional convective–dispersion equation for steady flow. Several examples are considered by using the analytical method, and the general effect of flow nonuniformity on dispersion is discussed. Comparisons of the analytical solution with numerical solutions of the exact equation indicate that the method will yield accurate results in many applications.

Although dispersion in porous media has been studied extensively in the laboratory [*Harleman and Rumer*, 1963; *List and Brooks*, 1967] and by theoretical analysis [*de Josselin de Jong*, 1958; *Bachmat*, 1966], the application of research results to field conditions remains limited mainly because of the complexity of the natural flows. Exact analytical solutions that describe dispersion are limited to flows with a constant convective velocity, but most applications involve complex velocity fields in which the velocity dependence of the dispersion coefficient may be important. Several numerical techniques [*Shamir and Harleman*, 1967; *Reddell and Sunada*, 1970; *Guymon*, 1970; *Guymon et al.*, 1970] have been applied to obtain solutions of the convective–dispersion equation, but the emphasis has been on uniform flows. Although the numerical techniques can in principle be extended to more complex flows, the practicality of these methods remains to be demonstrated for field conditions. When the limited precision of data that describe most field situations is considered, simple analytical estimates of dispersive effects may be as meaningful as detailed numerical solutions.

In this paper an approximate analytical technique is developed to treat longitudinal dispersion in steady flows with variations of velocity along a streamline. The analysis, which is based on the convective–dispersion equation with variable convective velocity and variable dispersion coefficient, involves transformation to a convected coordinate system and the use of a boundary layer approximation. The governing equation is then reduced to the simple diffusion equation, which can be solved directly. The effects of the variation of longitudinal velocity and dispersion coefficient are incorporated implicitly through two integral transformations. A general solution is thus obtained, and results for specific flow fields are found by evaluating two integrals of the velocity field. Several examples are considered and the limitations of the approximation are assessed.

STATEMENT OF THE PROBLEM

For saturated flow of an incompressible, binary solution in a homogeneous, isotropic, incompressible porous medium, the concentration C of a neutrally buoyant tracer in the absence of absorption or radioactive decay is described by [*Bachmat*, 1966; *de Josselin de Jong and Bossen*, 1961; *Scheidegger*, 1961]

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x_i}\left(D_{ik}\frac{\partial C}{\partial x_k} - u_i C\right) + D_m \frac{\partial}{\partial x_i}\left(\frac{\partial C}{\partial x_i}\right) \tag{1}$$

where x_i ($i = 1, 2, 3$) are Cartesian coordinates, D_{ik} is the dispersion coefficient tensor, u_i ($i = 1, 2, 3$) are the components of the seepage velocity vector $\mathbf{u}$, t is time, and D_m is the effective molecular diffusion coefficient (i.e., the molecular diffusion coefficient divided by the formation resistivity factor [*Klinkenburg*, 1951]).

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Bear [1961] and *Scheidegger* [1961] assumed the dispersion coefficient to be directly proportional to the seepage velocity and concluded that

$$D_{ik} = \alpha_{iklm} u_l u_m / u \tag{2}$$

where α_{iklm} is a fourth rank tensor and $u = |\mathbf{u}|$. If the x_1 axis is in the direction of the velocity vector $\mathbf{u}$, then $u_1 = u$ and $u_2 = u_3 = 0$ while [*Bachmat and Bear*, 1964]

$$D_{11} = a_I u$$

$$D_{22} = D_{33} = a_{II} u$$

$$D_{ik} = 0 \qquad (i \neq k)$$

where a_I and a_{II} are the longitudinal and lateral dispersivities, respectively. *List and Brooks* [1967] point out that while velocity power forms for the dispersion coefficient, as implied by equation 2, may not be valid for all ranges of velocities, certainly they do provide a reasonable approximation of the dispersion coefficient and its variation with seepage velocity. Experiments [*Harleman and Rumer*, 1963] and theoretical models [*de Josselin de Jong*, 1958] of the dispersion process have shown that the longitudinal dispersion coefficient is usually an order of magnitude larger than the lateral dispersion coefficient. Thus, in cases where lateral concentration gradients are of the same order or smaller than the longitudinal gradient, lateral dispersion will be a relatively minor effect. The following analysis will be restricted to such cases and the lateral dispersion will be neglected by setting $a_{II} = 0$.

If $ds^{(i)}$ is an element of arc length along the curvilinear coordinate line $y^{(i)}$ ($ds^{(i)} = h_i dy^{(i)}$, where h_i is a scale factor and $i = 1, 2, 3$) and the curvilinear coordinate system is chosen such that $u_1 = u$ and $u_2 = u_3 = 0$, then equation 1 with $a_{II} = 0$ reduces to [*Bachmat and Bear*, 1964]

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial s^{(1)}} = \frac{a_I}{h_2 h_3} \frac{\partial}{\partial s^{(1)}} \left[h_2 h_3 u \frac{\partial C}{\partial s^{(1)}} \right] + \frac{D_m}{h_2 h_3} \frac{\partial}{\partial s^{(1)}} \left[h_2 h_3 \frac{\partial C}{\partial s^{(1)}} \right] \tag{3}$$

For an incompressible solution, $\nabla \cdot \mathbf{u} = 0$. In orthogonal curvilinear coordinates with $u_2 = u_3 = 0$

$$\nabla \cdot \mathbf{u} = (h_2 h_3)^{-1} \frac{\partial}{\partial s^{(1)}} (h_2 h_3 u_1) \equiv 0 \tag{4}$$

Substituting equation 4 into (3) and simplifying

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial s} = \alpha u \frac{\partial^2 C}{\partial s^2} + D_m \left(\frac{\partial^2 C}{\partial s^2} - \frac{1}{u} \frac{\partial u}{\partial s} \frac{\partial C}{\partial s} \right) \tag{5}$$

where $s = s^{(1)}$, the arc length along the direction of flow, and $a_I = \alpha$.

The general longitudinal dispersion equation (5) is difficult to solve in general because the velocity u is a function of s. Therefore, an approximation is sought by boundary layer techniques.

METHOD OF APPROXIMATION

To develop a boundary layer solution, the following dimensionless variables are defined:

$$T = t/T_o$$

$$S = s/L_o \tag{6}$$

$$U = u/U_o$$

The normalizing dimensions T_o and L_o are assumed to be chosen such that $O(T) = O(S) = 1$. Steady flow is assumed; therefore, $L_o = U_o T_o$. Using (6), the convective dispersion equation (equation 5) transforms to

$$\frac{\partial C}{\partial T} + U \frac{\partial C}{\partial S} = \epsilon U \frac{\partial^2 C}{\partial S^2} + \epsilon D \left(\frac{\partial^2 C}{\partial S^2} - \frac{1}{U} \frac{\partial C}{\partial S} \frac{\partial U}{\partial S} \right) \tag{7}$$

where $\epsilon = \alpha / L_o$ and $D = D_m T_o / (\alpha L_o)$.

If molecular diffusion effects are of no greater magnitude than dispersive effects (i.e., $D \leq 1$), the orders of the left and right-hand sides of (7) are approximately 1 and ϵ, respectively. It would be expected that ϵ is small since any reasonable length scale characteristic of the flow field would be much larger than α, which is of the order of the mean grain diameter. Thus an asymptotic solution valid for small ϵ is sought.

To remove the nonconstant coefficient in the left-hand side of (7), the following variable is defined:

$$X(S) = \int_{S_o}^{S} U^{-1} \, dS \tag{8}$$

where $S_0 = S$ at $T = 0$ and $U = U(S)$. Physically, the variable $X(S)$ is a transformed spatial

coordinate that is equal to the dimensionless time required for the fluid to move from S_0 to S. Using (8) and considering C to be an explicit function of T and X, equation 7 transforms to

$$\frac{\partial C}{\partial T} + \frac{\partial C}{\partial X} = (\epsilon/U)\left(\frac{\partial^2 C}{\partial X^2} - \frac{\partial U}{\partial S}\frac{\partial C}{\partial X}\right)$$

$$+ (\epsilon D/U^2)\left(\frac{\partial^2 C}{\partial X^2} - \frac{\partial U}{\partial S}\frac{\partial C}{\partial X} - \frac{\partial U}{\partial S}\frac{\partial C}{\partial X}\right) \quad (9)$$

If ϵ is assumed to be identically zero, equation 9 reduces to

$$\frac{\partial C}{\partial T} + \frac{\partial C}{\partial X} = 0 \quad (10)$$

and the dispersive character of (9) is lost. Equation 10 defines only the convection of the flow, as can be seen from the general solution to (10) which is

$$C = f(X - T) \quad (11)$$

where f is an arbitrary function. If the concentration at $S = S_o$ is given as a function of time, i.e., $C = g(T)$ at $S = S_o$, the solution (11) becomes

$$C = g(T - X)$$

indicating that the concentration distribution does not change form but is simply convected through the flow field. In particular, if abrupt changes of concentration occur at $S = S_o$, i.e., $g(T)$ is discontinuous, these discontinuities will persist without any smoothing due to dispersion.

A discontinuity in concentration can be viewed as the hypothetical front that would exist between a tracer and native fluid if no dispersion or diffusion occurred. The location of the particular interface, which is at S_o when $T = O$, is denoted by S'. The time at which this convected front reaches $S = S'$ is given by $T = X(S = S')$. It is about this front that the tracer will be dispersed.

The behavior of the solution given by (11) as well as the loss of the second derivative in (9) when $\epsilon = 0$ is characteristic of singular perturbation problems [*Van Dyke*, 1964]. Also indicative of this problem is the disparity in the two characteristic lengths α and L_o as indicated by the presence of the small parameter ϵ. In the terminology of singular perturbation theory, equation 11 is the outer solution to (9). What must be sought is the inner solution, or boundary layer solution, of (9) that will be valid in the region around $T = X$, i.e., in the boundary layer region around the convected front, and will retain the dispersive character of (9).

To facilitate the development of the inner solution, a combined space–time variable H is introduced:

$$H = X - T \equiv \int_{S_o}^{S} \frac{dS}{U(S)} - \int_{S_o}^{S'} \frac{dS}{U(S)}$$

$$= \int_{S'}^{S} \frac{dS}{U(S)} \quad (12)$$

where S and S' are independent of time and space, respectively. Since $H = 0$ when $S = S'$, the location of the convected front, and $H > 0$ (<0) when $S > S'$ ($<S'$), H is a measure of the distance from the convected front.

Considering C to be an explicit function of H and T, equation 9 can be transformed to

$$\frac{\partial C}{\partial T} = (\epsilon/U)\left(\frac{\partial^2 C}{\partial H^2} - \frac{\partial U}{\partial S}\frac{\partial C}{\partial H}\right)$$

$$+ (\epsilon D/U^2)\left(\frac{\partial^2 C}{\partial H^2} - 2\frac{\partial U}{\partial S}\frac{\partial C}{\partial H}\right) \quad (13)$$

To retain the dispersive nature of (13) for small ϵ, a magnified, or stretched, boundary layer coordinate ζ is introduced:

$$H = \zeta\epsilon^{1/2} \quad (14)$$

where the exponent 1/2 is chosen so that concentration changes due to dispersion are of the same order of magnitude as temporal changes in concentration in the boundary layer region where $\zeta = O(1)$. Introducing (14) into (13) yields

$$\frac{\partial C}{\partial T} = (U^{-1} + DU^{-2})\frac{\partial^2 C}{\partial \zeta^2}$$

$$- \epsilon^{1/2}(U^{-1} + 2DU^{-2})\frac{\partial U}{\partial S}\frac{\partial C}{\partial \zeta} \quad (15)$$

Near singularities such as sources and sinks $\partial C/\partial S$ may not be $O(1)$, in which case (15) is not valid in the vicinity of such singularities. The coefficients U^{-1} and DU^{-2} are expanded in the Taylor series about $X = T$ (i.e., $H = O$ and $S = S'$).

$$U^{-1} = (U^{-1})_{S=S'} - \epsilon^{1/2}\zeta\left(U^{-2}\frac{\partial U}{\partial X}\right)_{S=S'} + \cdots$$

$$DU^{-2} = D[(U^{-2})_{S=S'} - 2\epsilon^{1/2}\zeta(U^{-3}\partial U/\partial X)_{S=S'} + \cdots] \quad (16)$$

Since ϵ is assumed small, a first order approximation is now made by neglecting terms of $0(\epsilon^{1/2})$. Thus, if the expansions of (16) are introduced into (15) and terms of $0(\epsilon^{1/2})$ and smaller are neglected, (15) reduces to

$$\frac{\partial C}{\partial T} = \frac{U(S') + D}{U^2(S')}\frac{\partial^2 C}{\partial \zeta^2} \quad (17)$$

where $U(S') = U(S)$ evaluated at $S = S'$ $(X = T)$.

If $\Omega = \Omega(T)$ is defined by

$$\Omega = \int_0^T \frac{U(S') + D}{U^2(S')}\,dT' \quad (17a)$$

or equivalently

$$\Omega = \int_{s_o}^{S'} \frac{U(S') + D}{U^3(S')}\,dS' \quad (18)$$

since $dS/dT = U(S)$, then (17) can be transformed to

$$\frac{\partial C}{\partial \Omega} = \frac{\partial^2 C}{\partial \zeta^2} \quad (19)$$

The equivalent dimensional form of (19) is

$$\frac{\partial C}{\partial \omega} = \alpha\frac{\partial^2 C}{\partial \eta^2} \quad (20)$$

where $\eta = x - t$, $t = x$ evaluated at $s = s'$,

$$x = \int_{s_o}^{s} u(s)^{-1}\,ds \quad (21)$$

and

$$\omega = \int_{s_o}^{s'} \frac{u(s') + D_m/\alpha}{u^3(s')}\,ds' \quad (22)$$

If a more general form of the dispersion coefficient

$$D_{11} = \alpha u^m \quad (22a)$$

is used, a similar analysis yields the same equivalent dimensional form as in (20) with ω given by

$$\omega = \int_{s_o}^{s} \frac{u^m(s') + D_m/\alpha}{u^3(s')}\,ds' \quad (23)$$

and $\eta = x - t$ with x as in (21).

It has thus been shown, through a series of coordinate transformations, that the general convective dispersion equations, (5) or (7), can be reduced to the simple heat conduction equation as given by (19) or (20), provided that the term $\epsilon^{1/2} = (\alpha/L_o)^{1/2} \ll 1$. Thus, many of the known solutions of the heat conduction equation may be applied to various types of dispersion problems.

SOME GENERAL SOLUTIONS

The solution of (20), which represents the point injection of a mass m per unit effective area of tracer over the surface $s = s_o$ at time $t = 0$, i.e.,

$$C = \frac{m}{\rho}\,\delta(s - s_o) \quad (24)$$

where δ is the Dirac delta function and ρ is the density of the solution, is [*Carslaw and Jaeger*, 1959]

$$C = \frac{m}{\rho u(s_o)}(4\pi\alpha\omega)^{-1/2}\exp(-\eta^2/4\alpha\omega) \quad (25)$$

This solution satisfies the condition that the total mass of tracer in the stream tube is conserved.

The solution for an arbitrary injection rate of mass at s_o is then obtained by superposition. The incremental increase in concentration dC, due to an incremental injection of tracer per unit of effective area dm at $s = s_o$ when $t = \lambda$ (i.e., $\eta = -\lambda$) is, using (25)

$$dC = \frac{dm}{\rho u(s_o)}[4\pi\alpha\omega(t, \lambda)]^{-1/2}\cdot\exp\left[-\frac{(\eta + \lambda)^2}{4\alpha\omega(t, \lambda)}\right] \quad (26)$$

where $\omega(t, \lambda) = \omega(t) - \omega(\lambda)$. If the rate of tracer injection is given by

$$\frac{dm}{dt} = \rho u(s_o)\phi(t) \quad (27)$$

the concentration at any time is, by superposition,

$$c = \int_0^t \phi(\lambda)[4\pi\alpha\omega(t, \lambda)]^{-1/2}\cdot\exp\left[-\frac{(\eta + \lambda)^2}{4\alpha\omega(t, \lambda)}\right]d\lambda \quad (28)$$

Note that the variable $\phi(t)$ has the physical interpretation

$$\phi(t) = \frac{dm/dt}{\rho u(s_o)}$$

$$= \frac{\text{mass of tracer injected/area/time}}{\text{mass of fluid flowing past injection point/area/time}}$$

If dispersion were completely neglected, the dimensionless ratio $\phi(t)$ would represent the concentration at the injection point $s = s_o$. It can be shown that, under the restrictions applied in developing (20), i.e., $\epsilon^{1/2} \ll 1$, the concentration at the injection point is given by

$$C(s_o, t) \equiv C_0(t) = \phi(t) \qquad (29)$$

except during a negligibly small dimensionless time period of $0(\epsilon)$ around any discontinuities in $\phi(t)$. Thus the general solution given in (28) also applies to the boundary value problem in which the time history of concentration at $s = s_o$ is prescribed.

The evaluation of the general integral solution given by (28) will, in general, depend on the flow field through the function $\omega(t, \lambda)$. For some functions $\phi(t)$, the integral can be approximated by applying the restriction $\epsilon^{1/2} \ll 1$ without introducing explicit forms for $\omega(t, \lambda)$, but these details will not be required here.

For the initial condition of a step change in concentration at the origin

$$C = C_0 \qquad s < s_o$$

$$C = 0 \qquad s > s_o \quad \text{at} \quad t = 0$$

with

$$C \to C_0 \quad \text{as} \quad s \to -\infty$$

$$C \to 0 \quad \text{as} \quad s \to \infty \qquad \text{for} \quad t > 0$$

the solution of (20) is

$$C(x, t) = \tfrac{1}{2} C_0 \operatorname{erfc} \{(x - t)/[4\alpha\omega(t)]^{1/2}\} \qquad (30)$$

This solution is, within the limitations of the approximation, also applicable in the case of a step change of concentration at the origin,

$$t = 0 \qquad C = 0 \qquad \text{for all} \qquad s \geq s_o$$

$$s = s_o \quad C = C_0 \quad \text{a constant} \quad \text{for} \quad t > 0$$

$$s \to \infty \qquad C \to 0$$

The solutions given by (25) and (30) provide substantial generality in that explicit evaluation of the characteristics of the flow field is not required to obtain the solution. The characteristics of the flow field are implicitly included through the integrals (21) and (22). These integrals are seen to depend solely on the velocity variation along the stream tube and may be evaluated analytically, graphically, or numerically. Although this analysis is valid only for steady flows, the method can be applied also to stepwise changes of flow with time. This approach will be illustrated by a pumping–recharge operation for a well in a confined aquifer.

SOME APPLICATIONS OF THE METHOD

Some features of dispersion in flow fields with variable velocity are illustrated by considering diverging flows in the form

$$u(s) = A/s^k \qquad (31)$$

where A and k are constants. This general form applies to uniform flow in one dimension, radial flow from a recharge well in a confined aquifer in two dimensions, and spherical flow from a point source in three dimensions. The values of the constants for each case are given in Table 1. The spherical flow situation may correspond to recharge from a well with a screened section that is small compared to the aquifer thickness.

The solutions are obtained by evaluating the integrals (21) and (22) for the velocity field (31) with $s_o = 0$. Molecular diffusion is neglected in this example. These integrations yield

$$x = \frac{s^{k+1}}{A(k+1)}$$

$$t = \frac{s'^{k+1}}{A(k+1)}$$

$$\omega = \frac{s'^{2k+1}}{(2k+1)A^2}$$

and for the step input solution (30) the results are of the form

$$C = \tfrac{1}{2} C_0 \operatorname{erfc}(\theta) \qquad (32)$$

where variable θ for each case is given in Table 1. The results are conveniently compared in terms of the natural variable for the uniform flow $\xi = (s - s')/(4\alpha s')^{1/2}$. The solutions for two and three-dimensional flows can be expressed in

TABLE 1. Summary of Solutions for Uniform, Radial, and Spherical Flows

Flow Configuration	A	k	Frontal Position $s'(t)$	Solution Variable θ (in Equation 32)
Uniform	U	0	At	$(s - s')/(4\alpha s')^{1/2}$
Radial	$q/(2\pi n)$	1	$(2At)^{1/2}$	$(s^2 - s'^2)/(16\alpha s'^3/3)^{1/2}$
Spherical	$Q/(4\pi n)$	2	$(3At)^{1/3}$	$(s^3 - s'^3)/(36\alpha s'^5/5)^{1/2}$

U is seepage velocity, q is source strength in two-dimensional radial flow, Q is source strength in three-dimensional spherical flow, and n is porosity.

terms of ξ with s'/α as a parameter; e.g., in the radial case

$$\theta = (3)^{1/2}\left[1 + \left(\frac{\alpha}{s'}\right)^{1/2}\xi\right]\xi$$

The comparison in Figure 1 of the three solutions for a constant value of s'/α illustrates the differences of dispersive effect when a front is displaced the same distance through media of equal dispersivity in the three different configurations. It is seen that the nonuniform diverging flows yield narrower dispersed zones than does the uniform flow. The concentration gradients in the case of uniform flow are symmetrical about the origin $\xi = 0$ whereas for finite s'/α the diverging flows show slightly steeper gradients ahead of the front. For very large s'/α the concentration distributions for the diverging flows also approach a cumulative normal form, and the thickness of the dispersed zone for uniform flow is greater than that for radial and spherical flows by factors of $(3)^{1/2}$ and $(5)^{1/2}$, respectively.

The general effects of flow nonuniformity can be analyzed in terms of the growth of the dispersed zone δ defined by

$$\delta^{-1} = -\frac{\partial C/C_0}{\partial s}\bigg|_{s=s'} \tag{33}$$

which from (30) is

$$\delta^2 = 4\pi\alpha\omega(s')u^2(s') \tag{34}$$

Using the integral expression for ω ((22) with $D_m = 0$),

$$\frac{d\delta^2}{ds} = 4\pi\alpha\left[1 + 2\omega u(s')\frac{du(s')}{ds'}\right]$$

Noting that ω is by definition positive for $s' > s_0$, and that the fluid acceleration $a = u\,du/ds$, it is

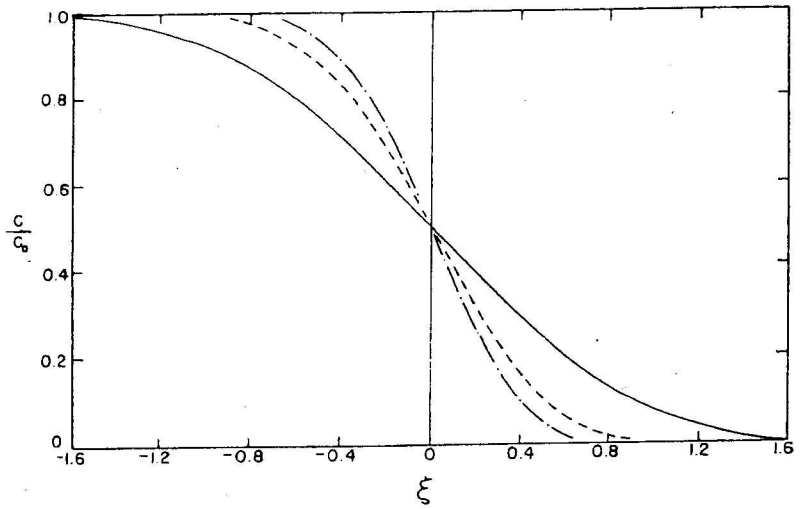

Fig. 1. Comparison of concentration distributions for uniform (solid line), radial (dashed line), and spherical (interrupted line with dot) flows from (32) and Table 1. $\xi = (s - s')/(4\alpha s')^{1/2}$, $s'/\alpha = 10^3$.

seen that, in general, accelerating flow ($a > 0$) will have greater dispersion than uniform flow and decelerating flows ($a < 0$) will have less.

An example, which demonstrates the effects of oscillatory variation of the velocity, is the flow field

$$u = u_o/(1 + \beta \sin ks)$$

where u_o, k, and β ($|\beta| < 1$) are constants. From (21) and (22) with $D_m = 0$ and $s_o = 0$.

$$x = \left[s + \frac{\beta}{k}(1 - \cos ks)\right] \Big/ u_o$$

$$\omega = \left[s' + \frac{2\beta}{k}(1 - \cos ks') + \frac{\beta^2}{2}\left(\frac{\sin ks' \cos ks'}{k} - s'\right)\right] \Big/ u_o^2$$

The solution for a step input is obtained by introducing x and ω in (30) and the thickness of the dispersed zone is given by (34). When this result is compared with that for a uniform flow ($\beta = 0$) at points where the two flows have the same frontal displacement ($ks' = 2n\pi$, $n = 1, 2, \cdots$), it is found that

$$\left(\frac{\delta}{\delta_0}\right)^2 = 1 + \beta^2/2$$

where δ corresponds to the oscillatory flow and δ_0 corresponds to the uniform flow. The oscillatory flow produces additional dispersion that is dependent on the amplitude of the oscillation but not on the wavelength.

A recharge–pumping cycle for a fully penetrating well in a confined aquifer will be analyzed, considering the effects of well radius r_w and molecular diffusion. During recharge at a rate Q_1 the velocity field is

$$u = A_1/r$$

$$A_1 = Q_1/(2\pi nb)$$

where r is the radial coordinate, n is the porosity, and b is the aquifer thickness. Using $s = r - r_w$, the integral for x (21) becomes

$$x_1 = \frac{r^2 - r_w^2}{2A_1} \tag{35}$$

and therefore

$$t_1 = \frac{r^{*2} - r_w^2}{2A_1} \tag{36}$$

where t_1 represents the time from the start of recharge and r^* is the frontal position. From (22)

$$\omega_1 = \frac{r^{*3} - r_w^3}{3A_1^2} + \frac{D_m}{\alpha}\frac{r^{*4} - r_w^4}{4A_1^3} \tag{37}$$

If water with concentration $C = 0$ is displaced by recharge water of concentration C_0, (30) applies and from (35), (36), and (37),

$$\frac{C}{C_0} = \frac{1}{2}\operatorname{erfc}\left(\frac{x_1 - t_1}{(4\alpha\omega_1)^{1/2}}\right) = \frac{1}{2} \cdot \operatorname{erfc}\left\{\frac{r^2 - r^{*2}}{\left[16\alpha\left(\frac{r^{*3} - r_w^3}{3} + \frac{D_m}{\alpha A_1}\frac{r^{*4} - r_w^4}{4}\right)\right]^{1/2}}\right\} \tag{38}$$

At time $t_1 = T_1$ ($t_2 = 0$), when the front is at $r = R_1$, pumping at a rate Q_2 is initiated; with $s = R_1 - r$ and

$$u = A_2/r$$

$$A_2 = Q_2/(2\pi nb)$$

it is found that

$$x_2 = (R_1^2 - r^2)/2A_2, \quad t_2 = (R_1^2 - r^{*2})/2A_2$$

$$\omega_2 = -\frac{r^{*3}}{3A_2} - \frac{D_m}{\alpha}\frac{r^{*4}}{4A_2^3} + C_2$$

where C_2 is a constant that is determined by matching with the recharge portion of the cycle. The solution during pumping is then

$$\frac{C}{C_0} = \frac{1}{2}\operatorname{erfc}\frac{t_2 - x_2}{(4\alpha\omega_2)^{1/2}} \tag{39}$$

where the change in sign of the numerator accounts for the change in flow direction. The matching of (38) and (39) when $t_1 = T_1$, $t_2 = 0$, $r^* = R_1$ requires that $\omega_1 A_1^2 = \omega_2 A_2^2$. The constant C_2 is found from this relationship and ω_2 becomes

$$\omega_2 = \frac{2R_1^3 - r^{*3} - r_w^3}{3A_2^2} + \frac{D_m}{4\alpha A_2^3} \cdot \left(R_1^4 + \frac{A_2}{A_1}(R_1^4 - r_w^4) - r^{*4}\right) \tag{40}$$

which, when introduced in (39), yields the concentration distribution during the pumping cycle,

$$\frac{C}{C_0} = \frac{1}{2}\,\text{erfc}\,\frac{r^2 - r^{*2}}{\left\{16\alpha\left[\dfrac{2R_1^3 - r^{*3} - r_w^3}{3} + \dfrac{D_m}{4\alpha A_2}\left(R_1^4 + \dfrac{A_2}{A_1}(R_1^4 - r_w^4) - r^{*4}\right)\right]\right\}^{1/2}} \tag{41}$$

Of particular interest is the concentration at the well, which is obtained by setting $r = r_w$ in (41). The result is conveniently expressed in terms of the volumetric variables, the volume recharged $\forall_1$, and the volume pumped $\forall_2 = Q_2 t_2$. When the effects of well radius and molecular diffusion are negligible, the concentration at the well is

$$\frac{C}{C_0} = \frac{1}{2}\,\text{erfc}\left\{\left(\frac{\forall_2}{\forall_1} - 1\right)\Big/\left[\frac{16}{3}\frac{\alpha}{R_1}\cdot\left(2 - \left|1 - \frac{\forall_2}{\forall_1}\right|^{1/2}\left(1 - \frac{\forall_2}{\forall_1}\right)\right)\right]^{1/2}\right\} \tag{42}$$

where the absolute value sign is required for the rate of change of concentration to be continuous through $\forall_2/\forall_1 = 0$. The analysis developed here can be generalized to include any number of recharge–pumping cycles.

EVALUATION OF THE APPROXIMATION

The restriction imposed in developing this approximation was that the parameter $\epsilon = \alpha/L_o$ be small in the sense that $\epsilon^{1/2} \ll 1$. The length scale L_0 should reflect the total distance traveled by the front; e.g., in the case of a recharge–pumping cycle for a well, L_o is the total distance traveled by the front rather than the net radial distance to the front. A physical interpretation of the condition $\epsilon^{1/2} \ll 1$ is obtained if it is noted that a dispersed zone thickness of the form of (33) will be of order $(\alpha L_o)^{1/2}$. The approximation will generally be valid when the dispersed zone thickness is small compared to the distance traveled by the front L_o. The approximation will not be valid very close ($\alpha/L_o = 0(1)$) to the point at which a step in concentration originates, and the accuracy of the approximation is expected to improve as the front moves away from the origin. Thus the approximation is not uniformly valid; the situation is similar to that in classical viscous boundary layer analysis, which fails near the leading edge of a flat plate.

In the case of radial flow, the results developed here can be compared with some obtained using other approximate methods. Using an ad hoc approximation wherein space derivatives were replaced by time derivatives in the dispersive term, *Gardner et al.* [1962] obtained results equivalent to (38) and (41) with $r_w = 0$. *Hoopes and Harleman* [1967] used a similar approach to obtain a result that is similar to (38) with $r_w = 0$, but differs slightly in that r rather than r^* appears in the denominator. *Dagan* [1971] developed perturbation solutions of the dispersion equation in the stream and potential function domain, and obtained a result which, although not identical, is equivalent to (38) in the limit as $\alpha/r^* \to 0$. These comparisons confirm the validity of the approximation proposed here when $\epsilon \ll 1$, but the explicit limitation must be obtained by comparison with solutions of the complete equation.

No exact analytical solutions of the convective dispersion equation (5) with variable velocity are known; therefore a comparison with numerical solutions was made. The implicit finite difference scheme developed by *Shamir and Harleman* [1967] was used to generate numerical solutions of the radial flow problem for a typical recharge–pumping cycle. For the numerical solution explicit boundary conditions must be specified at the well radius. During the recharge portion of the cycle, a constant concentration C_0 was specified at the well, and during pumping $\partial C/\partial r = 0$ was used.

Gershon and Nir [1969] compare solutions using several different boundary and initial conditions, for one-dimensional uniform flow, and find that all of the solutions become practically the same for values of s'/α greater than 100. For the approximate analysis developed here, these more precise boundary conditions are not justified because the approximation is weak in the region where the boundary conditions have an effect on the solutions, i.e., $s'/\alpha = 0(1)$. The comparison of numerical and approximate analytical solutions will thus reflect the combined effects of two approximations, one involving the reduction of the governing equation to the form in (20) and the other relating to the simplification of the boundary conditions as is implicit in the solution (30).

Numerical solutions were developed for the

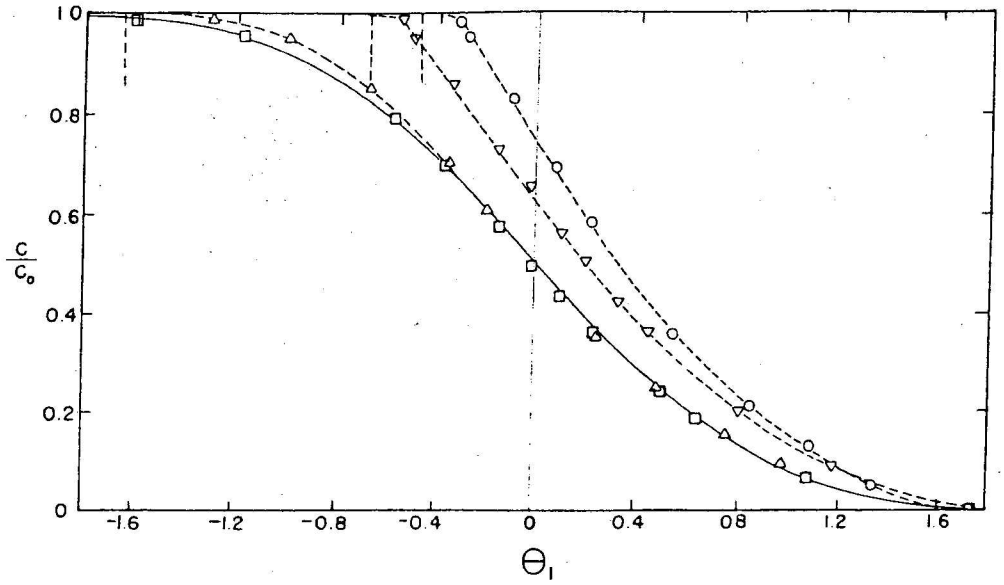

Fig. 2. Comparisons of numerical solutions (dashed line with circle, $r^*/\alpha = 3$; dashed line with inverted triangle, $r^*/\alpha = 4$; dashed line with triangle, $r^*/\alpha = 15$; square, $r^*/\alpha = 100$) and the approximate analytical solution (equation 38, solid line) during recharge. θ_1 is equal to the argument of the complimentary error function in (38). The vertical dashed line indicates the location of the well radius for each frontal position r^*/α.

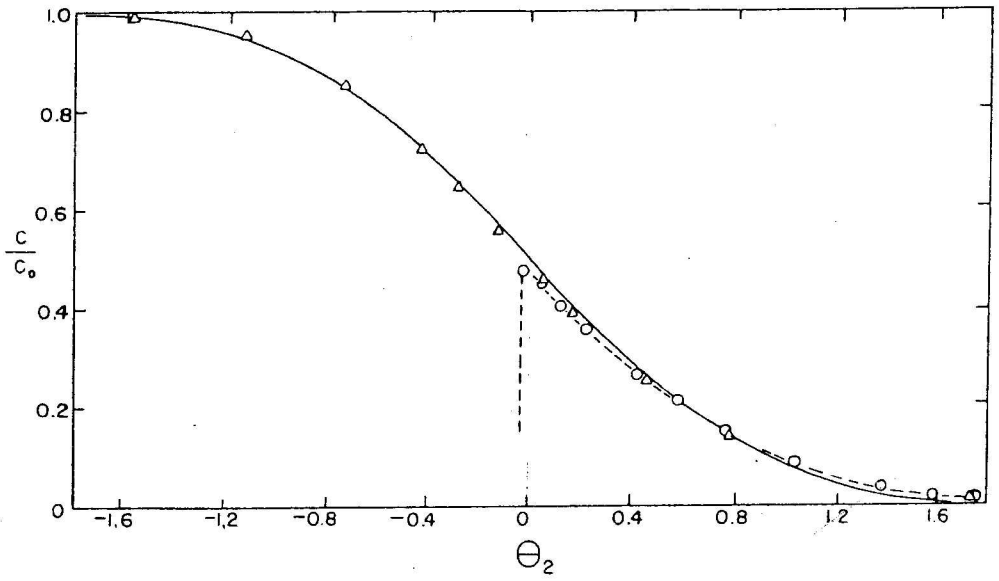

Fig. 3. Comparisons of numerical solutions (dashed line with circle, $r^*/\alpha = 6$; triangle, $r^*/\alpha = 80$) and the approximate analytical solution (41) (solid line) during pumping. θ_2 is equal to the argument of the complimentary error function in (41). The vertical dashed line indicates the location of the well radius for $r^*/\alpha = 6$.

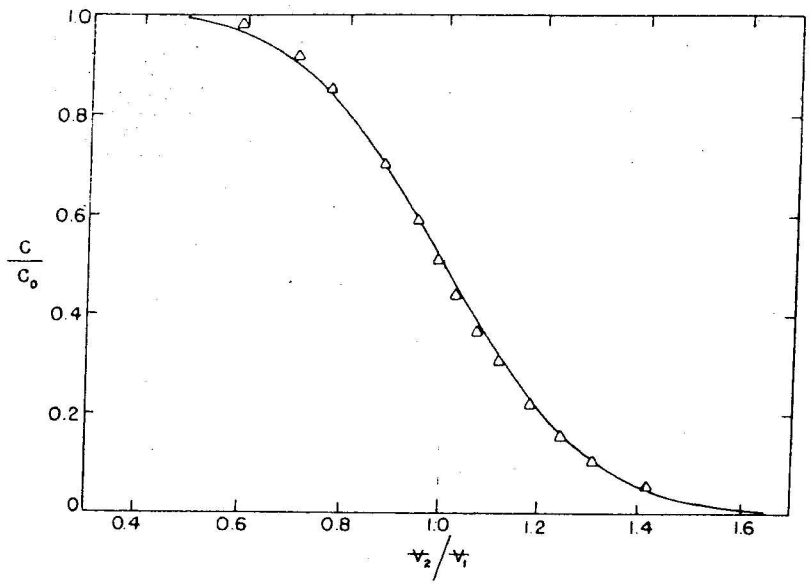

Fig. 4. Comparison of numerical solutions (triangle) and the approximate analytical solution (42) (solid line) at the well during pumping.

case $R_1/\alpha = 100$, $r_w/\alpha = 2.05$, $Q_2/Q_1 = 0.724$, $D_m/A_2 = 1.21 \times 10^{-5}$, and comparisons with the analytical approximation are shown in Figures 2, 3, and 4. During recharge (Figure 2) the approximate solution (38) yields good results for $r^*/\alpha = 100$, and even for $r^*/\alpha = 15$ shows reasonable agreement except at the higher concentrations. During pumping (Figure 3) very good agreement is noted at $r^*/\alpha = 80$ and even when the front is practically at the well ($r^*/\alpha = 6$) the differences are not large. The discrepancy near the well reflects the fact that boundary condition $\partial C/\partial r = 0$ was not applied in the analytical approximation. However, the time history of the concentration at the well during pumping (Figure 4) is represented quite well by the analytical result (42). Although the approximate solution yields some error in concentration distribution near the pumping well, the overall effect of such errors on the concentration in the well is small.

The results of these comparisons with numerical solutions in the radial flow case indicate that the approximate solution (30) will yield good results after the front has traveled a distance on the order of 100 times the dispersivity of the medium, and that usable estimates could be obtained for L_o as low as 15α. In terms of field applications, these restrictions are not serious because the overall scale of the flow is usually much larger than the dispersivity of the medium. The radial flow case, involving a singularity in the velocity at the origin, is a severe test of the approximate solution.

DISCUSSION

The analytical method developed here provides a simple and reliable technique by which the effects of flow nonuniformity on longitudinal dispersion in porous media can be estimated. The effects of nonuniformity are incorporated through two integrals of the velocity field (21) and (22), which can be evaluated from analytical expressions or numerical data. Several specific examples illustrated the application of the method, and it was found that, in general, accelerating flows will produce greater dispersion than decelerating flows.

This approach should be particularly useful in field applications involving complex flow patterns. Although the effects of natural inhomogeneity and anisotropy on dispersion require additional study, initial estimates of dispersion in a natural setting can be developed by using velocity data from numerical or analog modeling of the flow field or direct field observations. The method is currently being used in a study of groundwater quality on Long Island,

New York, wherein data from a Hele-Shaw model [*Collins and Gelhar*, 1970] are used to evaluate the flow field.

As was shown by the comparison of numerical and approximate solutions, the restrictions implicit in the approach do not seriously limit the applicability of the method. Good results are obtained after the dispersed zone has traveled a distance equal to 100 times the dispersivity of the medium, and this distance is typically small in relation to the flow field. This approximate method can also be used to analyze longitudinal dispersion in streams with variable cross section and dispersion coefficient.

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