ECE 521 HW#2 – Part1 Solution

1 a)

$$\begin{bmatrix} 10^{-5} & 10^{-5} & 1 \\ 10^{-5} & -10^{-5} & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2\times10^{-5} \\ -2\times10^{-5} \\ 1 \end{bmatrix}$$

To determine the exact solution I used partial pivoting and "infinite" precision

So the first pivot results in

$$\begin{bmatrix} 1 & 1 & 2 \\ 10^{-5} & -10^{-5} & 1 \\ 10^{-5} & 10^{-5} & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -2\times10^{-5} \\ 2\times10^{-5} \end{bmatrix}$$

Use GE

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -2\times10^{-5} & 1-2\times10^{-5} \\ 0 & 0 & 1-2\times10^{-5} \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -3\times10^{-5} \\ 10^{-5} \end{bmatrix}$$

We have an upper triangular form, from which

$$x_3 = \frac{10^{-5}}{1-2\times10^{-5}}$$

$$x_2 = 2$$

$$x_1 = -\frac{1}{1-2\times10^{-5}}$$

$\therefore$ the exact solution is $\begin{bmatrix} -\frac{1}{1-2\times10^{-5}} \\ 2 \\ \frac{10^{-5}}{1-2\times10^{-5}} \end{bmatrix}$

b) Partial pivoting with 3 digit arithmethic

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2\times10^{-5} & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3\times10^{-5} \\ 10^{-5} \end{bmatrix}$$

$\therefore$ the solution is $\begin{bmatrix} -1 \\ 2 \\ 10^{-5} \end{bmatrix}$

c) Complete pivoting $\Rightarrow$ pivot is a_{33}. Therefore

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & -10^{-5} & 10^{-5} \\ 1 & 10^{-5} & 10^{-5} \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -2\times10^{-5} \\ 10^{-5} \end{bmatrix}$$

<u>GE</u>

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -0.5 & -0.5 \\ 0 & -0.5 & -0.5 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -0.5 \\ -0.5 \end{bmatrix}$$

This is a singular matrix

$\Rightarrow$ cannot obtain a unique solution!

2. Markowitz

$$
\begin{array}{c|c}
\begin{bmatrix}
a_{11} & & a_{13} & a_{14} & \\
 & a_{22} & & & a_{25} \\
a_{31} & & a_{33} & & a_{35} \\
a_{41} & & & a_{44} & \\
 & & a_{53} & & a_{55}
\end{bmatrix} & \begin{matrix} 3 \\ 2 \\ 3 \\ 2 \\ 2 \end{matrix} \\
\begin{matrix} 3 & \;\;1 & \;\;3 & \;\;2 & \;\;3 \end{matrix} &
\end{array}
\Rightarrow
\begin{array}{c}
\left[\begin{array}{c|cccc}
a_{22} & & & & a_{25} \\ \hline
 & a_{11} & a_{13} & a_{14} & \\
 & a_{31} & a_{33} & & a_{35} \\
 & a_{41} & & a_{44} & \\
 & & a_{53} & & a_{55}
\end{array}\right] \\
\begin{matrix} 3 & \;\;3 & \;\;2 & \;\;2 \end{matrix}
\end{array}
$$

Markowitz $(a_{22}) = 0$.

0 Fill-ins ; Possible choices a_{44}, a_{55}

Taking a_{44} as pivot $\Rightarrow$ 0 Fill-ins

$$
\begin{array}{c|c}
\left[\begin{array}{c|ccc}
a_{44} & & a_{41} & \\ \hline
 & a_{33} & a_{31} & a_{35} \\
a_{14} & a_{13} & a_{11} & \\
 & a_{53} & & a_{55}
\end{array}\right] & \begin{matrix} \\ 3 \\ 2 \\ 2 \end{matrix} \\
\begin{matrix} \quad 3 & \;\;2 & \;\;2 \end{matrix} &
\end{array}
\quad \xRightarrow{a_{11} \text{ as pivot}} \quad
\begin{bmatrix}
a_{11} & a_{13} & \\
a_{31} & a_{33} & a_{35} \\
 & a_{53} & a_{55}
\end{bmatrix}
$$

0 Fill-ins

The next choices are a_{33} or a_{55} and again 0 Fill-ins are generated

$\therefore$ Pivot sequence a_{22}, a_{44}, a_{11}, a_{33}, a_{55}

Fill-ins 0

<u>Remark</u>

This wasn't a very interesting example since no Fill-ins were created. Furthermore, the tie-break rule didn't help either. So I arbitrarily selected the pivot.

3a) Define $f(x) = \frac{1}{x} - n = 0$

The solution is $x^* = \frac{1}{n}$ i.e. the reciprocal of n

Apply Newton's method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$f'(x_k) = -\frac{1}{x_k^2}$$

$$\therefore \quad x_{k+1} = x_k + x_k^2\left(\frac{1}{x_k} - n\right)$$

$$= 2x_k - n x_k^2 = x_k(2 - n x_k)$$

Thus the above recurrence relation will compute the reciprocal of n provided x_0 is close to $1/n$.

$n = \pi$; $\Rightarrow \frac{1}{n} = 0.318$

x_0	0.1	0.7
x_1	0.169	−0.139
x_2	0.248	−0.339
x_3	0.303	−1.039
x_4	0.318	−5.469

The iteration converges for $x_0 = 0.1$ but not for $x_0 = 0.7$.

b) Here $f(x) = Ax - b = 0$

$J = \frac{\partial f}{\partial x} = A$ i.e. the Jacobian matrix is A

Newton's method $\Rightarrow$ $x^{k+1} = x^k - J^{-1} f(x^k)$

$$= x^k - A^{-1}(Ax^k - b) = A^{-1}b$$

$\therefore$ the iteration converges in exactly 1 iteration regardless of the value of the initial guess x_0.

4. When plotting errors and function values in an iterative method, it is useful to look at a log plot

a) Observe quadratic convergence close to the solution. Also note the better the initial guess the faster the convergence

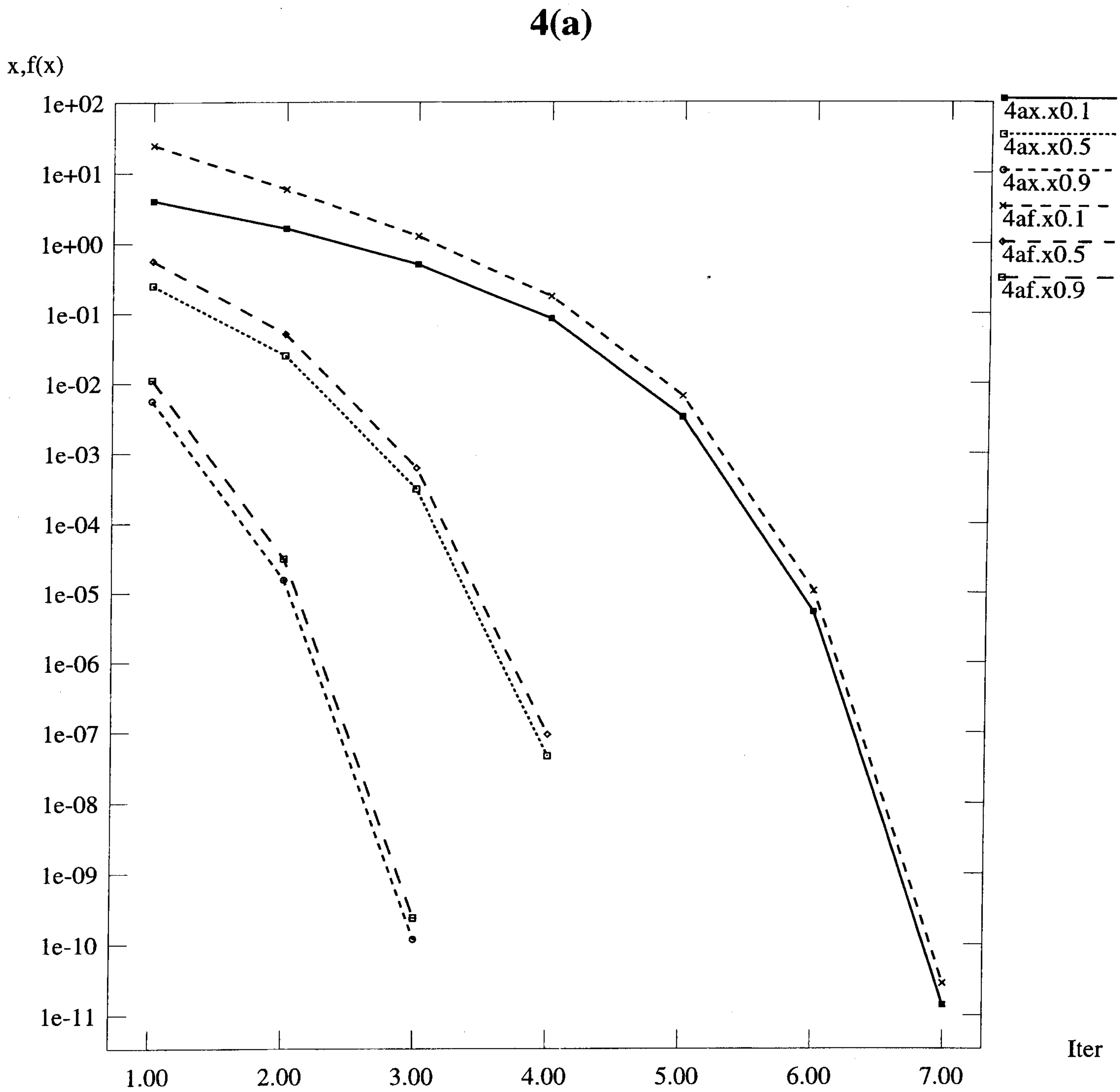
b) Here the convergence is linear

c) For $x_0 = 0.3$ & 0.5 one gets an overflow/Nan in the evaluation of exp(large number). The indicates the <u>need</u> for limiting or damping during the solution.

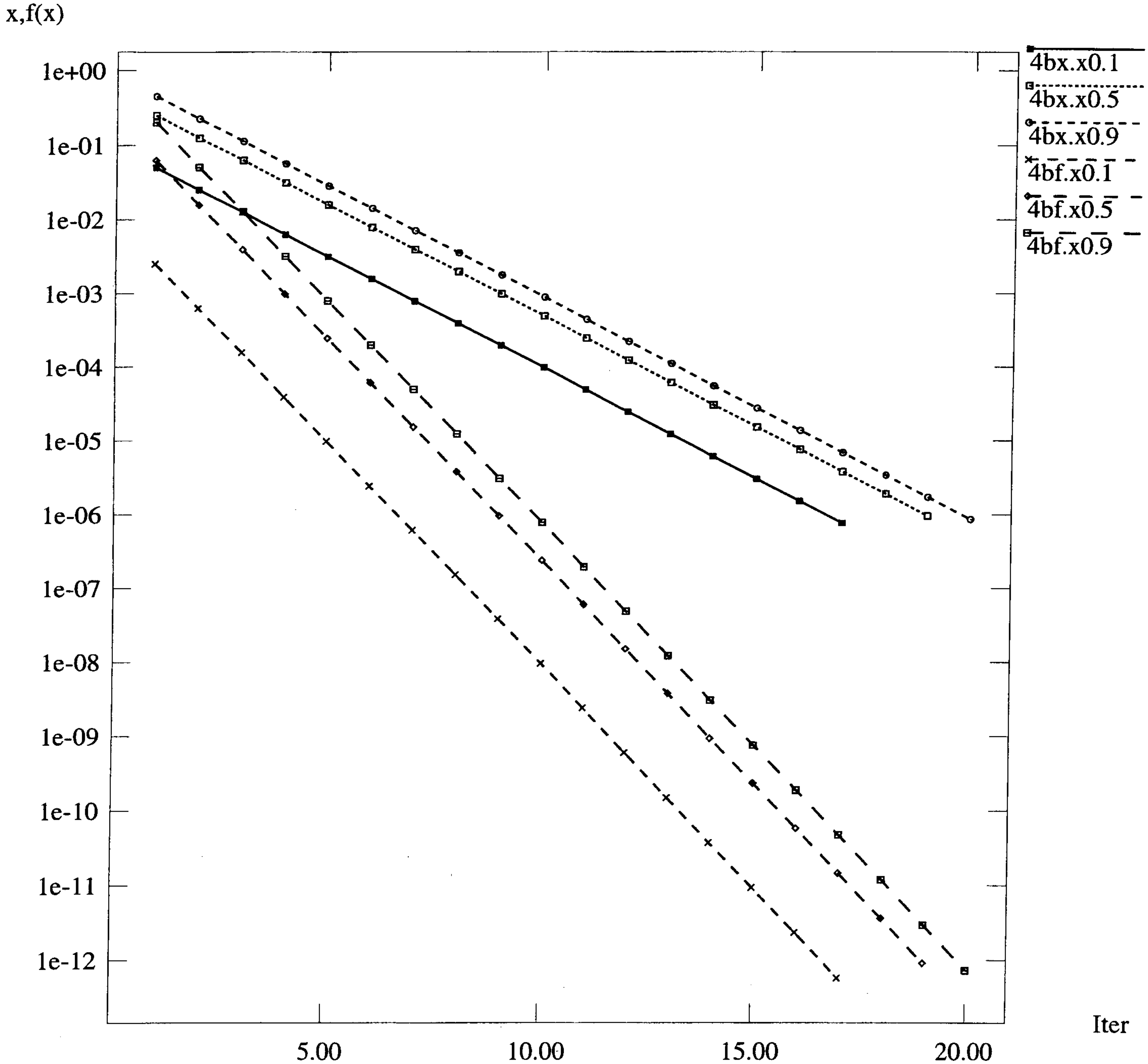
For $x_0 = 0.7$ & 0.8 the convergence is rapid with 0.7 being a better initial guess

d) Here the convergence is linear since an approximate derivative is used.

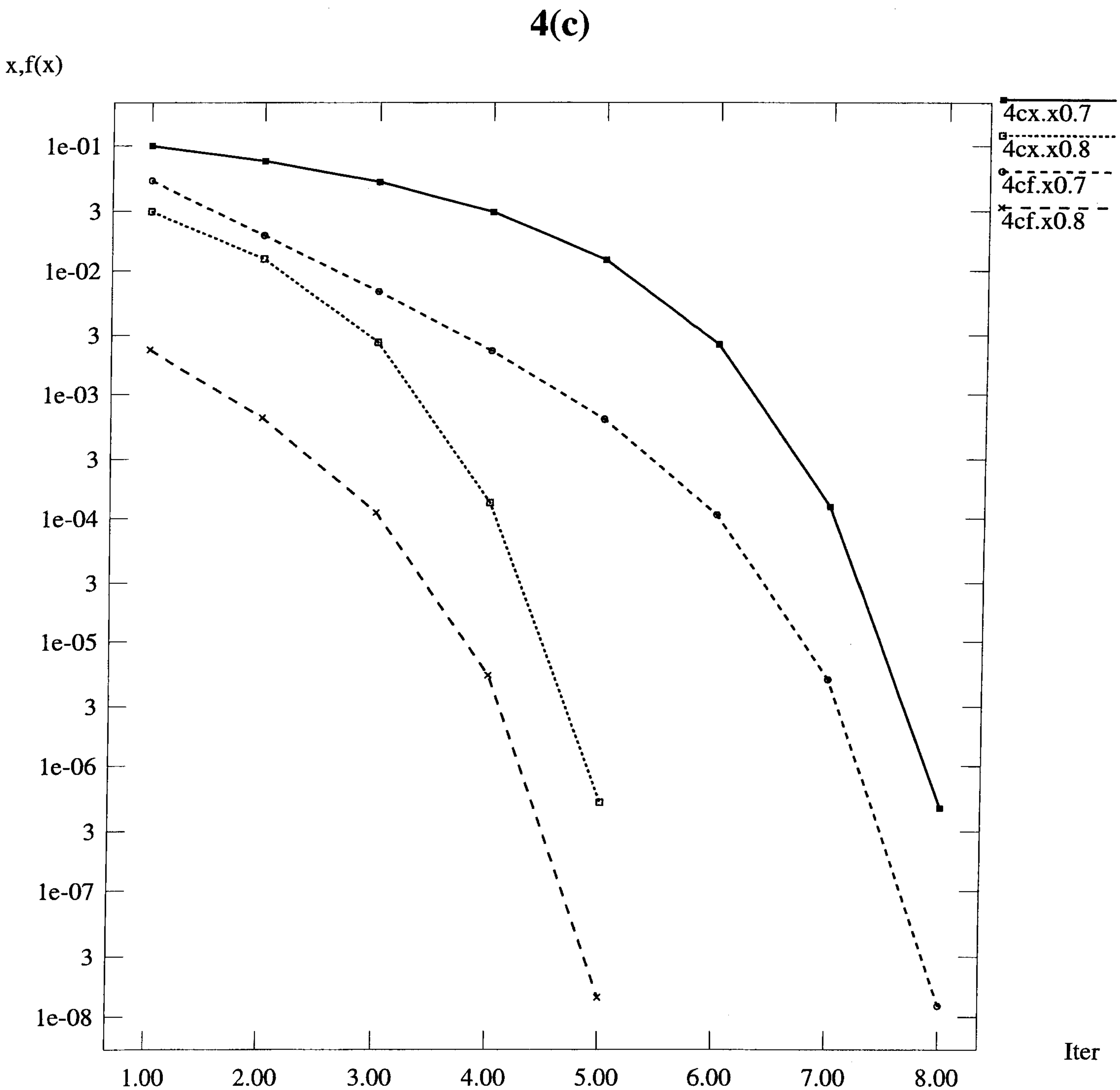
```
#include <stdio.h>
#include <math.h>
#include "macros.h"

#define MAXITER 30

main()
{
    newton(0.8, 0.74834);
}

newton(xold, sol)
double xold, sol;
{
    double f(), df();
    double abstol = 1e-6;
    double reltol = 1e-3;
    int numIter = 0;
    int conv = 0;
    double tol, x;
    while (numIter <= MAXITER && NOT conv) {
        numIter++;
        x = xold - f(xold)/df(xold);
        tol = abstol + reltol * MAX(ABS(x), ABS(xold));
        if (ABS(x - xold) < tol ) conv = 1;
        printf("%d %e %e\n", numIter, ABS(x-sol), ABS(f(x)));
        xold = x;
    }
}

double f(x)
double x;
{
    return( 1.0e-3 - 1.0e-16*(exp(40.0*x) -1.0));
}
double df(x)
double x;
{
    return( -40.0e-16*exp(40.0*x));
}
```

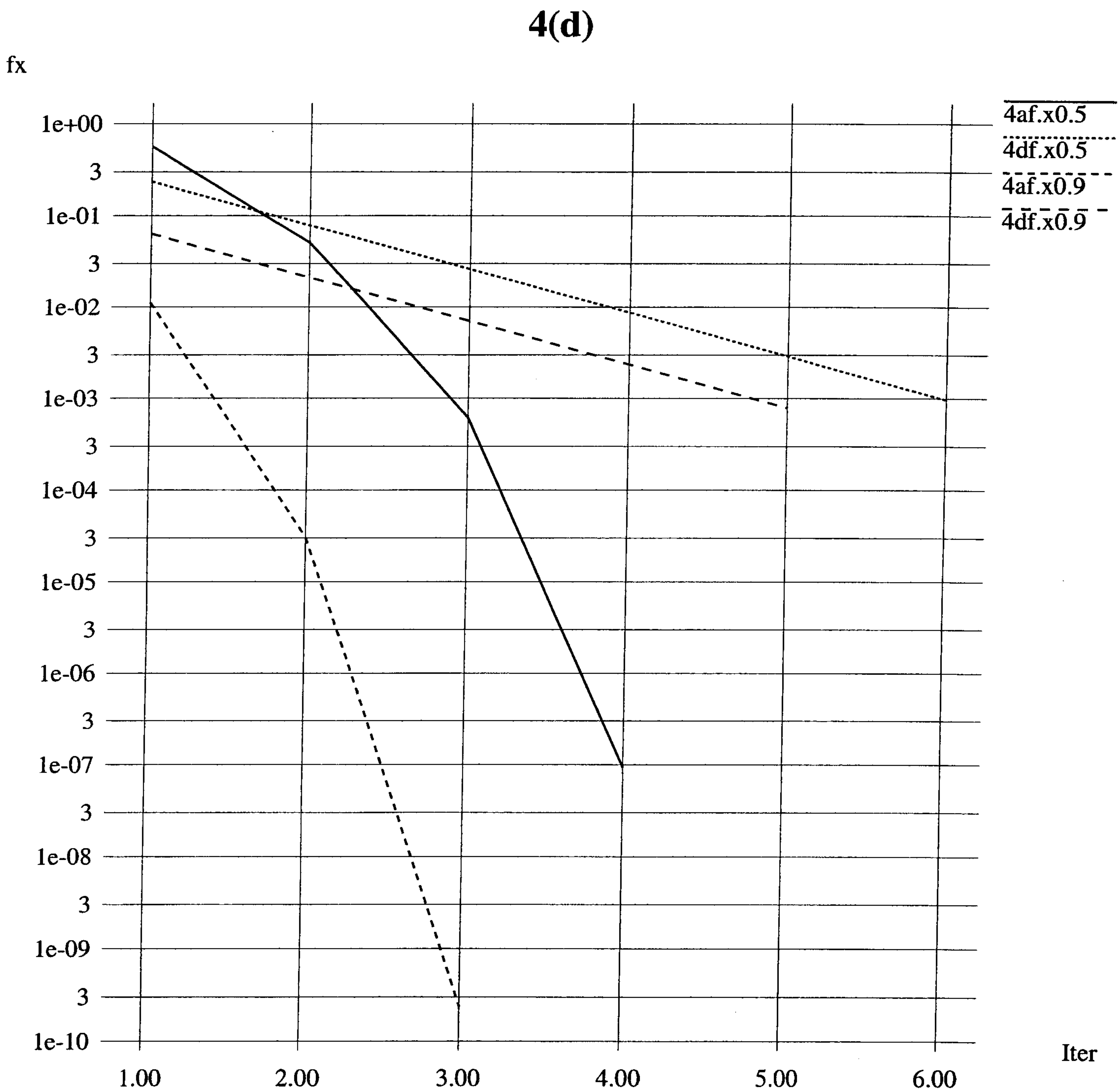

4(a)
x,f(x)
1e+02
1e+01
1e+00
1e-01
1e-02
1e-03
1e-04
1e-05
1e-06
1e-07
1e-08
1e-09
1e-10
1e-11
1.00
2.00
3.00
4.00
5.00
6.00
7.00
Iter
4ax.x0.1
4ax.x0.5
4ax.x0.9
4af.x0.1
4af.x0.5
4af.x0.9

4(b)



x,f(x)
1e+00
1e-01
1e-02
1e-03
1e-04
1e-05
1e-06
1e-07
1e-08
1e-09
1e-10
1e-11
1e-12
5.00
10.00
15.00
20.00
Iter
4bx.x0.1
4bx.x0.5
4bx.x0.9
4bf.x0.1
4bf.x0.5
4bf.x0.9


4(c)
x,f(x)
1e-01
3
1e-02
3
1e-03
3
1e-04
3
1e-05
3
1e-06
3
1e-07
3
1e-08
1.00
2.00
3.00
4.00
5.00
6.00
7.00
8.00
Iter
4cx.x0.7
4cx.x0.8
4cf.x0.7
4cf.x0.8


4(d)
fx
1e+00
3
1e-01
3
1e-02
3
1e-03
3
1e-04
3
1e-05
3
1e-06
3
1e-07
3
1e-08
3
1e-09
3
1e-10
1.00
2.00
3.00
4.00
5.00
6.00
Iter
4af.x0.5
4df.x0.5
4af.x0.9
4df.x0.9