ECE 521	Lecture #5	Oct. 5, 2016
Note Title		10/5/2016

HW # 1 Due Oct. 12

Submission: HW#1 - 1 hard copy

HW #1 - Part 2

hardcopy
- source code of files that you have modified (*.c, *.h)
- Outputs for the test*.ckt
  - netlist printout
  - circuit matrix & RHS
- Input & Output for testn, testt, test o

softcopy: upload zip file with all *.c, *.h, *.ckt, Makefile files to the teach server ENGR

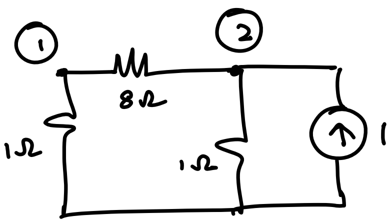
Solution of Linear equations $Ax = b$

direct method: GE or LU decomposition

$\underbrace{\text{GE}}_{\text{back solve}}$ or $\underbrace{\text{LU decomposition}}_{\text{forward/back solve}}$

iterative methods:

Relaxation method



$$\begin{bmatrix} 9/8 & -1/8 \\ -1/8 & 9/8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Using GE the solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.9 \end{bmatrix}$

Relaxation method idea:

Guess x_2^0, solve for x_1^1

Use x_1^1 (x_1^0) and solve x_2^1

Repeat the process

$x_2^0 = 0;$ $\quad x_1^1 = 0 \qquad x_1^k = \frac{1}{8} x_2^{k-1}$

$x_2^1 = 8/9 \qquad x_2^k = \frac{8}{9}\left(1 + \frac{1}{8} x_1^k\right)$

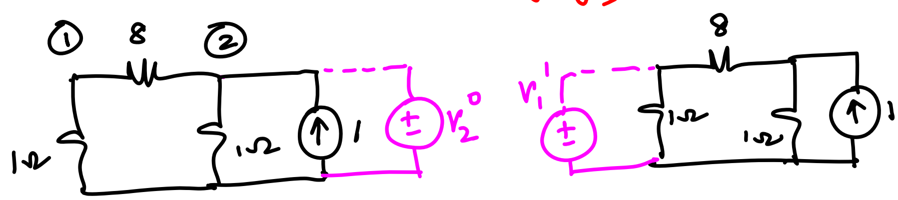
$x_1^2 = \frac{8}{81} = 0.0988$

$x_2^2 = 0.8999$

$x_1^3 = 0.09999$

$x_2^3 \simeq 0.9$

So the iterative process is progressing towards the correct solution (converging)



Suppose $A = \begin{bmatrix} \frac{1}{8} & -9/8 \\ -9/8 & 1/8 \end{bmatrix}$

The solution from GE is $\begin{bmatrix} -0.9 \\ -0.1 \end{bmatrix}$

Iteration equations: $x_1^k = 9 x_2^{k-1}$

$x_2^k = 8 + 9 x_1^{k-1}$

Start with $x_2^0 = 0;$ $\quad x_1^1 = 0$

$x_2^1 = 8$

$x_1^2 = 72$

The process diverges quickly

Questions: 1) Under what conditions does the iterative scheme converge

2) How fast does it converge

3) If the scheme converges to some $\hat{x}$

does $\hat{x}$ solve the original problem

Let x^* be the solution of $Ax = b$

Let x^i be the ith iterate of the iterative method

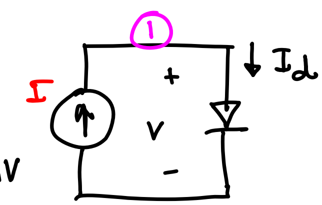
Define $e^i = x^i - x^*$ error of ith iterate

Convergence if $\|e^i\|_2 \to 0$ as i increases independent of x^0 (i.e., the initial guess)

This particular scheme converges when the matrix is diagonally dominant

Nonlinear circuits

$$I_d = I_s\left(e^{V/V_{th}} - 1\right)$$



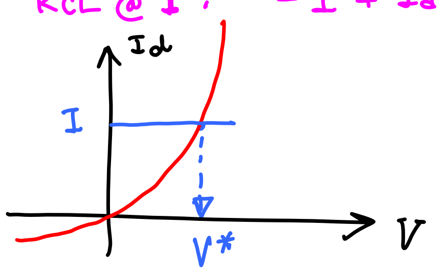
V_{th} = thermal voltage = 25.8 mV at room temp

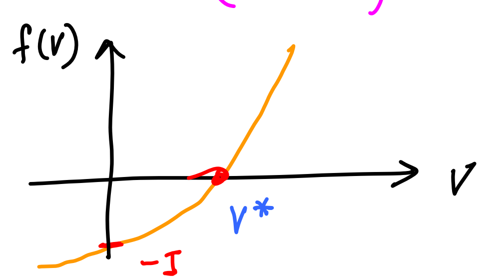
Given a value of I how do we determine V.

$$V = V_{th} \ln\left(\frac{I_d}{I_s} + 1\right) = V_{th} \ln\left(\frac{I}{I_s} + 1\right)$$

How do we solve this problem numerically

KCL @ 1 : $-I + I_d = 0 = -I + I_s\left(e^{V/V_{th}} - 1\right) = f(V) = 0$





An iterative process $\Rightarrow$ sequence of $\{v^{(k)}\}$
such that $v^{(k)} \to v^*$ as k increases
starting from an initial guess $v^{(0)}$

Suppose v^0 is close to the solution v^*, then
a Taylor's series expansion gives

$$0 = f(v^*) = f(v^0) + \frac{\partial f}{\partial v}\bigg|_{v^0} (v^* - v^0) + \frac{1}{2} \frac{\partial^2 f}{\partial v^2}\bigg|_{\hat{v}} (v^* - v^0)^2$$

where $\hat{v} \in [v^0, v^*]$

Since v^0 is close to v^*

$$f(v^0) + \frac{\partial f}{\partial v}\bigg|_{v^0} (v^* - v^0) \simeq 0$$

$$\frac{\partial f}{\partial v}\bigg|_{v^0} (v^* - v^0) = -f(v^0)$$

Newton's method is based on the above idea
the iterates are generated by a linearization
of the function at each iteration

$$|x^{k+1} - x^*| \leq c\,|x^k - x^*|^2$$

quadratic convergence

$$|x^{k+1} - x^*| \leq c\,|x^k - x^*|$$

$c < 1$

linear convergence

If $|c(x^0 - x^*)| \leq \gamma < 1$

$$|x^1 - x^*| \leq c\,|x^0 - x^*|^2 = \underbrace{c\,|x^0 - x^*|}_{\gamma}\,|x^0 - x^*| \leq \gamma\,|x^0 - x^*|$$

$$|x^2 - x^*| \leq c\, |x^1 - x^*|^2$$
$$\leq c\, |x^1 - x^*|\, |x^1 - x^*|$$
$$\vdots$$
$$|x^k - x^*| \leq \gamma^k\, |x^0 - x^*|$$

If $\frac{\partial^2 f}{\partial x^2}$ is bounded & $\frac{\partial f}{\partial x}$ is bounded away from zero then Newton's method converges to the solution given an initial guess that is close to the solution

Multidimensional Case

$\underline{f}(x)$ and $\underline{x}$ are n-vectors

$$\left.\begin{aligned} f_1(x_1, x_2, \ldots x_n) &= 0 \\ f_2(x_1, x_2 \ldots x_n) &= 0 \\ &\vdots \\ f_n(x_1, x_2, \ldots x_n) &= 0 \end{aligned}\right\}$$ KCLs at the different nodes

1-unknown case $\quad \left.\frac{\partial f}{\partial x}\right|_{x^k} (x^{k+1} - x^k) = -f(x^k)$

$$\left.\frac{\partial f}{\partial x}\right|_{x^k} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{x^k} = \text{Jacobian matrix} = J(x^k)$$

Newton's Equation is now

$$J(x^k)\left(x^{k+1}-x^k\right) = -f(x^k)$$

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Iteration Equation: $x^{k+1} = x^k - J^{-1}(x^k) f(x^k)$

Suppose: 1) $\| J^{-1}(x) \| \leq \beta$ bounded inverse

2) $\| J(x) - J(y) \| \leq L \| x - y \| \quad \forall x, y$

Lipschitz continuous

Then $x^{(k)}$ converges to x^* (the solution)

provided $x^{(0)}$ is sufficiently close to x^*

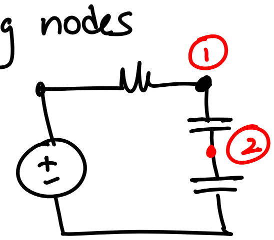
$$\| x^{(k+1)} - x^* \| \leq \alpha \| x^{(k)} - x^* \|^2$$

(quadratic convergence)

What does this all mean?

- all device model equations must be continuous with continuous derivatives.
  - derivative calculations are accurate.
- Provide a good initial guess $x^{(0)}$
- There should be no floating nodes

A floating node $\Rightarrow$ singular Jacobian



In SPICE: an error

"No DC path to ground from Node 2"

Review of Solution of Linear Equations Ax = b

- **Gaussian Elimination**
  - LU factorization ($A=LU$) followed by forward and backward solves
  - Pivoting for accuracy
  - Error mechanisms (round-off)
    - **Ill-conditioning**
    - **Numerical stability**
  - Complexity: $O(N^3)$
- **Gaussian Elimination for sparse matrices**
  - Improved computational cost: $O(N^{1.1}\text{-}N^{1.5})$ and reduced storage
  - Data structures
  - Pivoting for sparsity (Markowitz reordering) and accuracy

Solving Linear Systems

- **Direct methods: find the exact solution in a finite number of steps**
  - Gaussian Elimination
- **Iterative methods: produce a sequence of approximate solutions that hopefully converge to the exact solution**
  - Stationary
    - **Gauss-Jacobi**
    - **Gauss-Seidel**
    - **SOR (Successive Overrelaxation Method)**
  - Non Stationary
    - **GCR, CG, GMRES…..**

From: A. Nardi

Iterative Methods

Iterative methods can be expressed in the general form:

$$x^{(k)} = F(x^{(k-1)})$$

where k is the iteration index: k = 1, 2, 3, …

Hopefully: $x^{(k)} \rightarrow x^*$ (solution of my problem)

- **Will the method converge?**
- **If so, how quickly?**

Classification of Iterative Methods

Stationary:

$x^{(k+1)} = Gx^{(k)} + c$

where *G* and *c* do not depend on iteration count *(k)*

Non Stationary:

$x^{(k+1)} = x^{(k)} + a_k p^{(k)}$

where computation uses information that changes at each iteration

From: A. Nardi

Nonlinear Equations – Iterative Methods

- **Start from an initial value x^0**
- **Generate a sequence of iterates x^{n-1}, x^n, x^{n+1} which hopefully converges to the solution x^***
- **Iterates are generated according to an iteration function F: $x^{n+1}=F(x^n)$**

Ask

- When does it converge to the correct solution ?
- What is the convergence rate ?

From: A. Nardi

Newton-Raphson (NR) Method

Consists of linearizing the system

Want to solve f(x)=0 → Replace f(x) with its linearized version and solve

$$f(x) = f(x^*) + \frac{df}{dx}(x^*)(x - x^*) \qquad \textit{Taylor Series}$$

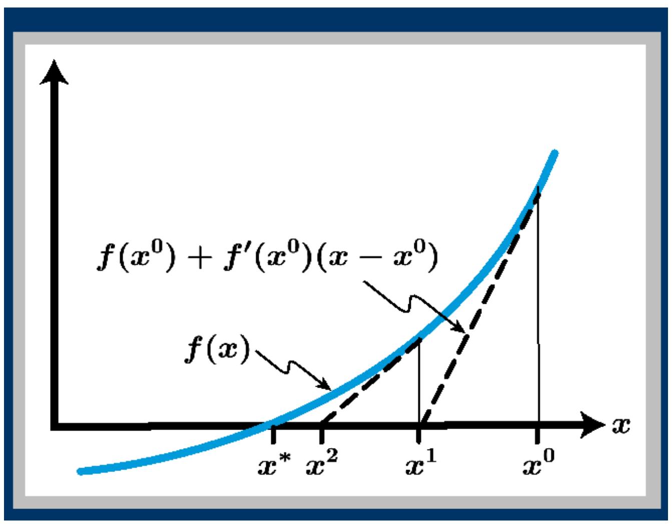
$$f(x^{k+1}) = f(x^k) + \frac{df}{dx}(x^k)(x^{k+1} - x^k)$$

$$\Rightarrow x^{k+1} = x^k - \left[\frac{df}{dx}(x^k)\right]^{-1} f(x^k) \qquad \textit{Iteration function}$$

Note: at each step need to evaluate *f* and *f'*

From: A. Nardi

Newton-Raphson Method – Graphical View



From: A. Nardi

Newton-Raphson Method Algorithm

Define iteration

Do $k = 0$ to ….

$$x^{k+1} = x^k - \left[\frac{df}{dx}(x^k)\right]^{-1} f(x^k)$$

until convergence

- **How about convergence?**
- **An iteration $\{x^{(k)}\}$ is said to converge with order q if there exists a vector norm such that for each $k \geq N$:**

$$\left\|x^{k+1} - \hat{x}\right\| \leq \alpha \left\|x^k - \hat{x}\right\|^q \text{ where } \alpha < 1 \text{ for } q = 1$$

From: A. Nardi

Convergence of Newton Method

$$0 = f(x^*) = f(x^k) + \frac{df}{dx}(x^k)(x^* - x^k) + \frac{d^2 f}{dx^2}(\tilde{x})(x^* - x^k)^2$$

$$\underbrace{\text{some } \tilde{x} \in [x^k, x^*]}_{\text{Remainder form: Taylor series}}$$

But

$$0 = f(x^k) + \frac{df}{dx}(x^k)(x^{k+1} - x^k)$$ by Newton definition

From: A. Nardi

Convergence of Newton Method

Subtracting $\frac{df}{dx}(x^k)(x^{k+1} - x^*) = \frac{d^2 f}{d^2 x}(\tilde{x})(x^k - x^*)^2$

Dividing through $(x^{k+1} - x^*) = [\frac{df}{dx}(x^k)]^{-1} \frac{d^2 f}{d^2 x}(\tilde{x})(x^k - x^*)^2$

Let $\left| [\frac{df}{dx}(x^k)]^{-1} \frac{d^2 f}{d^2 x}(\tilde{x}) \right| = K^k$

then $\left| x^{k+1} - x^* \right| \le K^k \left| x^k - x^* \right|^2$

Convergence is quadratic

From: A. Nardi

Convergence of Newton Method

<u>Local Convergence Theorem</u>

If

$a)\ \frac{df}{dx}$ *bounded away from zero*

$b)\ \frac{d^2 f}{dx^2}$ *bounded*

$\Big\}$ K *is bounded*

Then Newton's method converges given a sufficiently close initial guess (and convergence is quadratic)

From: A. Nardi

Example

EXAMPLE : $f(x) = x^3 - 2, \quad x^* = \sqrt[3]{2} \approx 1.259921$

k	x^k	$\lvert x^k - x^* \rvert$
0	10.0	8.740
1	6.673333	5.413
.	.	.
8	1.261665	1.744e − 03
9	1.259924	2.410e − 06
10	1.259921	4.609e − 12

Asymptotically,

$|x^{k+1} - x^*| \approx C|x^k - x^*|^\alpha$

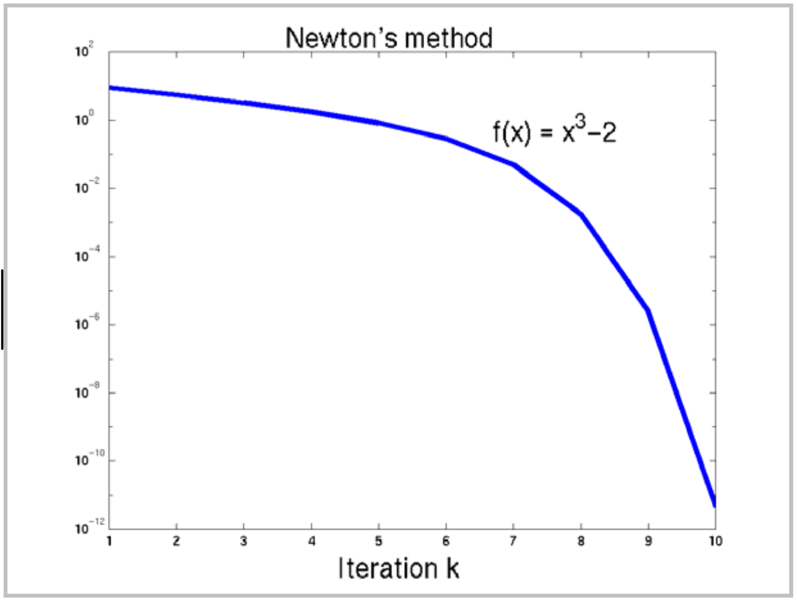
$C = 0.7951$

$\alpha = 2.000$ Quadratic

From: J. White

Example

$$\left|x^k - x^*\right|$$



From: J. White

Example 1

$$f(x) = x^2 - 1 = 0, \quad find \ \ x \ \ (x^* = 1)$$

$$\frac{df}{dx}(x^k) = 2\,x^k$$

$$2x^k(x^{k+1} - x^k) = -\left(\left(x^k\right)^2 - 1\right)$$

$$2x^k(x^{k+1} - x^*) + 2x^k(x^* - x^k) = -\left(\left(x^k\right)^2 - \left(x^*\right)^2\right)$$

$$or \ \ (x^{k+1} - x^*) = \frac{1}{2x^k}(x^k - x^*)^2$$

Convergence is quadratic

From: A. Nardi

Example 2

$$f(x) = x^2 = 0, \quad x^* = 0$$

$$\frac{df}{dx}(x^k) = 2\,x^k$$

Note : $\left(\frac{df}{dx}\right)^{-1}$ not bounded away from zero

$$\Rightarrow 2x^k(x^{k+1} - 0) = (x^k - 0)^2$$

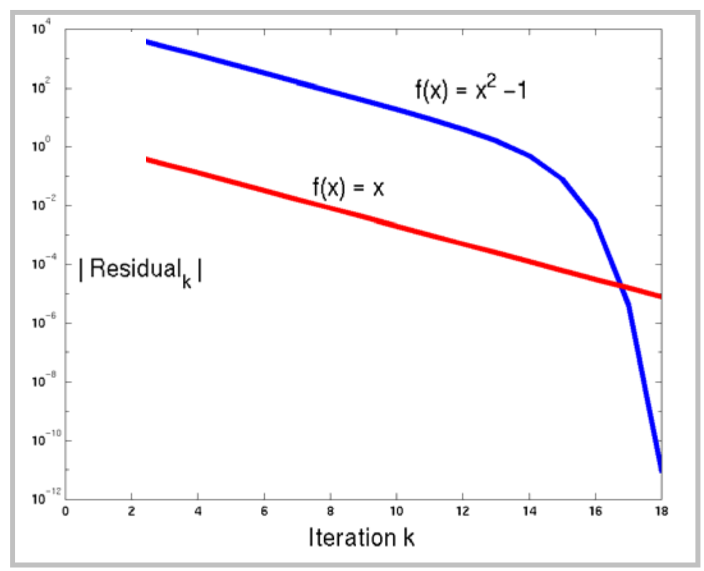
$$x^{k+1} - 0 = \frac{1}{2}\left(x^k - 0\right) \qquad \text{for } x^k \neq x^* = 0$$

$$or \ \ (x_{k+1} - x^*) = \frac{1}{2}(x_k - x^*)$$

Convergence is linear

From: A. Nardi

Convergence of Examples 1, 2



From: J. White

Convergence Check for Newton Method

$x^0 =$ Initial Guess, $k = 0$

Repeat {

$$\frac{\partial f(x^k)}{\partial x}\left(x^{k+1} - x^k\right) = -f\left(x^k\right)$$

$$k = k + 1$$

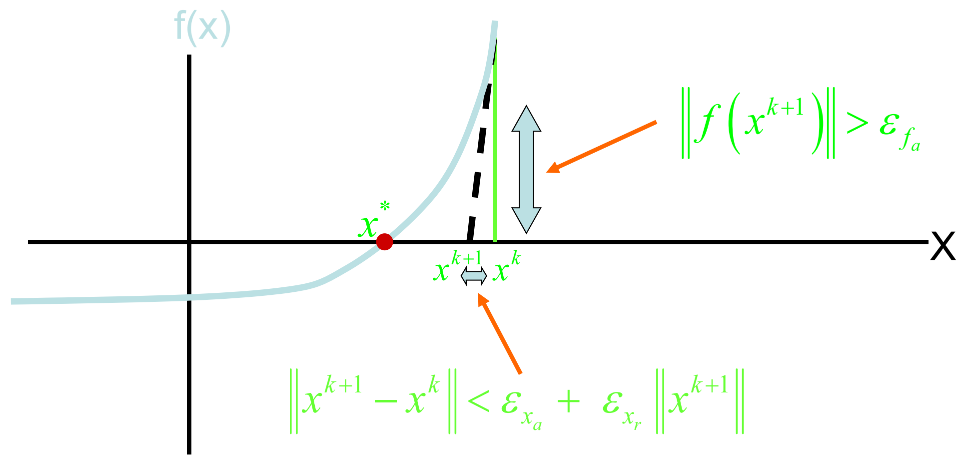
} Until ?

$\left\| x^{k+1} - x^k \right\| < threshold\,?$ $\left\| f\left(x^{k+1}\right) \right\| < threshold\,?$

From: A. Nardi

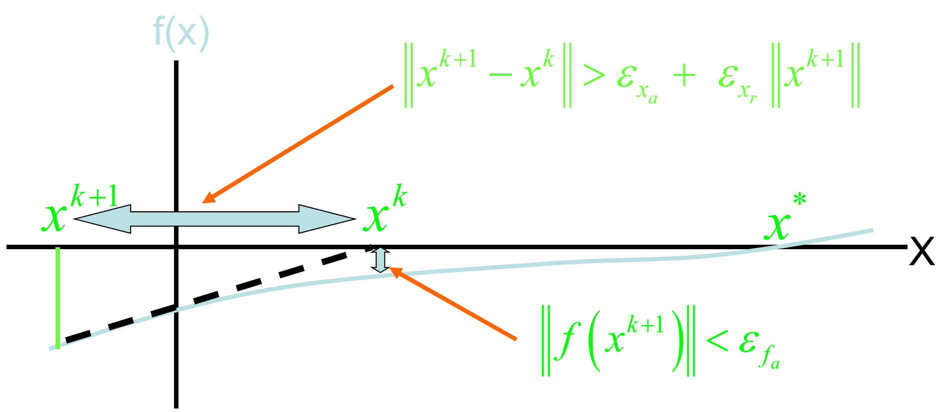
Check $\left\| x^{k+1} - x^k \right\|$ close to zero value

Need to check f(x) to avoid false convergence



From: A. Nardi

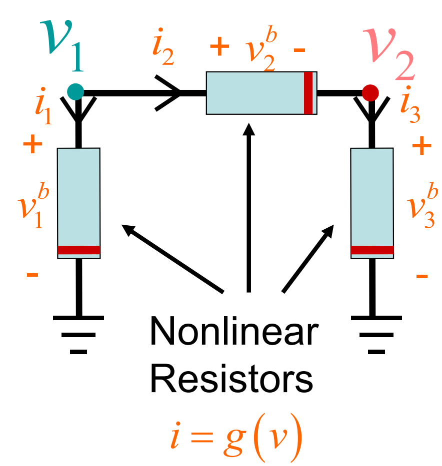
Check $\left\| f(x) \right\|$ close to zero value



Need to check delta-x to avoid false convergence

From: A. Nardi

Nonlinear Problems – Multidimensional Example



<u>Nodal Analysis</u>

At Node 1: $i_1 + i_2 = 0$

$$\Rightarrow g(v_1) + g(v_1 - v_2) = 0$$

At Node 2: $i_3 - i_2 = 0$

$$\Rightarrow g(v_3) - g(v_1 - v_2) = 0$$

Two coupled nonlinear equations in two unknowns

From: A. Nardi