ECE 521 Lecture # 10 Oct. 24, 2016

Note Title 10/24/2016

Newton Loop

oldSol ← 0

spClear ( ); Clear Rhs

cktLood (matrix, Rhs)

spFactor (matrix)

SpSolve (") → Sol

$$f(x) = 0$$

$$J(x^k)(x^{k+1} - x^k) = -f(x^k)$$

This formulation is being used

$$J(x^k)\, x^{k+1} = \boxed{J(x^k)\, x^k - f(x^k)}$$

$$|x^{k+1} - x^k| < \epsilon_A + \epsilon_R \max(|x^{k+1}|, |x^k|)$$

$\epsilon_A = 10^{-6}$, $\epsilon_R = 10^{-3}$

Copy Sol to old Sol

old Sol [i] = Sol [i]

for (i = 0; i ≤ Num Eqn; i++)

Rhs [i] = 0.0;

Linear equation: $f_1 = x_1 + x_2 + x_3 = 0$

$$\frac{\partial f_1}{\partial x} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1^{k+1} \\ x_2^{k+1} \\ x_3^{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1^k \\ x_2^k \\ x_3^k \end{bmatrix} - \begin{bmatrix} x_1^k + x_2^k + x_3^k \end{bmatrix}$$

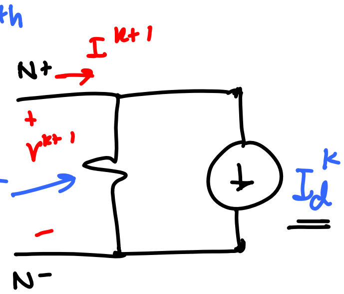
$$= 0$$

Diode example:

$$i = I_s\left(e^{v/V_{th}} - 1\right)$$

Node Voltages

$$\frac{\partial i}{\partial v} = \frac{I_s}{V_{th}} e^{v/V_{th}}$$



$$G_d^k = \left.\frac{\partial i}{\partial v}\right|_{v^k}$$

$$I_d^k = -G_d^k v^k + i(v^k)$$

$v =$ NodeVoltages[N+] - NodeVoltages[N-];

Companion model → $J(x^k)$; $J(x^k)x^k - f(x^k)$

$$i^{k+1} = \underline{i(v^k)} + \left.\frac{\partial i}{\partial v}\right|_{v^k}\left(v^{k+1} - v^k\right)$$

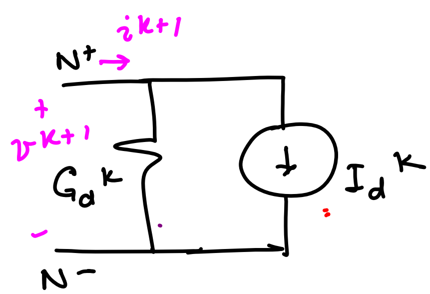
$$\left.\frac{\partial i}{\partial v}\right|_{v^k} v^{k+1} + \boxed{i(v^k) - \left.\frac{\partial i}{\partial v}\right| v^k}$$

MATRIX

$$\begin{array}{c|cc} & V_{N+} & V_{N-} \\ \hline N+ & +G_d & -G_d \\ N- & -G_d & +G_d \end{array}$$

RHS

$$\begin{bmatrix} -I_d^k \\ +I_d^k \end{bmatrix}$$



Integration Methods

FE, BE, TR (one step methods)

FE: $x_n = x_{n-1} + h\dot{x}_{n-1}$

BE: $x_n = x_{n-1} + h\dot{x}_n$

TR: $x_n = x_{n-1} + \frac{h}{2}\left(\dot{x}_n + \dot{x}_{n-1}\right)$

Linear multistep method (p-step method)

$$\sum_{i=0}^{p} \alpha_i x_{n-i} + h\sum_{i=0}^{p} \beta_i \dot{x}_{n-i} = 0$$

|LTE| bounded by LE

$$LE_n = \sum_{i=0}^{p} \alpha_i\, x(t_{n-i}) + h\sum_{i=0}^{p} \beta_i\, \dot{x}(t_{n-i})$$

$$= \sum_{i=0}^{p} \alpha_i\, x(t_n - ih) + h\sum_{i=0}^{p} \beta_i\, \dot{x}(t_n - ih)$$

Once α_i, β_i are selected i.e, a particular integration method then only 'h' can be used to control the error

$$LE_n = E[x(t), h]$$

$$E[x,h] = E[x,0] + E^{(1)}[x,0]h + E^{(2)}[x,0]\frac{h^2}{2} + \cdots + E^{(k+1)}[x,0]\frac{h^{k+1}}{(k+1)!} + O(h^{k+2})$$

$$E[x,h] = \sum_{i=0}^{p} \alpha_i\, x(t_n - ih) + h\sum_{i=0}^{p} \beta_i\, \dot{x}(t_n - ih)$$

$$E^{(1)}[x,h] = \sum_{i=0}^{p} \alpha_i\, \dot{x}(t_n - ih)(-i) + \sum_{i=0}^{p} \beta_i\, \dot{x}(t_n - ih) + h\sum_{i=0}^{p} \beta_i\, \ddot{x}(t_n - ih)(-i)$$

$$E^{(1)}[x,0] = \sum_{i=0}^{p} \alpha_i\, \dot{x}(t_n)(-i) + \sum_{i=0}^{p} \beta_i\, \dot{x}(t_n)$$

$$E[x,0] = \sum_{i=0}^{p} \alpha_i\, x(t_n)$$

A multistep method is said to be of order k if the method is exact for polynomials of degree $\leq k$

$$x(t) = c,\ t,\ t^{1},\ \ldots\ t^{k}$$

$$x^{(k+1)}(t) = 0 \quad \text{for a polynomial of degree } k$$

Exactness constraints

$$0 = E[x,0] \Rightarrow \sum_{i=0}^{p} \alpha_i = 0$$

$$0 = E^{(1)}[x,0] \Rightarrow \sum_{i=0}^{p} \left[\alpha_i(-i) + \beta_i\right] = 0$$

$$\vdots$$

$$0 = E^{(k)}[x,0] \Rightarrow \sum_{i=0}^{p} \left[\alpha_i(-i)^{k} + k\beta_i(-i)^{k-1}\right] = 0$$

$$0 \neq E^{(k+1)}[x,0] \Rightarrow \sum_{i=0}^{p} \left[\alpha_i(-i)^{k+1} + (k+1)\beta_i(-i)^{k}\right] \neq 0$$

Example BE $\quad x_n = x_{n-1} + h\dot{x}_n$

$$x_n - x_{n-1} - h\dot{x}_n = 0$$

1-step $\quad \alpha_0 = 1,\ \alpha_1 = -1,\ \beta_0 = -1,\ \beta_1 = 0$

$$E[x,0] = \sum_{i=0}^{p} \alpha_i = \alpha_0 + \alpha_1 = 1 + -1 = 0$$

$$E^{(1)}[x,0] = \sum_{i=0}^{1} \alpha_i(-i) + \beta_i = \alpha_0(0) + \alpha_1(-1) + \beta_0 + \beta_1$$

$$= (-1)(-1) - 1 = 0$$

$$E^{(2)}[x,0] = \sum_{i=0}^{1} \alpha_i(-i)^2 + 2\beta_i(-i)$$

$$\alpha_0(0)^2 + \alpha_1(-1)^2 + 2\beta_0(0) + 2\beta_1(-1)$$

$$-1(1) = -1 \neq 0$$

$\Rightarrow$ BE is a first order method

TR : $x_n = x_{n-1} + \frac{h}{2}\left[\dot{x}_n + \dot{x}_{n-1}\right]$

1-step method

$$x_n - x_{n-1} - h\frac{1}{2}\dot{x}_n - h\frac{1}{2}\dot{x}_{n-1} = 0$$

$$\alpha_0 = 1, \quad \alpha_1 = -1, \quad \beta_0 = -\frac{1}{2}, \quad \beta_1 = -\frac{1}{2}$$

$$E[x,0] = 0, \quad E^{(1)}[x,0] = 0, \quad E^{(2)}[x,0] = 0$$

$$E^{(3)}[x,0] \neq 0$$

$\Rightarrow$ TR is a 2nd order method

Synthesis of Linear multistep integration methods

p step method of order k $\alpha_0 = 1$

$$\underbrace{\sum_{i=0}^{p} \alpha_i x_{n-i}}_{p \text{ coeff}} + h \underbrace{\sum_{i=0}^{p} \beta_i \dot{x}_{n-i}}_{\beta_0, \beta_1, \ldots \beta_p \} \, p+1 \text{ coeff}}$$

Need to determine $p + p + 1 = 2p+1$ Coeff

Exactness constraints $k+1$ equations

$2p+1$ unknowns & $k+1$ equations

$$2p+1 \geq k+1 \Rightarrow p \geq \frac{k}{2}$$

TR: $k = 2, \quad p = 1$

Algorithm for choosing coeff

1) Choose order of method $- k$ (accuracy)
2) Choose number of steps $- p$ $\left(\geq \frac{k}{2}\right)$
3) Write the $(k+1)$ exactness constraints
4) For $p > \frac{k}{2}$ $(2p+1) - (k+1) = 2p - k$ coeff can be assigned arbitrarily (or some other crit)

Example Suppose $p = k$ then k coeff need additional constraints

Take $\beta_1, \beta_2, \dots \beta_p = 0$

Then the integration method is

$$\sum_{i=0}^{k} \alpha_i x_{n-i} + h\beta_0 \dot{x}_n = 0 \qquad (\alpha_0 = 1)$$

This is the k-th order backward differentiation formula (BDF) also referred to as Gear's methods

Example $p = k = 2$ 2nd-order Gear's method

$\alpha_0 = 1, \alpha_1, \alpha_2, \beta_0$

$$E[x,0] = \sum_{i=0}^{2} \alpha_i = 0 \quad = \quad 1 + \alpha_1 + \alpha_2 = 0$$

$$E^{(1)}[x,0] = \sum_{i=0}^{2} \alpha_i(-i) + \beta_0 \qquad \alpha_1(-1) + \alpha_2(-2) + \beta_0 = 0$$

$$E^{(2)}[x,0] = \sum_{i=0}^{2} \alpha_i(-i)^2 + 2\beta_0 \cancel{(-i)}^{0}$$

$$\alpha_1(-1)^2 + \alpha_2(-2)^2 = 0$$

$$\alpha_1 = -\frac{4}{3}, \quad \alpha_2 = \frac{1}{3}, \quad \beta_0 = -\frac{2}{3}$$

So the integration method is

$$x_n - \frac{4}{3}x_{n-1} + \frac{1}{3}x_{n-2} - \frac{2}{3}h\dot{x}_n = 0$$

$$x_n = \frac{4}{3}x_{n-1} - \frac{1}{3}x_{n-2} + \frac{2}{3}h\underline{\dot{x}_n}$$

$$\dot{x}_n = \frac{1}{h}\left[\frac{3}{2}\underline{\underline{x_n}} - \underbrace{2x_{n-1} + \frac{1}{2}x_{n-2}}\right]$$

$$= \alpha x_n + \beta$$

Local Error for a kth order method

$$E[x,0] = 0 = E^{(1)}[x,0] \dots = E^{(k)}[x,0]$$

$$E^{(k+1)}[x,0] \neq 0$$

$$LE_n = E[x,h] = \cancel{E[x,0]}^{0} + \cancel{E^{(1)}[x,0]}^{0}h + \cancel{E^{(2)}[x,0]}^{0}\frac{h^2}{2}$$

$$+ \cdots + \underbrace{E^{(k+1)}[x,0]\frac{h^{k+1}}{(k+1)!} + O(h^{k+2})}$$

$$LE \cong E^{(k+1)}[x,0]\,\frac{h^{k+1}}{(k+1)!}$$

$$= \frac{h^{k+1}}{(k+1)!}\sum_{i=0}^{p}\left[\alpha_i(-i)^{k+1} + (k+1)\beta_i(-i)^{k}\right]x^{(k+1)}(t_n)$$

$$= C_{k+1}\,\underline{\underline{h^{k+1}}}\;x^{(k+1)}(t_n)$$

$$C_{k+1} = \frac{1}{(k+1)!}\sum_{i=0}^{p}\left[\alpha_i(-i)^{k+1} + (k+1)\beta_i(-i)^{k}\right]$$

$\underline{BE}$:

$$x_n = x_{n-1} + h\dot{x}_n$$

$$x_n - x_{n-1} - h\dot{x}_n = 0$$

$$\alpha_0 = 1,\quad \alpha_1 = -1,\quad \beta_0 = -1,\quad \beta_1 = 0$$

$$C_2 = \frac{1}{2!}\sum_{i=0}^{1}\left[\alpha_i(-i)^2 + 2\beta_i(-i)\right]$$

$$= \frac{1}{2}\left[\alpha_0\cancel{(0)^2}^{0} + \alpha_1(-1)^2 + 2\beta_0\cancel{(0)}^{0} + 2\cancel{\beta_1}^{0}(-1)\right]$$

$$= -\frac{1}{2}$$

LE for BE: $-\frac{1}{2}h^2\ddot{x}(t_n)$

LE for TR: $-\frac{1}{12}h^3\dddot{x}(t_n)$

Two integration methods of order k

Compare leading coefficient of LE for error

LE for Gear's second order method

$$= \frac{1}{6} h^3 \dddot{x}(t_n)$$

More on Local Error

LMS integration method (p-step method)

(1) $$\sum_{i=0}^{p} \alpha_i x_{n-i} + h \sum_{i=0}^{p} \beta_i \dot{x}_{n-i} = 0$$

$$LE = \frac{1}{(k+1)!} \left[\sum_{i=0}^{p} \alpha_i (-i)^{k+1} + (k+1) \beta_i (-i)^k \right] h^{k+1} x^{(k+1)}(t_n)$$

New integration method as

(2) $$\sum_{i=0}^{p} \alpha_i' x_{n-i} + h \sum_{i=0}^{p} \beta_i' \dot{x}_{n-i} = 0$$

$$\alpha_i' = \frac{\alpha_i}{\eta} \quad ; \quad \beta_i' = \frac{\beta_i}{\eta}$$

This scaling would change the LE wrongly suggesting that a simple scaling will make the method more accurate

Need a normalization such that LE is the same regardless of whether (1) or (2) is used

Normalization: $\sum_{i=0}^{p} \beta_i = -1$

BE: $x_n - x_{n-1} - h \dot{x}_n = 0$ $\quad \beta_0 = -1, \ \beta_1 = 0$

TR: $x_n - x_{n-1} - \frac{h}{2} \dot{x}_n - \frac{h}{2} \dot{x}_{n-1} = 0$; $\quad \beta_0 = -\frac{1}{2}, \ \beta_1 = -\frac{1}{2}$

One could have written the TR method as:

$$\frac{x_n}{2} - \frac{x_{n-1}}{2} - \frac{h}{4}\dot{x}_n - \frac{h}{4}\dot{x}_{n-1} = 0$$

$\beta_0 = -\frac{1}{4}$, $\beta_1 = -\frac{1}{4}$

Implicit Integration Methods

$$\dot{x} = f(x)$$

$$\sum_{i=0}^{p}\left[\alpha_i x_{n-i} + h\beta_i \dot{x}_{n-i}\right] = 0 \qquad \beta_0 \neq 0$$

$$\underline{\alpha_0 x_n} + \sum_{i=1}^{p} \alpha_i x_{n-i} + \underline{h\beta_0 \dot{x}_n} + h\sum_{i=1}^{p} \beta_i \dot{x}_{n-i} = 0$$

$$\alpha_0 x_n + h\beta_0 \underset{f(x_n)}{\underset{\parallel}{\dot{x}_n}} + \sum_{i=1}^{p} \alpha_i x_{n-i} + h\beta_i \underset{f(x_{n-i})}{\underset{\parallel}{\dot{x}_{n-i}}} = 0$$

$$\alpha_0 x_n + h\beta_0 f(x_n) + \underbrace{\text{value calculated from previous time points}}_{b} = 0$$

$$F(x_n) = \alpha_0 x_n + h\beta_0 f(x_n) + b = 0$$

This is a nonlinear equation in x_n

$\Rightarrow$ solve using Newton's method

$$J(x_n^k)\left(x_n^{k+1} - x_n^k\right) = -F(x_n^k)$$