ECE 521 Lecture # 12 Oct. 31, 2016

Note Title 10/31/2016

HW # 3 Part 2 Due Monday Nov. 7

HW # 2 Graded handed back

part 2 100 pts

→ Stamps

→ abs( ) fabs( ) <math.h> double fabs( )

ABS( )

→ Array bounds

Solution, old Solution NumEqns+1

for (i = 0 ; i $\leq$ NumEqns ; i++

i < NumEqns+1 i++

→ Convergence checks

x^{k+1}, x^k

$$\left| x^{k+1}[i] - x^k[i] \right| \leq \text{ABSTOL} + \text{RELTOL} \times \text{MAX}\left(\left|x^{k+1}[i]\right|, \left|x^k[i]\right| \right)$$

Newton Loop

for (numIter $\leq$ MAXITER or NOT Converged) {

}

Stability of integration methods

$$\underbrace{\left(z^p + \alpha_1 z^{p-1} + \cdots + \alpha_p\right)}_{P_N(z)} + \sigma \underbrace{\left(\beta_0 z^p + \beta_1 z^{p-1} + \cdots + \beta_p\right)}_{P_D(z)} = 0$$

$$\sigma = -\frac{P_N(z)}{P_D(z)}$$

For large $h \Rightarrow \sigma \rightarrow -\infty$

$\Rightarrow P_D(z) \rightarrow 0$

$$\Rightarrow \beta_0 z^p + \beta_1 z^{p-1} + \cdots + \beta_p = 0$$

What if we set $\beta_1, \beta_2, \cdots \beta_p = 0$

$$\Rightarrow \beta_0 z^p = 0$$

The roots are $z=0$ of multiplicity p

There are p roots, all at the origin $\Rightarrow$ stability region

includes $h = \infty$

This is the idea behind Gear's methods

(or backward differentiation formula)

$p = K$ $$\sum_{i=0}^{K} \alpha_i x_{n-i} + h \beta_0 \dot{x}_n = 0$$

$K = 1$: $$\alpha_0 x_n + \alpha_1 x_{n-1} + h \beta_0 \dot{x}_n = 0$$

BE method: $\alpha_0 = 1$, $\alpha_1 = -1$, $\beta_0 = -1$

(1st order Gear's method or BDF)

Variable time steps

h is no longer uniform

$$\sum_{i=0}^{p} \alpha_i x_{n-i} + h \beta_i \dot{x}_{n-i} = 0$$ LMS method

with $\sum_{i=0}^{p} \beta_i = -1$ for LE calculations

Example 2nd order Gear's method (BDF)

$$\sum_{i=0}^{2} \alpha_i x_{n-i} - h_n \dot{x}_n = 0$$

$$\alpha_0 x_n + \alpha_1 x_{n-1} + \alpha_2 x_{n-2} - h_n \dot{x}_n = 0$$

For a 2nd order method $\Rightarrow$ exact for polynomials of degree ≤ 2

Take $P(t) = (t-t_n)^k$ $\quad k = 0, 1, 2$

$k=0$: $x(t) = 1 \quad \dot{x}(t) = 0$

$k=1$: $x(t) = (t-t_n) \quad \dot{x}(t) = 1$

$k=2$: $x(t) = (t-t_n)^2 \quad \dot{x}(t) = 2(t-t_n)$

For LE calculation we use exact solution

$k=0$: $\alpha_0 x(t_n) + \alpha_1 x(t_{n-1}) + \alpha_2 x(t_{n-2}) - h\dot{x}(t_n) = 0$

$$\alpha_0 + \alpha_1 + \alpha_2 = 0$$

$k=1$: $\alpha_0(0) + \alpha_1 \underbrace{(t_{n-1} - t_n)}_{(-h_n)} + \alpha_2 (t_{n-2} - t_n) - h_n = 0$ $\quad -(h_n + h_{n-1})$

$k=2$:

$$\alpha_0 = h_n \left(\frac{1}{h_n} + \frac{1}{h_n + h_{n-1}} \right)$$

All coefficients are dependent on the timestep

In General for k-th BDF $(1 \leq \text{order} \leq k)$

$$\alpha_0 = h_n \sum_{i=1}^{k} \frac{1}{t_n - t_{n-i}}$$

$1 \le j \le k: \quad \alpha_j = \frac{-h_n}{(t_n - t_{n-j})} \prod_{\substack{i=1 \\ i \ne j}}^{k} \frac{t_n - t_{n-i}}{(t_{n-j} - t_{n-i})}$

Ex: 2nd-order BDF

$$\alpha_0 = h_n \sum_{i=1}^{2} \frac{1}{t_n - t_{n-i}} = h_n \left[\frac{1}{t_n - t_{n-1}} + \frac{1}{t_n - t_{n-2}} \right]$$

$$= h_n \left[\frac{1}{h_n} + \frac{1}{h_n + h_{n-1}} \right]$$

$$\alpha_1 = \frac{-h_n}{(t_n - t_{n-1})} \prod_{\substack{i=1 \\ i \ne 1}}^{2} \frac{t_n - t_{n-i}}{t_{n-1} - t_{n-i}}$$

$$= \frac{-h_n}{h_n} \frac{t_n - t_{n-2}}{t_{n-1} - t_{n-2}} = -\frac{h_n + h_{n-1}}{h_{n-1}}$$

For Kth order BDF LE

$$C_{k+1} = \frac{1}{(k+1)!} \frac{1}{h_n^k} \prod_{i=1}^{k} (t_n - t_{n-i})$$

where $LE = C_{k+1} \, h_n^{k+1} \, x^{(k+1)}(t_n)$

Let us consider TR

Uniform h: $x_n = x_{n-1} + \frac{h}{2} (\dot{x}_n + \dot{x}_{n-1})$

Non uniform $\alpha_0 x_n + \alpha_1 x_{n-1} + h_n \beta_0 \dot{x}_n + h_n \beta_1 \dot{x}_{n-1} = 0$

$$\sum_{i=0}^{1} \beta_0 = -1$$

For variable time steps:

$$x_n - x_{n-1} - \frac{h_n}{2} (\dot{x}_n + \dot{x}_{n-1}) = 0$$

same as TR with uniform h

$$LE = \frac{-h_n^3}{12} \dddot{x}(t_n)$$

Timestep control

How do we determine the stepsize h_n?

Goal: choose as fewer timepoints as possible given an error constraint

$\Rightarrow$ minimize # of timepoints

Assume E_n is a bound on the absolute error

$$|LE_n| \leq E_n$$

$$|LE_n| = \left| C_{k+1} h_n^{k+1} x^{(k+1)}(t_n) \right| \leq E_n$$

$$h_n \leq \left[\frac{E_n}{\left| C_{k+1} x^{(k+1)}(t_n) \right|} \right]^{\frac{1}{k+1}}$$

E_n is given, C_{k+1} is known

what about $x^{(k+1)}(t_n)$?

In SPICE divided differences are used

$$DD_1 = \frac{x_n - x_{n-1}}{h_n} \simeq \dot{x}_n$$

$$DD_2 = \frac{DD_1(t_n) - DD_1(t_{n-1})}{h_n + h_{n-1}} \simeq \frac{\ddot{x}_n}{2!}$$

$$\vdots$$

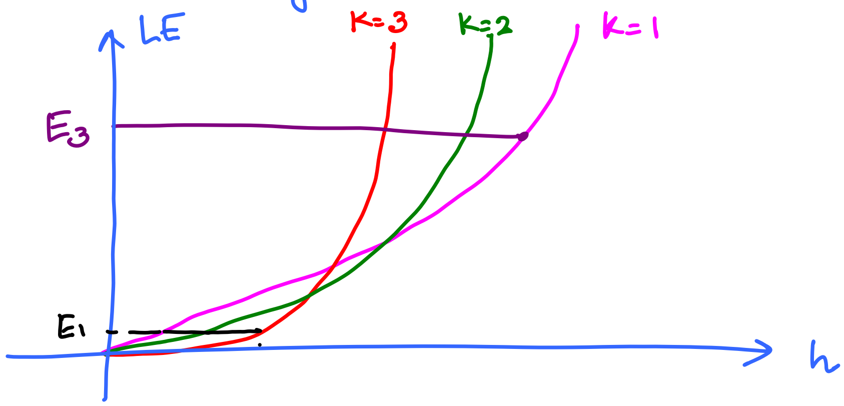
$$DD_{k+1} = \frac{DD_k(t_n) - DD_k(t_{n-1})}{\sum_{i=0}^{k} h_{n-i}} \simeq \frac{x_n^{(k+1)}}{(k+1)!}$$

Order Control

If we have a method of order k

(BDF: $1 \leq k \leq 6$)

→ Choose order which gives you the largest timestep



In SPICE the default method is TR

(METHOD = GEAR MAXORD = N $(2 \le N \le 6)$)

Heuristic rules for timestep control

- Do not change timestep too often
- Change order only if improvement is worthwhile i.e. at least 2h

- Attempt change of stepsize and order only if $|LE| < E_n$ (error is large)

In SPICE: $E_n = \epsilon_A + \epsilon_R \, MAX(|x_n|, |x_{n-1}|)$

ABSTOL (currents)
VNTOL (voltages)
CHGTOL (charges)

Linear Multistep Methods – Stability

$$\sum_{i=0}^{p} \alpha_i x_{n-i} + h\beta_i \dot{x}_{n-i} = 0; \text{ Testproblem } \frac{d}{dt}x(t) = \lambda x(t), x(0) = 1 \qquad (1)$$

$$\sum_{i=0}^{p} (\alpha_i + \beta_i h\lambda) x_{n-i} = \sum_{i=0}^{p} (\alpha_i + \sigma\beta_i) x_{n-i} = 0 \qquad (2)$$

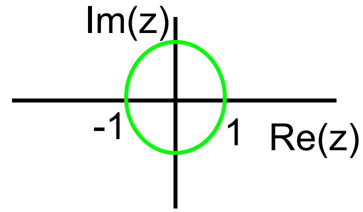
- A method is stable if all solutions of the associated difference equation (2) obtained by setting σ=0 remain bounded as $n \to \infty$
- The region of absolute stability of a method is the set of σ (complex) such that all solutions of (2) remain bounded as $n \to \infty$
- Note: A method is stable if its region of absolute stability contains the origin, i.e., σ=0

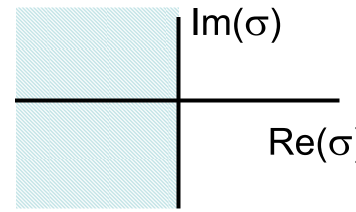
Stability

The region of absolute stability of a method is the set of σ such that all the roots of $\sum_{i=0}^{p} (\alpha_i + \sigma\beta_i) z^{p-i} = 0$ are inside or on the complex unit circle, i.e., $|z| \le 1$, and the roots for which $|z| = 1$ are of multiplicity 1

A method is A-stable if the region of absolute stability contains the entire left-half plane (Re(σ)<0)

TR is an A-stable method





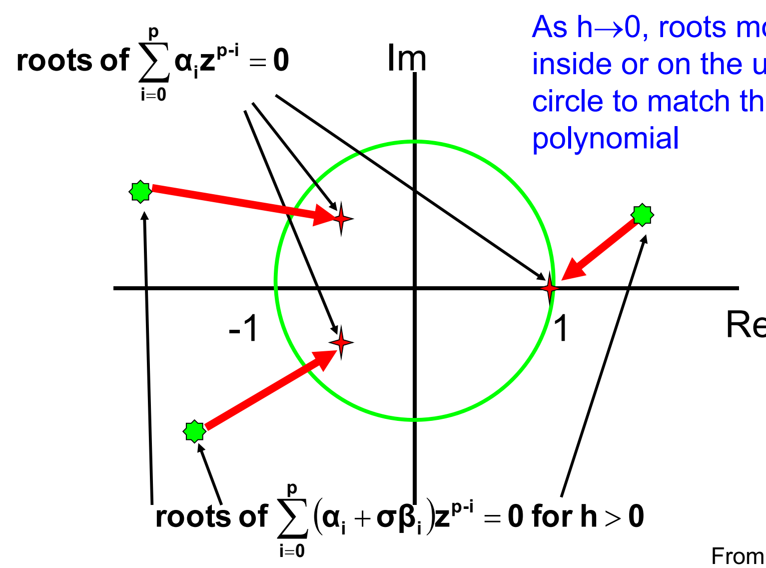
From: A. Nardi

Stability

- **Each method is associated with two polynomials of coefficients α and β:**
  - α: associated with past function values (x_{n-i})
  - β: associated with past derivative values (x'_{n-i})
- **Stability: roots of α polynomial must satisfy $|z| \le 1$ and be of multiplicity 1 for $|z|=1$**
- **Absolute stability: roots of (α+ $\sigma\beta$) polynomial must satisfy $|z| \le 1$ and be of multiplicity 1 for $|z|=1$**

From: A. Nardi

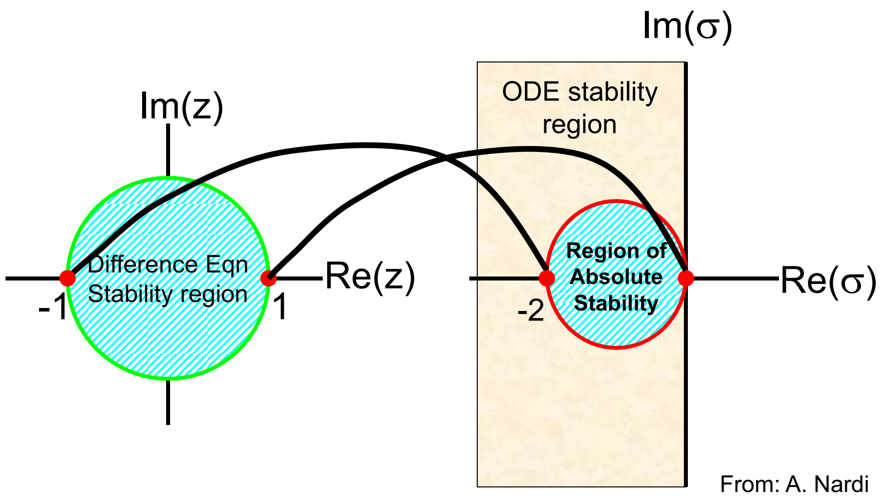
Stability and Region of Absolute Stability



From: A. Nardi

FE Region of Absolute Stability

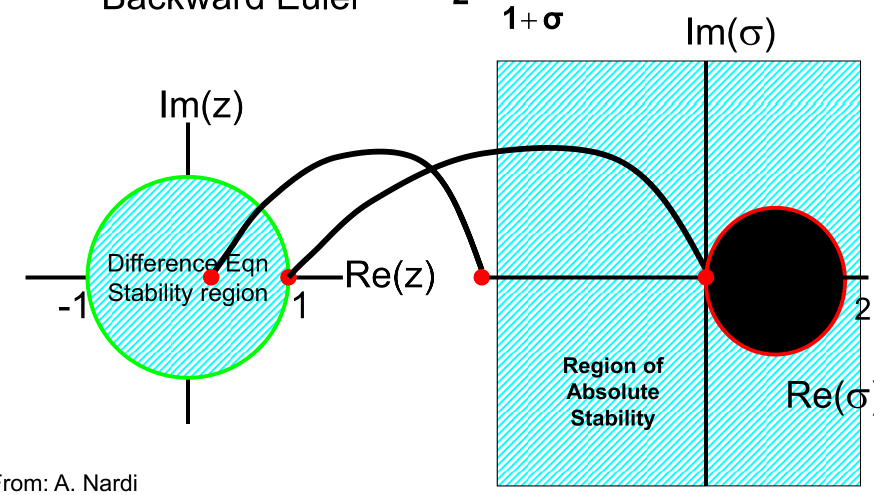
Forward Euler $\quad \mathbf{z = 1 + \sigma}$



From: A. Nardi

BE Region of Absolute Stability

Backward Euler $\quad \mathbf{z = \dfrac{1}{1+\sigma}}$



From: A. Nardi

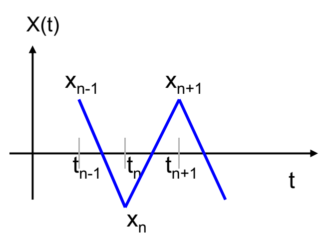
L-Stability

- **An A-stable method is L-stable if Re(λ)<0 implies** $\lim_{h\to\infty} \mathbf{x_n = 0}$ **for all** $\mathbf{x_{n-1}}$

$$\text{Consider TR } \mathrm{Re}(\lambda) < 0: x_n - x_{n-1} - \frac{h}{2}(\dot{x}_n + \dot{x}_{n-1}) = 0$$

$$\left(1 - \frac{\lambda h}{2}\right) x_n - \left(1 + \frac{\lambda h}{2}\right) x_{n-1} = 0$$

$$\Rightarrow x_n = \frac{\left(1 + \frac{\lambda h}{2}\right)}{\left(1 - \frac{\lambda h}{2}\right)} x_{n-1} \Rightarrow \lim_{h\to\infty} x_n = -x_{n-1}$$



i.e., there is ringing

TR in not an L-stable method

Finding the Region of Absolute Stability

$$(1 + \sigma\beta_0) z^p + (\alpha_1 + \sigma\beta_1) z^{p-1} + \ldots (\alpha_p + \sigma\beta_p) = 0$$

For what values of σ do all the p roots of this polynomial satisfy the stability condition?

$$z^p + \alpha_1 z^{p-1} + \ldots \alpha_p + \sigma\left(\beta_0 z^p + \beta_1 z^{p-1} + \ldots + \beta_p\right) = 0$$

Methods

1. Choose σ, compute roots, test, repeat for all σ (BAD)
2. Solve for $\sigma = -P_N(z)/P_D(z)$

$$P_N(z) = z^p + \alpha_1 z^{p-1} + \ldots \alpha_p$$

$$P_D(z) = \beta_0 z^p + \beta_1 z^{p-1} + \ldots \beta_p$$

$$\text{Consider } S = \left\{ \sigma \,\middle|\, \sigma = -\frac{P_N(z)}{P_D(z)}, |z| \le 1 \right\} \text{ and let z vary in } |z| \le 1$$

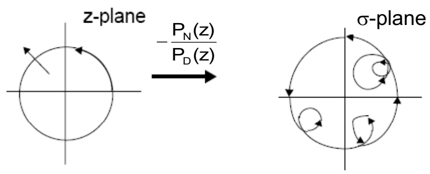
May get same σ values for two or more different z's, one with $|z| \le 1$ and one with $|z| > 1$

⇒ S is a superset of the region of absolute stability

Boundary Locus Γ_σ

Boundary locus Γ_σ is the contour $-\dfrac{P_N(z)}{P_D(z)}$ when $|z| = 1$, i.e., $z = e^{j\theta}$,

$0 \le \theta \le 2\pi$. It is a map from the z-plane to the σ-plane



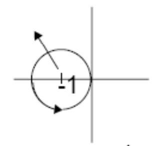
Basic Results from Theory of Complex Variables

- Mapping is conformal, i.e., angle preserving
- Γ_σ separates the σ-plane into disjoint sets. In each set, the number of roots outside the unit circle is constant
- The boundary of the stability region is a subset of Γ_σ

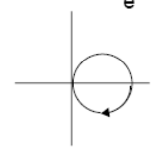
Region of Absolute Stability

Move counterclockwise along $|z|=1$, then you see one more root to the outside of the unit circle for those σ to the right than for those σ to the left when traversing Γ_σ

<u>F.E.:</u> $q\,(z = e^{i\theta}) = e^{i\theta}-1$

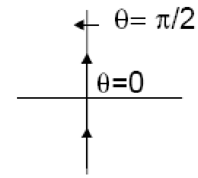


<u>B.E.:</u> $q\,(z = e^{i\theta}) = 1-\frac{1}{e^{i\theta}} = 1 - e^{-i\theta}$



<u>T.R.:</u> $x_n = x_{n-1} + \frac{q}{2}(x_{n-1} + x_n)$

$$\Rightarrow q = \frac{2\,(z-1)}{z+1} = \frac{2\,(e^{i\theta}-1)}{e^{i\theta}+1}$$

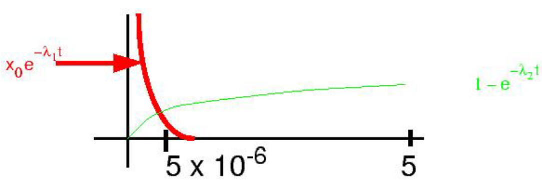


From: A. Sangiovanni-Vincentelli

Large Timestep Issues
Stiff Problems

$$\begin{cases} \dfrac{dx(t)}{dt} = -\lambda_1(x - s(t)) + \dfrac{ds(t)}{dt} \\ x(0) = x_0 \end{cases} \quad \text{where} \quad s(t) = 1 - e^{-\lambda_2 t}, \quad \lambda_1 = 10^6, \lambda_2 = 1$$

Exact solution: $x(t) = x_0 e^{-\lambda_1 t} + 1 - e^{-\lambda_2 t}$



For $t \ge 5 \cdot 10^{-6}$ $\quad x_0 e^{-\lambda_1 t} \approx 0$

For $t \ge 5$ $\quad 1 - e^{-\lambda_2 t} \approx 1$

Interval of interest is [0,5]

Uniform step size (for accuracy)

$\Rightarrow \Delta t \le 10^{-6}$

$\Rightarrow 5\times10^{6}$ steps !!!

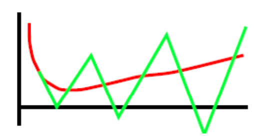
From: A. Nardi

One Possible Strategy – Variable Time Steps

- **Take 5 steps of size 10^{-6} for accuracy during initial phase and then 5 steps of size 1**
- **With FE cannot use h > 2×10^{-6} ($\lambda= 10^6$)**

Stiff problem:

1. **Natural time constants**
2. **Input time constants**
3. **Interval of interest**

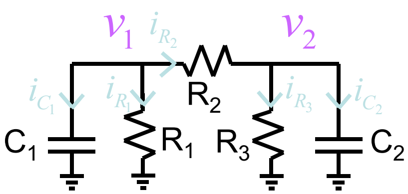


If these are widely separated, then the problem is stiff

From: A. Nardi

Application Example

On-chip Signal Transmission– 2x2 example



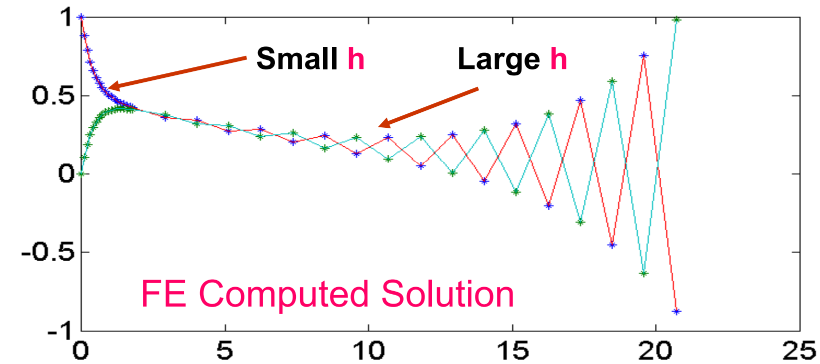
Let $C_1 = C_2 = 1,\ R_1 = R_3 = 10,\ R_2 = 1$

Time Constants 1/0.1 sec and 1/2.1 sec

$$\frac{dx}{dt} = \underbrace{\begin{bmatrix} -1.1 & 1.0 \\ 1.0 & -1.1 \end{bmatrix}}_{A} x$$

From: A. Nardi

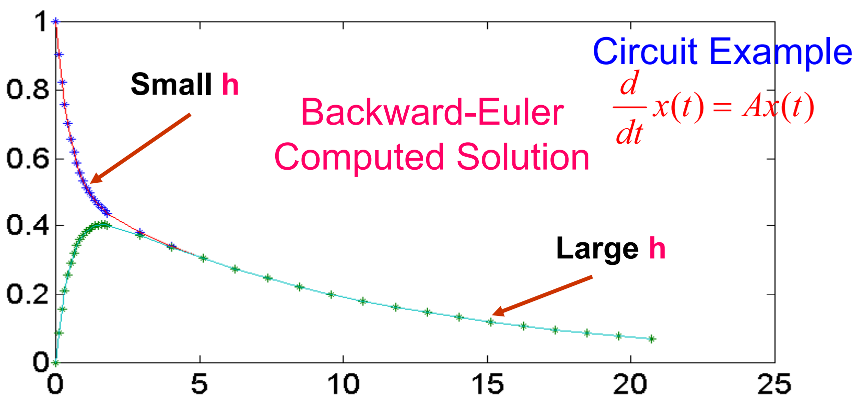
FE on Two Time-constant Circuit



Forward-Euler is accurate for small timesteps, but goes unstable when timestep is increased

From: A. Nardi

BE on Two Time-constant Circuit



With BE can use small timesteps for fast dynamics and then switch to large timesteps for the slow decay

From: A. Nardi

Summary of Stiff Problems

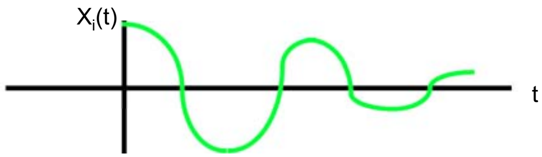
- **The analysis of stiff circuits requires the use of variable step sizes**
- **Not all the linear multistep methods can be efficiently used to integrate stiff equations**
- **To be able to choose h based only on accuracy considerations, the region of absolute stability should allow a large h for large time constants, without being constrained by the small time constants**
- **A-stable methods satisfy this requirement**

From: A. Nardi

Requirements of Stiffly Stable Integration Methods

$$\dot{x}_i(t) = \lambda_i x_i(t) \qquad i = 1,2,\ldots,n,\ \lambda_i = a_i + jb_i$$

$$x_i(t) = C_i e^{a_i t} \cos b_i t$$

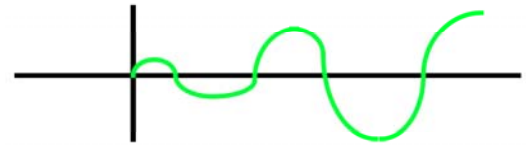


For accuracy want at least 8 points per cycle

$$\Rightarrow h \le \frac{1}{8}\left(\frac{2\pi}{b_i}\right) \text{ or } hb_i \le \frac{\pi}{4} \Rightarrow \mathrm{Im}(\sigma) \le \frac{\pi}{4}$$

Requirements of Stiffly Stable Integration Methods

Want a region of absolute stability which gives a stable algorithm for initial transient



$$\Rightarrow a_i h > 0,\ a_i h < \mu$$

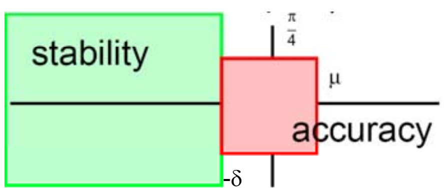
$$\mathrm{Re}(\sigma) = a_i h$$

$$\text{Require}: 0 \le \mathrm{Re}(\sigma) \le \mu$$

Requirements of Stiffly Stable Integration Methods

Require a small h to capture fast transient

$$\lambda_{fast} = a_{fast} + jb_{fast} \Rightarrow \text{Take}: \delta > \frac{5}{a_{fast}},\ a_{fast} > 0$$



Remarks:

- There is a region $\mathrm{Re}(\sigma)<-\delta$ that is absolutely stable
- For $0 < \mathrm{Im}(\sigma) < \pi/4$ the region is of absolute stability and the algorithm is accurate
- For $0 < \mathrm{Re}(\sigma) < \mu$ the region is stable and the algorithm is accurate

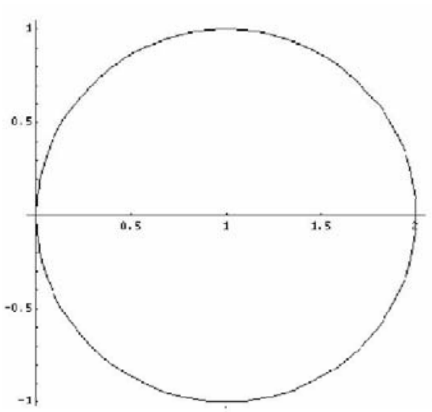
Backward Differentiation Formula - BDF (Gear Methods)

$$\sum_{i=0}^{k} \alpha_i x_{n-i} + h\beta_0 \dot{x}_n = 0 \qquad \text{where} \quad \beta_0 \ne 0$$

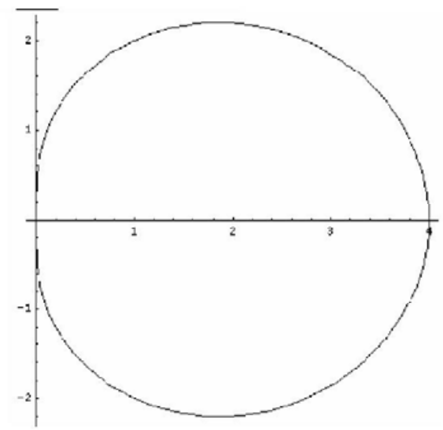
- Gear's first order method is BE
- It can be shown that:
  - Gear's methods up to order 6 are stiffly stable and are well-suited for stiff ODEs
  - Gear's methods of order higher than 6 are not stiffly stable
- Less stringent than A-stable

From: A. Nardi

Gear's Method Region of Absolute Stability (outside the closed curve)



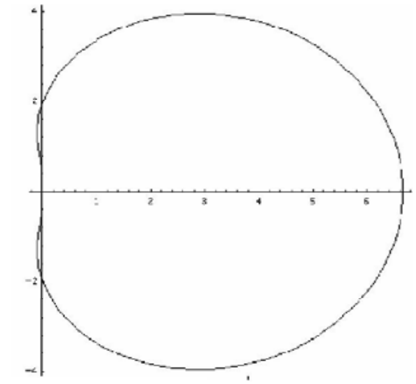
k=1



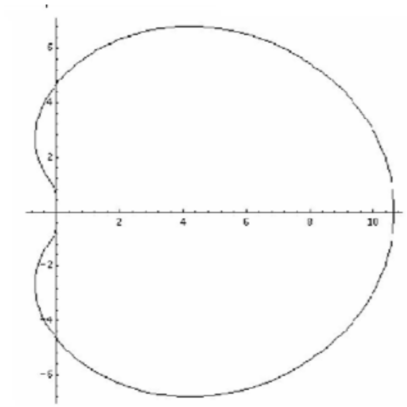
k=2

From: A. Sangiovanni-Vincentelli

Gear's Method Region of Absolute Stability (outside the closed curve)



k=3



k=4

From: A. Sangiovanni-Vincentelli

Observations on Stiff Stability

- **FE: timestep is limited by stability and not by accuracy**
- **BE: A-stable, any timestep could be used**
- **TR: most accurate A-stable multistep method**
- **Gear: stiffly stable method (up to order 6)**

- **The analysis of stiff circuits requires the use of variable timestep**

From: A. Nardi