Sensitivity analysis

(small changes in parameter values)

- sensitivity circuits

Original circuit Sensitivity circuit

Sensitivity circuit has the same topology as the original circuit

$\hat{v}_b, \hat{i}_b, \hat{v}_n$ are the sensitivities w.r.t. 'p'

For linear algebraic systems: $Ax = b$

A is real or complex

We are interested in the sensitivity of x to parameter p $\longrightarrow \frac{\partial x}{\partial p}$

$$Ax = b$$

$$\frac{\partial A}{\partial p} x + A \frac{\partial x}{\partial p} = \frac{\partial b}{\partial p}$$

$$A \frac{\partial x}{\partial p} = -\frac{\partial A}{\partial p} x + \frac{\partial b}{\partial p}$$

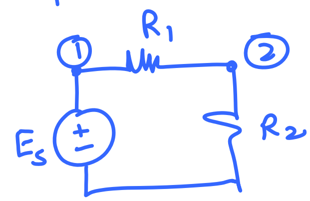
$$\frac{\partial x}{\partial p} = A^{-1} \left[-\frac{\partial A}{\partial p} x + \frac{\partial b}{\partial p} \right]$$

Using LU factorization and forward back solves

Sensitivity with parameters $p_1, p_2, \dots p_m$ can be determined by forward back solves when A is available in its LU factored form

Most often we are interested in the sensitivity of some output (ϕ) with respect to parameters

$\frac{\partial V_2}{\partial R_2}$



$$V_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_E \end{bmatrix}$$

$$\phi = c^T x \qquad \text{What is } \frac{\partial \phi}{\partial p} \text{ ?}$$

$$\frac{\partial \phi}{\partial p} = c^T \frac{\partial x}{\partial p} = \underbrace{-c^T A^{-1}}_{x_a^T} \left(\frac{\partial A}{\partial p} x - \frac{\partial b}{\partial p} \right)$$

$$x_a^T A = -c^T A^{-1} A$$

$$(x_a^T A)^T = (-c^T)^T \Rightarrow A^T x_a = -c$$

Once x_a is known from the solution of → then sensitivity can be easily calculated

$$\frac{\partial \phi}{\partial p} = x_a^T \left(\frac{\partial A}{\partial p} x - \frac{\partial b}{\partial p} \right)$$

where $A^T x_a = -c$ ← adjoint vector

Suppose we are solving $Ax = b$

Decompose A into its LU factors

$$L\underbrace{Ux}_{Y} = b \qquad Ly = b \rightarrow (FS)$$

$$Ux = Y \longrightarrow (BS)$$

$$(LU)^T x_a = -c \Rightarrow U^T \underbrace{L^T x_a}_{Z} = -c$$

$$U^T Z = -c$$

$$L^T x_a = Z$$

Calculation of x_a requires forward/back subst

$$\frac{\partial \phi}{\partial p_i} = x_a^T \left[\frac{\partial A}{\partial p_i} x - \frac{\partial b}{\partial p_i} \right]$$

Observation:

Suppose we want to evaluate a scalar output for multiple RHS vectors, then the adjoint network has an advantage

$$A x_i = b_i \qquad \phi_i = c^T x_i$$

$$x_i = A^{-1} b_i \Rightarrow \phi_i = (c^T A^{-1}) b_i$$

$$= -(-c^T A^{-1})\, b_i$$

$$x_a = -c^T A^{-1} \Rightarrow \phi_i = (-x_a)^T b_i$$

Adjoint analysis is used for small-signal noise analysis

$$A x = b$$

Direct sensitivity calculation →

$$A \frac{\partial x}{\partial p} = -\left[\frac{\partial A}{\partial p} x - \frac{\partial b}{\partial p} \right]$$

Adjoint method:

$$\phi = c^T x$$

$$A^T x_a = -c$$

$$\frac{\partial \phi}{\partial p} = x_a \left[\frac{\partial A}{\partial p} x - \frac{\partial b}{\partial p} \right]$$

Sensitivity analysis for nonlinear circuits

1) DC sensitivity

$$f(x) = 0$$

$$\frac{\partial f}{\partial x} \left(x^{k+1} - x^k \right) = - f(x^k)$$

Solution is x^*

$$f(x, p) = 0$$

$$\left.\frac{\partial f}{\partial x}\right|_{x^*} \frac{\partial x}{\partial p} + \frac{\partial f}{\partial p} = 0$$

$\frac{\partial x}{\partial p}$ are the sensitivities

$\frac{\partial x}{\partial p}$ can be obtained by forward/back solves

$$\left.\frac{\partial f}{\partial x}\right|_{x^*} \frac{\partial x}{\partial p} = -\frac{\partial f}{\partial p}$$

The derivatives of $\frac{\partial f}{\partial p}$ are needed!

2) AC analysis → linear sensitivity analysis

tricky if 'p' affects the dc operating point

3) Transient sensitivity

$$f(x, \dot{x}, p) = 0$$

At the DC operating point $f(x, p) = 0$

$\left.\frac{\partial x}{\partial p}\right|_{dc.op}$ can be computed

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial f}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial p} + \frac{\partial f}{\partial p} = 0$$

Let $z = \frac{\partial x}{\partial p}$ which is the desired sensitivity

$$\frac{\partial \dot{x}}{\partial p} = \frac{\partial}{\partial p}\frac{dx}{dt} = \frac{d}{dt}\frac{\partial x}{\partial p} = \frac{d}{dt} z = \dot{z}$$

$$\frac{\partial f}{\partial x} z + \frac{\partial f}{\partial \dot{x}} \dot{z} + \frac{\partial f}{\partial p} = 0$$

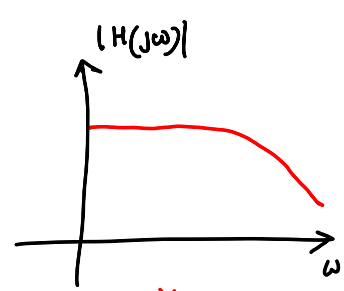
This is a DAE for z which is the sensitivity vector

Pole/zero analysis

small-signal AC analysis

$$H(s) = \frac{N(s)}{D(s)}$$

$$= K \frac{\prod_{i=1}^{n} (s - z_i)}{\prod_{i=1}^{m} (s - p_i)}$$



z_i are the zeros

p_i are the poles

- stability of the circuit
- pole/zero information ⇒ frequency response
  ⇒ time domain response using inverse Laplace transfor

For small-signal ac analysis

$$\underbrace{A(s)}_{G+sC}\, x = b \implies x = A^{-1}(s)\, b$$
$$= (G+sC)^{-1}\, b$$

Suppose the output of interest is

$$y = \ell^T x = \ell^T (G+sC)^{-1}\, b$$
$$= \ell^T \frac{\operatorname{adj}(G+sC)}{\det(G+sC)}\, b$$

poles of the transfer function are given by the values of s for which

$$\det(G+sC) = 0$$

What about the zeros?

$$A(s)x = b$$
$$-\ell^T x + y = 0$$

$$\begin{bmatrix} A(s) & 0 \\ -\ell^T & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

$$\therefore\ Y = \frac{\det \begin{bmatrix} A(s) & b \\ -\ell^T & 0 \end{bmatrix}}{\det \begin{bmatrix} A(s) & 0 \\ -\ell^T & 1 \end{bmatrix}} = \frac{\quad}{\det A(s)}$$

The zeros are the values of s for which

$$\det \underbrace{\begin{bmatrix} A(s) & b \\ -\ell^T & 0 \end{bmatrix}}_{A_a(s)} = 0$$

How do we determine the roots of these determinants?

1) Muller's method
2) QZ algorithm

The determinant is a polynomial in s and we are interested in its roots

$$f(s) = a_0 + a_1 s + a_2 s^2 + \cdots + a_n s^n$$

Muller's method is an iterative method

- approximate polynomial by quadratic $b_0 + b_1 s + b_2 s^2$

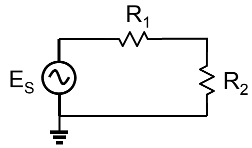
- start with values of s: s_i, s_{i-1}, s_{i-2}; f_i, f_{i-1}, f_{i-2}
- Find b_0, b_1, b_2 for the quadratic & solve quadratic → take the smallest s_{i+1}

- Repeat process to termination $|f(s)|$ small
- Once a root has been found, its effect is factored out and the process is repeated on the reduced polynomial

$$g(s) = \frac{f(s)}{s - p_1}$$

Example of Sensitivity Analysis

$$\begin{bmatrix} \frac{1}{R_1} & -\frac{1}{R_1} & 1 \\ -\frac{1}{R_1} & \frac{1}{R_1}+\frac{1}{R_2} & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_E \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ E_s \end{bmatrix}$$



Rewrite in lower triangular form

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{R_1} & \frac{1}{R_1}+\frac{1}{R_2} & 0 \\ \frac{1}{R_1} & -\frac{1}{R_1} & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_E \end{bmatrix} = \begin{bmatrix} E_s \\ 0 \\ 0 \end{bmatrix} \Rightarrow \quad x = \begin{bmatrix} E_s \\ \frac{R_2}{R_1+R_2}E_s \\ -\frac{E_s}{R_1+R_2} \end{bmatrix}$$

Sensitivity Calculation

Suppose $p = R_2$

$$\frac{\partial A}{\partial R_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{R_2^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \frac{\partial b}{\partial R_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

RHS is $(\partial A/\partial p)x - \partial b/\partial p$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{R_2^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} E_s \\ \frac{R_2}{R_1+R_2}E_s \\ \frac{E_S}{R_1+R_2} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{E_S}{R_2(R_1+R_2)} \\ 0 \end{bmatrix}$$

Sensitivity Calculation

To determine $\partial x/\partial p$, Solve $A\partial x/\partial p = -[(\partial A/\partial p)x - \partial b/\partial p]$

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{R_1} & \frac{1}{R_1}+\frac{1}{R_2} & 0 \\ \frac{1}{R_1} & -\frac{1}{R_1} & 1 \end{bmatrix} \begin{bmatrix} \partial V_1/\partial R_2 \\ \partial V_2/\partial R_2 \\ \partial I_E/\partial R_2 \end{bmatrix} = -\begin{bmatrix} 0 \\ -\frac{E_S}{R_2(R_1+R_2)} \\ 0 \end{bmatrix}$$

Therefore,

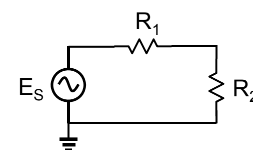
$$\frac{\partial V_1}{\partial R_2} = 0$$

$$\frac{\partial V_2}{\partial R_2} = \frac{R_1 E_s}{(R_1+R_2)^2}$$

$$\frac{\partial I_E}{\partial R_2} = \frac{E_s}{(R_1+R_2)^2}$$

Example of Adjoint Analysis

$$\begin{bmatrix} \frac{1}{R_1} & -\frac{1}{R_1} & 1 \\ -\frac{1}{R_1} & \frac{1}{R_1}+\frac{1}{R_2} & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_E \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ E_s \end{bmatrix}$$



Rewrite in lower triangular form

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{R_1} & \frac{1}{R_1}+\frac{1}{R_2} & 0 \\ \frac{1}{R_1} & -\frac{1}{R_1} & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_E \end{bmatrix} = \begin{bmatrix} E_s \\ 0 \\ 0 \end{bmatrix} \Rightarrow \quad x = \begin{bmatrix} E_s \\ \frac{R_2}{R_1+R_2}E_s \\ -\frac{E_s}{R_1+R_2} \end{bmatrix}$$

Adjoint Analysis

The adjoint system is given by A^T:

$$A^T = \begin{bmatrix} 1 & -\frac{1}{R_1} & \frac{1}{R_1} \\ 0 & \frac{1}{R_1}+\frac{1}{R_2} & -\frac{1}{R_1} \\ 0 & 0 & 1 \end{bmatrix}$$

The output is V_2:

$$V_2 = \varphi = \underbrace{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}}_{c^T} \begin{bmatrix} V_1 \\ V_2 \\ I_E \end{bmatrix}$$

Solve $A^T x_a = -c$

$$\begin{bmatrix} 1 & -\frac{1}{R_1} & \frac{1}{R_1} \\ 0 & \frac{1}{R_1}+\frac{1}{R_2} & -\frac{1}{R_1} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_{1a} \\ V_{2a} \\ I_{Ea} \end{bmatrix} = -\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The solution is x_a (the adjoint vector):

$$x_a = \begin{bmatrix} -\frac{R_2}{R_1+R_2} \\ -\frac{R_1 R_2}{R_1+R_2} \\ 0 \end{bmatrix}$$

Sensitivity Calculation Using Adjoint Method

Suppose $p = R_2$

$$\frac{\partial A}{\partial R_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{R_2^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \frac{\partial b}{\partial R_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

RHS is $(\partial A/\partial p)x - \partial b/\partial p$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{R_2^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} E_s \\ \frac{R_2}{R_1+R_2} E_s \\ \frac{E_s}{R_1+R_2} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{E_s}{R_2(R_1+R_2)} \\ 0 \end{bmatrix}$$

Sensitivity Calculation

To determine $\partial\varphi/\partial p$, Calculate = $x_a^T[(\partial A/\partial p)x - \partial b/\partial p]$

$$\frac{\partial \varphi}{\partial R_2} = \begin{bmatrix} -\frac{R_2}{R_1+R_2} & -\frac{R_1 R_2}{R_1+R_2} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{E_s}{R_2(R_1+R_2)} \\ 0 \end{bmatrix} = \frac{R_1 E_s}{(R_1+R_2)^2}$$

Suppose we wanted $\partial\varphi/\partial E_s$, then

$$\frac{\partial A}{\partial E_s} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \frac{\partial b}{\partial E_s} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \frac{\partial A}{\partial E_s} x - \frac{\partial b}{\partial E_s} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

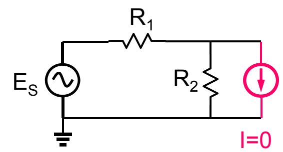
$$\frac{\partial \varphi}{\partial E_s} = \begin{bmatrix} -\frac{R_2}{R_1+R_2} & -\frac{R_1 R_2}{R_1+R_2} & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \frac{R_2}{R_1+R_2}$$

NO additional matrix solve required!

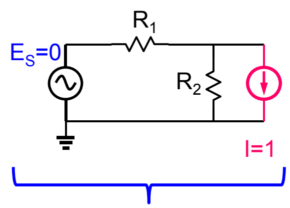
Circuit Interpretation of Adjoint Method

$$A^T x_a = -c$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{R_1} & \frac{1}{R_1}+\frac{1}{R_2} & 0 \\ \frac{1}{R_1} & -\frac{1}{R_1} & 1 \end{bmatrix} \quad b = \begin{bmatrix} E_s \\ 0 \\ 0 \end{bmatrix}$$



$$A^T = \begin{bmatrix} 1 & -\frac{1}{R_1} & \frac{1}{R_1} \\ 0 & \frac{1}{R_1}+\frac{1}{R_2} & -\frac{1}{R_1} \\ 0 & 0 & 1 \end{bmatrix} \quad c = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$



Adjoint Network

Noise Analysis

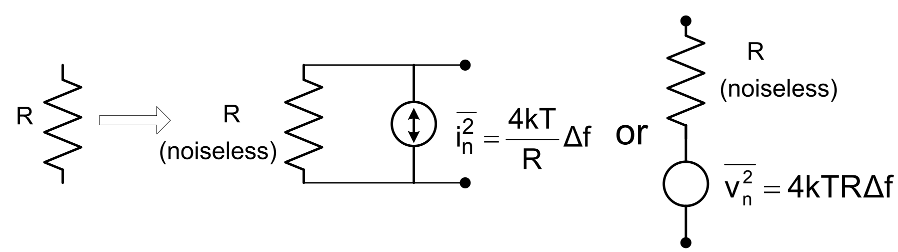
- **Semiconductor devices (Diodes, BJT, MOS, R) introduce noise in a circuit**
- **We are interested in noise phenomena caused by small current and voltage fluctuations generated within the devices**
  - These fluctuations give rise to uncertainty in current/voltage at nodes of a circuit
- **Noise is present due to the fact that charge is not continuous but is carried in discrete amounts $q=1.6\times10^{-19}$ C (quantization of charge)**

Nyquist, Johnson, Schottky were pioneers who explained the origins of noise

Thermal Noise

- **Also called Johnson or Nyquist noise**
- **Associated with random thermal motion of electrons within the physical body of a resistor**
- **Thermally agitated charge carriers in a conductor result in a randomly varying current, hence, a random voltage**
- **Thermal origin ⇒ directly related to temperature. As temperature approaches 0 absolute, thermal noise also approaches 0**

Resistor Thermal Noise



Since the noise signal has random phase, and is defined solely in terms of its mean square value, it has <u>no polarity</u>

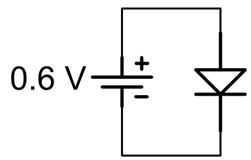
Shot Noise

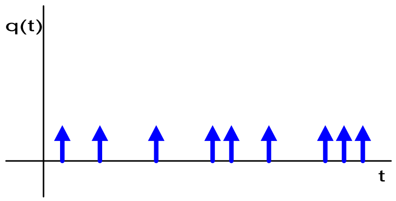
- Described and explained by Schottky in 1918
- Associated with flow of carriers across a barrier (pn junction)

Consider a diode:

Some carriers get enough energy to cross over the barrier

Since they cross the barrier randomly arrival of charge is random





$$\overline{i^2} = 2qI_d\Delta f$$

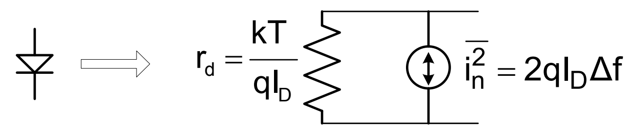
Diode: Example + Small-Signal Model

$$I_D = 1\text{mA}, \quad \Delta f = 1\text{MHz}$$

$$\overline{i_D^2} = 2q(1\text{mA})\Delta f = 2 \times 1.6 \times 10^{-19} \times 10^{-3} \times 10^{6}$$

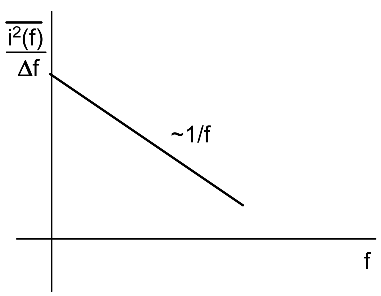
$$= 3.2 \times 10^{-16}\,\text{A}^2$$

$$\therefore \quad \text{rms value} = \sqrt{\overline{i_D^2}} = 1.7 \times 10^{-8}\,\text{A} = 17\text{nA}$$



Flicker Noise or 1/f Noise (Pink Noise)

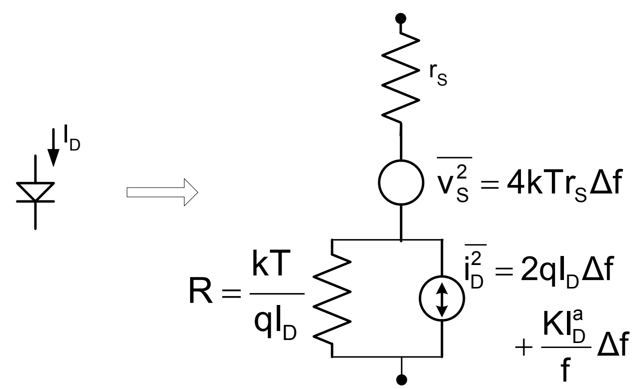
- Origins of flicker noise are varied
- Present in diodes, BJTs, and MOSFETs
- Tends to be associated with surface states
  - Surface states capture and release carriers in a random manner



$$\overline{i^2} = k_1 \frac{I^a}{f^b}\Delta f$$

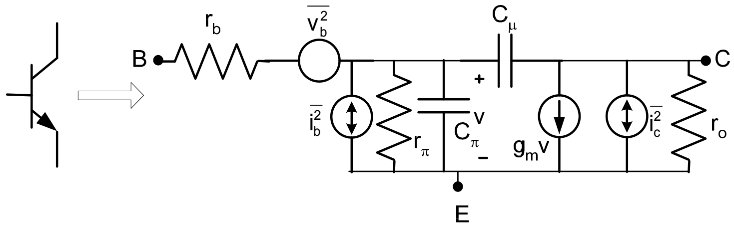
I = direct current
a = a constant 0.5 < a < 2
b = a constant ~ 1 ⇒ "1/f noise"
k_1 varies with device

Small-Signal Diode Noise Model



r_s = series ohmic resistance (a physical resistor)

Small-Signal BJT Noise Model



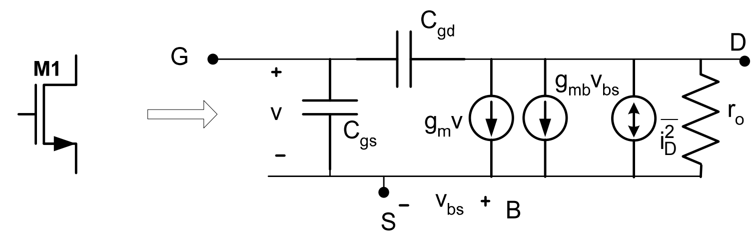
$$\overline{v_b^2} = 4kTr_b\Delta f$$

$$\overline{i_b^2} = 2qI_B\Delta f + K_1\frac{I_B^a}{f}\Delta f$$

$$\overline{i_c^2} = 2qI_C\Delta f$$

r_b = series base resistance (a physical resistor)

Small-Signal MOSFET Noise Model



$$\overline{i_D^2} = 4kT\gamma g_{d0}\Delta f + \frac{K}{f}\frac{g_m^2}{WLC_{ox}^2}\Delta f$$

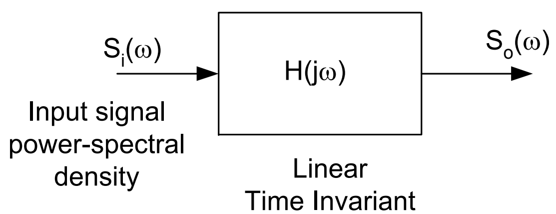
where $g_{d0} = g_{ds}\big|_{V_{ds}=0}$

$\gamma = 1$ at zero V_{DS} and 2/3 in saturation for long channel devices

$$g_{ds}\big|_{V_{ds}=0} = g_{m,sat}$$

$$4kT\gamma g_{d0}\Delta f = 4kT\left(\frac{2}{3}\right)g_m$$

Circuit Noise Calculations



$$S_o(\omega) = S_i(\omega)\left[H(j\omega)H^*(j\omega)\right]$$

$$= S_i(\omega)\left|H(j\omega)\right|^2$$

$$\therefore \sqrt{S_o(\omega)} = \sqrt{S_i(\omega)}\left|H(j\omega)\right|$$

Analysis Procedure

Assume noise sources are uncorrelated (separate, independent physical mechanisms)

For each source $\overline{i_n^2}$ $(\overline{v_n^2})$:

1)Replace with a sinusoidal, deterministic source of value i_n (v_n)

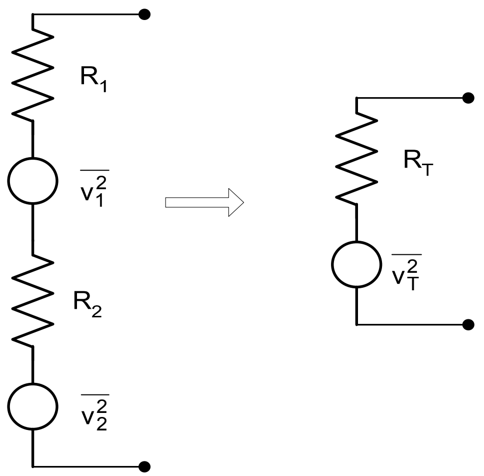
$$\sqrt{\overline{i_n^2}} \quad \left(\sqrt{\overline{v_n^2}}\right)$$

2)Calculate $v_{on}(\omega) = i_n(\omega)\left|H(j\omega)\right|$

$\frac{\overline{v_{on}^2}}{\Delta f} =$ output voltage spectral density due to i_n^2

$$\overline{v_{o_{total}}^2} = \overline{v_{on1}^2} + \overline{v_{on2}^2} + \ldots$$

Resistor Example



$$\overline{v_T^2} = \overline{v_1^2} + \overline{v_2^2} = 4kT(R_1 + R_2)\Delta f$$

BJT Example (Ignore C_μ and r_o)

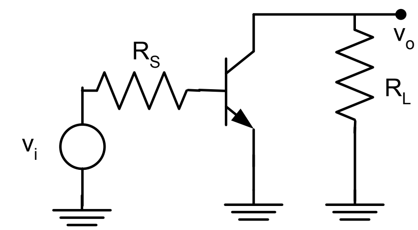
$$\overline{v_S^2} = 4kTR_S\Delta f$$

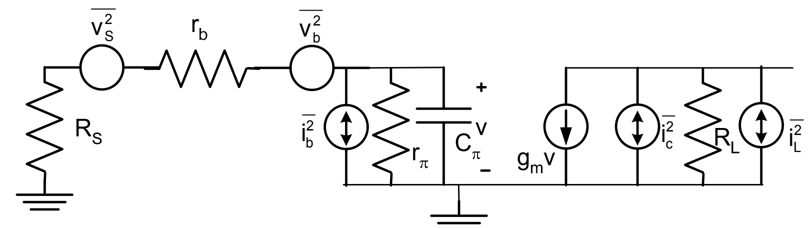
$$\overline{v_b^2} = 4kTr_b\Delta f$$

$$\overline{i_b^2} = 2qI_B\Delta f \text{ (ignore 1/f noise)}$$

$$\overline{i_c^2} = 2qI_C\Delta f$$

$$\overline{i_L^2} = 4kT\frac{1}{R_L}\Delta f$$

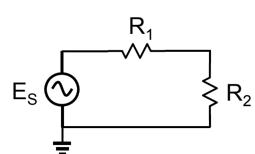




Calculate output noise from each noise source
⇒ Same circuit with different source vector (RHS)

Example of Noise Analysis

$$\begin{bmatrix} \frac{1}{R_1} & -\frac{1}{R_1} & 1 \\ -\frac{1}{R_1} & \frac{1}{R_1}+\frac{1}{R_2} & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_E \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ E_s \end{bmatrix}$$



Rewrite in lower triangular form

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{R_1} & \frac{1}{R_1}+\frac{1}{R_2} & 0 \\ \frac{1}{R_1} & -\frac{1}{R_1} & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_E \end{bmatrix} = \begin{bmatrix} E_s \\ 0 \\ 0 \end{bmatrix}$$

Adjoint System

The adjoint system is given by A^T:

$$A^T = \begin{bmatrix} 1 & -\frac{1}{R_1} & \frac{1}{R_1} \\ 0 & \frac{1}{R_1}+\frac{1}{R_2} & -\frac{1}{R_1} \\ 0 & 0 & 1 \end{bmatrix}$$

The output is V_2:

$$V_2 = \varphi = \underbrace{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}}_{C^T} \begin{bmatrix} V_1 \\ V_2 \\ I_E \end{bmatrix}$$

Solve $A^T x_a = -c$

$$\begin{bmatrix} 1 & -\frac{1}{R_1} & \frac{1}{R_1} \\ 0 & \frac{1}{R_1}+\frac{1}{R_2} & -\frac{1}{R_1} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_{1a} \\ V_{2a} \\ I_{Ea} \end{bmatrix} = -\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The solution is x_a (the adjoint vector):

$$x_a = \begin{bmatrix} -\frac{R_2}{R_1+R_2} \\ -\frac{R_1R_2}{R_1+R_2} \\ 0 \end{bmatrix}$$

Compute Noise From Each Noise Source

$\varphi=-(x_a)^T S_i$

Output noise due to R_1

$$v_{o,R_1} = -\begin{bmatrix} -\frac{R_2}{R_1+R_2} & -\frac{R_1R_2}{R_1+R_2} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -i_{R1} \\ i_{R1} \end{bmatrix} = -\frac{R_1R_2}{R_1+R_2} i_{R1}$$

Output noise due to R_2

$$v_{o,R_2} = -\begin{bmatrix} -\frac{R_2}{R_1+R_2} & -\frac{R_1R_2}{R_1+R_2} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ i_{R2} \\ 0 \end{bmatrix} = \frac{R_1R_2}{R_1+R_2} i_{R2}$$

Total output noise

$$\overline{v_o^2} = \left(\frac{R_1R_2}{R_1+R_2}\right)^2 \left(\overline{i_{R1}^2} + \overline{i_{R2}^2}\right)$$