Test (11/21/16)

Total # Pages 7
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Name SOLUTION

1. (20 points) _______
2. (30 points) _______
3. (20 points) _______
4. (30 points) _______

Total (100 points) _______

GOOD LUCK
1. The two-port shown below is defined in terms of $ABCD$ or transmission parameters.

The $ABCD$-parameters are defined as (note the direction of $I_2$)
\[ V_{p1} = AV_{p2} + BI_2 \]
\[ I_1 = CV_{p2} + DI_2 \]

Write the smallest MNA stamp (matrix and RHS) for this element. (20 points)

For nodal eqns current leaving node is positive
\[ \therefore I_A = -I_2 \]
\[ \implies V_{p1} = V_1 - V_2 = A(V_3 - V_4) - B I_A \]
\[ I_1 = C(V_3 - V_4) - D I_A \]

For the smallest MNA stamp use $I_A$ as the unknown current

\[
\begin{bmatrix}
V_1 & V_2 & V_3 & V_4 & I_A \\
KCL_@1 & & & & & 0 \\
KCL_@2 & & & & & 0 \\
KCL_@3 & & & & & 0 \\
KCL_@4 & & & & & 0 \\
BCR & & & & & 0 \\
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]
2. For the circuit shown below answer the following questions:

\[ I_s(t) \]

\[ G_2 \]

\[ v_1 \]

\[ v_2 \]

\[ q = 3v_1 + v_2^2 \]

\[ I = v_1^2 \]

a) Write the equations that describe the dynamic behavior of the circuit. Take \( q \), the capacitor branch charge, as an unknown. (5 points)

\[
\begin{align*}
    v_1^2 + G_2(v_1 - v_2) &= I_s(t) \\
    G_2(v_2 - v_1) + \frac{dq}{dt} &= 0 \\
    -q + 3v_1 + v_2^2 &= 0
\end{align*}
\]

b) Draw the companion model for the complete circuit at timepoint \( t_n \). Assume an integration method described by \( \dot{x}_n = \alpha x_n + \beta \) is used. (5 points)

\[
\begin{align*}
    \dot{x}_n &= \alpha x_n + \beta \\
    \dot{q}_n &= \alpha q_n + \beta
\end{align*}
\]

\[ I_s(t_n) \]

\[ G_2 \]

\[ v_1_n \]

\[ v_2_n \]

\[ \alpha q_n \]

\[ \beta \]

\[ \dot{i}_n = v_{1n} \]

\[ q_n = 3v_{1n} + v_{2n}^2 \]
c) Draw the companion model for the complete circuit at timepoint $t_n$ and iteration $k+1$ when Newton's method is used to solve the nonlinear equations. (10 points)

\[ G_1^k = 2V_1^k \]
\[ I_1^k = i_n(V_1^k) - (2V_1^k) V_1^k \]
\[ I_2^k = \alpha q(V_1^k) - 3\alpha V_1^k - (2\alpha V_2^k) V_2^k \]

\[
\begin{bmatrix}
V_1 & V_2 & q \\
KCL @ 1 & 2V_1^k + G_2 & -G_2 & 0 \\
KCL @ 2 & -G_2 & + G_2 & \alpha \\
BCR - q & 3 & 2V_2^k & -1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
V_1 & V_2 \\
estyuk & 2V_1^k + G_2 & -G_2 \\
1 & -G_2 + 3\alpha & + G_2 + 2\alpha V_2^k \\
2 & \\
\end{bmatrix}
\]
3. The $\Gamma_\sigma$ contour for a $7^{th}$ order integration method is shown below. In each box fill in the number of roots outside the unit circle ($|z| = 1$) and shade the region of absolute stability. (20 points)
4. A linear multistep integration method is given by

\[ x_n = x_{n-1} + \alpha h \dot{x}_n + \beta h \ddot{x}_{n-1} \]

\[ 0 \leq \beta \leq \frac{1}{2} \]

a) What is the relation between \( \beta \) and \( \alpha \) for the given method to be a first-order integration method? (5 points)

\[ x_n - x_{n-1} - \alpha h \dot{x}_n - \beta h \ddot{x}_{n-1} = 0 \]

\[ \Rightarrow \alpha_0 = 1, \quad \alpha_1 = -1, \quad \beta_0 = -\alpha, \quad \beta_1 = -\beta \]

Exactness constraints:

\[ \alpha_0 + \alpha_1 = 1 - 1 = 0 \quad \text{satisfied} \]

\[ -\alpha_1 + \beta_0 + \beta_1 = 0 = 1 - \alpha - \beta = 0 \quad \text{or} \quad \alpha + \beta = 1 \]

\[ \alpha_1 + 2\beta_1(-1) \neq 0 \Rightarrow -1 + 2\beta \neq 0 \Rightarrow \beta \neq \frac{1}{2} \]

\[ \therefore \text{condition is} \quad \alpha = 1 - \beta, \quad \beta \neq \frac{1}{2} \]

or \[ 0 \leq \beta < \frac{1}{2} \]

b) What is the relation between \( \beta \) and \( \alpha \) for the given method to be a second-order integration method? (5 points)

For second order, require

\[ \alpha_1 + 2\beta_1(-1) = 0 \Rightarrow -1 + 2\beta = 0 \quad \text{or} \quad \beta = \frac{1}{2} \]

Check:

\[ -\alpha_1 + 3\beta_1 = 1 - 3\beta \neq 0 \]

\[ \therefore \text{second order for} \quad \beta = \frac{1}{2} \Rightarrow \alpha = \frac{1}{2} \]

i.e. TR method
c) For the condition of part (a) what is the local error. (10 points)

First-order method \( \alpha = 1 - \beta, \ \beta \neq \frac{1}{2} \)

\[
LE = \frac{c_2}{2} h^2 \dddot{x}(t_n)
\]

where \( c_2 = \alpha_1 - 2 \beta_1 \)

Since \( \sum_{i=0}^{1} \beta_i = \beta_0 + \beta_1 = -\alpha - \beta = -1 \)

\[= -1 + 2\beta \]

i.e. already normalized

\[
\therefore LE = \frac{-1 + 2\beta}{2} h^2 \dddot{x}(t_n)
\]

or \[-\frac{(1 - 2\beta)}{2} h^2 \dddot{x}(t_n) \text{ since } \beta < \frac{1}{2}\]

d) For the condition of part (a) and \( \beta = 1/4 \), write the equation for the region of absolute stability for the given method. You don't need to solve the equation or draw a plot. (10 points)

\[
\beta = \frac{1}{4} \Rightarrow \alpha = 1 - \beta = \frac{3}{4}
\]

\[
\Rightarrow x_n - x_{n-1} - \frac{3}{4} h \ddot{x}_n - \frac{1}{4} h \dddot{x}_{n-1} = 0
\]

\[
x' = \lambda x
\]

\[
\Rightarrow x_n - x_{n-1} - \frac{3}{4} h \lambda x_n - \frac{1}{4} h \lambda x_{n-1} = 0
\]

\[
\left(1 - \frac{3}{4} \sigma\right) x_n - \left(1 + \frac{1}{4} \sigma\right) x_{n-1} = 0 \quad \sigma = \lambda h
\]

\[
\text{or} \quad z = \frac{1 + \frac{1}{4} \sigma}{1 - \frac{3}{4} \sigma} = \frac{4 + \sigma}{4 - 3\sigma}
\]

require \( |z| < 1 \Rightarrow \left| \frac{4 + \sigma}{4 - 3\sigma} \right| < 1 \)

(singl e root)