## Properties of Laplace Transform and Transform Pairs (Defined for \( t \geq 0; f(t) = 0 \) for \( t < 0 \))

<table>
<thead>
<tr>
<th>Property</th>
<th>( f(t) )</th>
<th>( F(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiplication by a constant</td>
<td>( Kf(t) )</td>
<td>( KF(s) )</td>
</tr>
<tr>
<td>Addition/subtraction</td>
<td>( f_1(t) + f_2(t) - f_3(t) + \ldots )</td>
<td>( F_1(s) + F_2(s) - F_3(s) + \ldots )</td>
</tr>
<tr>
<td>First derivative (time)</td>
<td>( \frac{df(t)}{dt} )</td>
<td>( sF(s) - f(0^-) )</td>
</tr>
<tr>
<td>Second derivative (time)</td>
<td>( \frac{d^2 f(t)}{dt^2} )</td>
<td>( s^2 F(s) - sf(0^-) - \frac{df(0^-)}{dt} )</td>
</tr>
<tr>
<td>( n )th derivative (time)</td>
<td>( \frac{d^n f(t)}{dt^n} )</td>
<td>( s^n F(s) - s^{n-1} f(0^-) - s^{n-2} \frac{df(0^-)}{dt} - \ldots - s d^n f(0^-) )</td>
</tr>
<tr>
<td>Time integral</td>
<td>( \int_0^t f(\tau) d\tau )</td>
<td>( \frac{F(s)}{s} )</td>
</tr>
<tr>
<td>Time shift</td>
<td>( f(t - a)u(t - a), a &gt; 0 )</td>
<td>( e^{-as}F(s) )</td>
</tr>
<tr>
<td>Frequency shift</td>
<td>( e^{-at}f(t) )</td>
<td>( F(s + a) )</td>
</tr>
<tr>
<td>Scaling</td>
<td>( f(at), a &gt; 0 )</td>
<td>( \frac{1}{a} F\left(\frac{s}{a}\right) )</td>
</tr>
<tr>
<td>First derivative (s)</td>
<td>( tf(t) )</td>
<td>( -\frac{dF(s)}{ds} )</td>
</tr>
<tr>
<td>( n )th derivative (s)</td>
<td>( t^n f(t) )</td>
<td>( (-1)^n \frac{d^n F(s)}{ds^n} )</td>
</tr>
<tr>
<td>( s ) integral</td>
<td>( f(t) )</td>
<td>( \int_0^\infty F(u) du )</td>
</tr>
</tbody>
</table>

### Type of Function | \( f(t) \) (\( t > 0 \)) | \( F(s) \) |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(impulse)</td>
<td>( \delta(t) )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>(step)</td>
<td>( u(t) )</td>
<td>( \frac{1}{s} )</td>
</tr>
<tr>
<td>(ramp)</td>
<td>( t )</td>
<td>( \frac{1}{s^2} )</td>
</tr>
<tr>
<td>(exponential)</td>
<td>( e^{-at} )</td>
<td>( \frac{1}{s + a} )</td>
</tr>
<tr>
<td>(sine)</td>
<td>( \sin \omega t )</td>
<td>( \frac{\omega}{s^2 + \omega^2} )</td>
</tr>
<tr>
<td>(cosine)</td>
<td>( \cos \omega t )</td>
<td>( \frac{s}{s^2 + \omega^2} )</td>
</tr>
<tr>
<td>(damped ramp)</td>
<td>( te^{-at} )</td>
<td>( \frac{1}{(s + a)^2} )</td>
</tr>
<tr>
<td>(damped sine)</td>
<td>( e^{-at} \sin \omega t )</td>
<td>( \frac{\omega}{(s + a)^2 + \omega^2} )</td>
</tr>
<tr>
<td>(damped cosine)</td>
<td>( e^{-at} \cos \omega t )</td>
<td>( \frac{s + a}{(s + a)^2 + \omega^2} )</td>
</tr>
</tbody>
</table>
Laplace transform

The step function \( K u(t) \) is defined as

\[
K u(t) = \begin{cases} 
0, & t < 0, \\
K, & t > 0. 
\end{cases}
\]

If \( K \) is 1, the function is the unit step.

The impulse function is defined as

\[
\int_{-\infty}^{\infty} K \delta(t)\,dt = K \\
\delta(t) = 0, \quad t \neq 0.
\]

If \( K \) is 1, the function is the unit impulse.

Properties of the impulse function

- If \( f(t) \) is continuous at \( t=a \) then \( f(t)\delta(t-a) = f(a)\delta(t-a) \)

\[
\int_{-\infty}^{\infty} f(t)\delta(t-a)\,dt = f(a)
\]

- Sampling or sifting property

The ramp function is defined as

\[
r(t) = tu(t) = \int_{0}^{t} u(t)\,dt
\]

Partial Fraction Expansion

If \( F(s) \) is a proper rational function \( (n > m) \)

\[
F(s) = K \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}
\]

Distinct poles: \( F(s) \) can be expressed as

\[
F(s) = \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \cdots + \frac{k_n}{s-p_n}
\]

where \( k_i (i=1, ..., n) \) is given by \( k_i = (s-p_i)F(s)\bigg|_{s=p_i} \) (Cover-up algorithm)

Repeated poles: (pole \( p_i \) of multiplicity \( r \)) the terms corresponding to the repeated pole are

\[
F(s) = \cdots + \frac{k_0}{(s-p_1)^r} + \frac{k_1}{(s-p_1)^{r-1}} + \cdots + \frac{k_{r-1}}{(s-p_1)}
\]

where \( k_0 \) and \( k_l (l=1, ..., r-1) \) are given by

\[
k_0 = (s-p_1)^r F(s)\bigg|_{s=p_i},
\]

\[
k_l = \frac{1}{l!} \frac{d^l}{ds^l} (s-p_1)^r F(s)\bigg|_{s=p_i}
\]
Quotient-rule for derivatives:

\[
\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}
\]

Cramer’s Rule for solving equations:
To solving a set of \( n \)-equations given in matrix form as \( Ax = d \), i.e.,
\[
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
\vdots \\
x_n
\end{bmatrix} =
\begin{bmatrix}
d_1 \\
\vdots \\
d_n
\end{bmatrix}
\]

Let the determinant of \( A \) be denoted by \( D \) (\( D = \text{determinant (A)} \))
\[
D = \begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix}
\]

To solve for \( x_k \), compute
\[
D_k = \begin{vmatrix}
a_{11} & \cdots & a_{1(k-1)} & d_1 & a_{1(k+1)} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a_{n1} & \cdots & a_{n(k-1)} & d_n & a_{n(k+1)} & \cdots & a_{nn}
\end{vmatrix}
\]

This is the determinant of the original matrix with the \( k \)-th column replaced by the \( d \) vector. Then solve for \( x_k = \frac{D_k}{D} \). The process can be repeated for \( 1 \leq k \leq n \) to solve for all the unknowns.

Useful Inverse Transform Pairs:

<table>
<thead>
<tr>
<th>Type of Roots</th>
<th>( F(s) )</th>
<th>( f(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distinct real</td>
<td>( \frac{K}{s + a} )</td>
<td>( Ke^{-at}u(t) )</td>
</tr>
<tr>
<td>Repeated real</td>
<td>( \frac{K}{(s + a)^2} )</td>
<td>( Kte^{-at}u(t) )</td>
</tr>
<tr>
<td>Distinct complex</td>
<td>( \frac{K}{s + \alpha - j\beta} + \frac{K^*}{s + \alpha + j\beta} )</td>
<td>( 2</td>
</tr>
<tr>
<td>Repeated complex</td>
<td>( \frac{K}{(s + \alpha - j\beta)^2} + \frac{K^*}{(s + \alpha + j\beta)^2} )</td>
<td>( 2t</td>
</tr>
</tbody>
</table>

Initial value theorem
\[
\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)
\]

Final value theorem
\[
\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)
\]

The initial-value theorem is based on the assumption that \( f(t) \) contains no impulse functions. The final value theorem is valid only if the poles of \( sF(s) \) lie in the left half of the \( s \) plane.
Circuit Element Current Voltage Relationships in Time- and s-Domains

<table>
<thead>
<tr>
<th>Time-Domain</th>
<th>s-Domain</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Resistor</strong></td>
<td>![Resistor Diagram]</td>
</tr>
<tr>
<td>![Resistor Symbol] $v = Ri$</td>
<td>![Resistor Symbol] $V = RI$</td>
</tr>
<tr>
<td><strong>Inductor</strong></td>
<td>![Inductor Diagram]</td>
</tr>
<tr>
<td>![Inductor Symbol] $v_L = L \frac{di_L}{dt}$</td>
<td>![Inductor Symbol] $V_L = sL I_L - L i_L(0^-)$</td>
</tr>
<tr>
<td>$i_L = \frac{1}{L} \int_{0^-}^{t} v_L dt + i_L(0^-)$</td>
<td>$I_L = \frac{V_L}{sL} + \frac{i_L(0^-)}{s}$</td>
</tr>
<tr>
<td><strong>Capacitor</strong></td>
<td>![Capacitor Diagram]</td>
</tr>
<tr>
<td>![Capacitor Symbol] $v_C = \frac{1}{C} \int_{0^-}^{t} i_C dt + v_C(0^-)$</td>
<td>![Capacitor Symbol] $V_C = \frac{I_C}{sC} + \frac{v_C(0^-)}{s}$</td>
</tr>
<tr>
<td>$i_C = C \frac{dv_C}{dt}$</td>
<td>$I_C = sC V_C - C v_C(0^-)$</td>
</tr>
</tbody>
</table>

**Ohm’s law in the s-domain $V = ZI$**

$Z$ is the s-domain impedance of the element. A resistor has an impedance of $R$ ohms, an inductor has an impedance of $sL$ ohms, and a capacitor has an impedance of $1/sC$ ohms.

The admittance is the reciprocal of the impedance $Y = 1/Z$

**Solution of quadratic equation**

For a quadratic equation $ax^2 + bx + c = 0$

The solution is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
Transfer function $H(s)$ is the ratio of the output ($Y(s)$) to the input ($X(s)$) for zero initial conditions.

$$H(s) = \frac{Y(s)}{X(s)}$$

or $Y(s) = H(s)X(s)$

**Impulse response** is the output response for a unit impulse ($\delta(t)$) as the input.

$$X(s) = 1$$

$$.\ Y(s) = H(s) \implies H(s)$$ is the impulse response transform (s-domain)

$$h(t) = L^{-1}[H(s)]$$ is the impulse response waveform (t-domain)

**Step response** is the output response for a unit step ($u(t)$) as the input.

$$X(s) = \frac{1}{s}$$

$$.\ Y(s) = G(s) = H(s)\frac{1}{s}$$ where $G(s)$ is the step response transform (s-domain)

$$g(t) = L^{-1}[G(s)]$$ is the step response waveform (t-domain)

**Relation between Step and Impulse responses**

$$G(s) = \frac{H(s)}{s}$$ (s-domain) $$g(t) = \int_{0}^{t} h(\tau)d\tau$$ (t-domain)

**Stable circuit**

A linear circuit is stable when all the poles of $H(s)$ are in the left-half of the s-plane. Alternatively, the circuit is stable when $h(t)$ decays to zero as $t \to \infty$.

**DC steady-state response** for a stable circuit with transfer function $H(s)$

For $x(t) = u(t)$, $y(t) = g(t)$

$$y(t \to \infty) = y(\infty) = g(\infty) = H(s)\big|_{s=0} = H(0)$$

**Sinusoidal steady-state response** for a stable circuit with transfer function $H(s)$

For $x(t) = A\cos(\omega t)$

$$y_{ss}(t) = A|H(j\omega)|\cos(\omega t + \angle H(j\omega))$$

**Convolution integral**

$$Y(s) = H(s)X(s) \implies y(t) = h(t) * x(t) = \int_{0}^{t} h(t - \tau)x(\tau)d\tau$$

**Graphical Convolucion**

1. Reflect $h(\tau)$ across the $\tau=0$ (vertical) axis to obtain $h(-\tau)$.
2. Shift $h(t-\tau)$ is $h(-\tau)$ shifted to the right by $t$ seconds.
3. Multiply $h(t-\tau)$ and $x(\tau)$, i.e., $h(t-\tau)x(\tau)$
4. Integrate $\int_{0}^{t} h(t - \tau)x(\tau)d\tau$
Bode Plots

Given $H(j\omega) = |H(j\omega)|\angle H(j\omega)$ the plot of $|H(j\omega)|$ in dBs as a function of $\omega$ in log scale is the Bode plot for magnitude (gain); the plot of $\angle H(j\omega)$ as a function of $\omega$ in log scale is the Bode plot for phase.

**dBs (decibels)**

$$|H(j\omega)|_{\text{dB}} = 20\log_{10}|H(j\omega)|$$

<table>
<thead>
<tr>
<th>Number</th>
<th>$10^{-3}$</th>
<th>$10^{-2}$</th>
<th>$10^{-1}$</th>
<th>1</th>
<th>10</th>
<th>$10^{1}$</th>
<th>$10^{2}$</th>
<th>$10^{3}$</th>
<th>2</th>
<th>$1/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>log$_{10}$</td>
<td>-3</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>0.3</td>
<td>-0.3</td>
<td></td>
</tr>
<tr>
<td>dBs</td>
<td>-60</td>
<td>-40</td>
<td>-20</td>
<td>0</td>
<td>20</td>
<td>40</td>
<td>60</td>
<td>6</td>
<td>-6</td>
<td></td>
</tr>
</tbody>
</table>

### Bode Plot Summary

Summary of Bode straight-line magnitude and phase plots.

<table>
<thead>
<tr>
<th>Factor</th>
<th>Magnitude</th>
<th>Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K$</td>
<td>$20\log_{10}K$</td>
<td>$0^\circ$</td>
</tr>
<tr>
<td>$(j\omega)^N$</td>
<td>$20N$ dB/decade</td>
<td>$90N^\circ$</td>
</tr>
<tr>
<td>$\frac{1}{(j\omega)^N}$</td>
<td>$-20N$ dB/decade</td>
<td>$-90N^\circ$</td>
</tr>
<tr>
<td>$(1 + \frac{j\omega}{z})^N$</td>
<td>$20N$ dB/decade</td>
<td>$0^\circ$, $90N^\circ$</td>
</tr>
<tr>
<td>$\frac{1}{(1 + j\omega/p)^N}$</td>
<td>$-20N$ dB/decade</td>
<td>$-90N^\circ$</td>
</tr>
</tbody>
</table>