Transfer Functions

\[ Y(s) = H(s) \cdot X(s) \]

\[ = \frac{N(s)}{(s-p_1)(s-p_2) \cdots (s-p_N)} \quad \frac{N_2(s)}{(s-p_{N+1})(s-p_{N+2}) \cdots (s-p_{N+E})} \]

Partial Fraction Expansion

\[ Y(s) = \sum_{i=1}^{N} \frac{K_i}{(s-p_i)} + \sum_{j=1}^{E} \frac{K_j}{(s-p_{N+i})} \]

\[ y(t) = \sum_{i=1}^{N} k_i e^{p_i t} u(t) + \sum_{j=1}^{E} k_j e^{p_{N+i} t} u(t) \]

Stability of a circuit

A linear circuit is stable if its natural response decays to zero as \( t \to \infty \)
All natural poles are in the left half s-plane

\[ H_1(s) = \frac{1}{s+1} \quad \text{Stable circuit} \]

\[ H_2(s) = \frac{1}{s-2} \quad \text{Unstable circuit} \]

\[ \mathcal{L}^{-1} \left[ H_1(s) \right] = e^{-t} u(t) \]

\[ \mathcal{L}^{-1} \left[ H_2(s) \right] = 2^t u(t) \]

### Impulse Response

Zero state output response of a circuit to a unit impulse applied at the input at \( t = 0 \) \((s(t))\)

\[ x(s) = 1 \]

\[ y(s) = H(s) x(s) = H(s) \]

\[ y(t) = \mathcal{L}^{-1} \left[ y(s) \right] = \mathcal{L}^{-1} \left[ H(s) \right] = h(t) \]

\[ y(s) = \frac{K_1}{s-p_1} + \frac{K_2}{s-p_2} + \ldots + \frac{K_n}{s-p_n} \]

\[ h(t) = \left[ k_1 e^{p_1 t} + k_2 e^{p_2 t} + \ldots + k_n e^{p_n t} \right] u(t) \]

For a stable circuit \( h(t) \rightarrow 0 \) as \( t \rightarrow \infty \)

What if \( h(t) \) is unavailable as measured data

\[ y(s) = H(s) X(s) \]

\[ y(t) = h(t) \ast x(t) \]

Convolution
Step Response

Zero-state (zero initial condition) output response of a circuit when the input is a unit step function applied at time \( t = 0 \) \((u(t))\):

\[
\gamma(s) = H(s)X(s) = H(s) \frac{1}{s} = G(s)
\]

\[
= \frac{N(s)}{(s-p_1)(s-p_2)\cdots(s-p_N)} \frac{1}{s}
\]

\[
= \frac{K_1}{s-p_1} + \frac{K_2}{s-p_2} + \cdots + \frac{K_N}{s-p_N} + \frac{K_0}{s}
\]

\[
g(t) = \mathcal{L}^{-1}[G(s)] = \left[ k_1 e^{p_1 t} + k_2 e^{p_2 t} + \cdots + k_N e^{p_N t} \right] u(t) + K_0 u(t)
\]

For a stable circuit, the natural response \( \rightarrow 0 \) as \( t \rightarrow \infty \):

\[
\lim_{t \rightarrow \infty} g(t) = K_0 \quad \text{dc steady state value}
\]

\[
\lim_{t \rightarrow \infty} g(t) = \lim_{s \rightarrow 0} sG(s) \quad \text{Final value theorem}
\]

\[
= \lim_{s \rightarrow 0} \frac{H(s)}{s} \frac{1}{s} = H(0)
\]

Application Example

The step response is useful for specifying the performance of both analog and digital systems.
$T_d = \text{delay time}$

$T_r = \text{rise time}$

Clock Distribution

Network in a large digital integrated circuit

CLK is generated at one location and distributed on chip to other blocks

Clock skew = $T_{db} - T_{da}$

Goal of a clock distribution network is to minimize the clock skew

Observation:

$G(s) = \frac{H(s)}{s}$

$q(t)$ is the integral of $h(t)$

$G(s) = \mathcal{L}\left[\int_0^t h(t) \, dt\right] = \frac{H(s)}{s}$

Step response is the integral of the impulse response.

Sinusoidal Steady-state response

The input is a Sinusoid and we are interested in the response as $t \to \infty$
\[ x(t) = A \cos \omega t \quad u(t) \]
\[ X(s) = A \frac{s}{s^2 + \omega^2} \]
\[ Y(s) = H(s) \cdot X(s) \]
\[ = \frac{N_i(s)}{(s-p_1)(s-p_2)\ldots(s-p_n)} + \frac{As}{s^2 + \omega^2} \]
\[ = \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \ldots + \frac{k_n}{s-p_n} + \frac{K}{s-j\omega} + \frac{K^*}{s+j\omega} \]
\[ Y(t) = \left[ k_1 e^{pt} + k_2 e^{pt_2} + \ldots + k_ne^{pt_n} \right] u(t) + 2|K| \cos (\omega t + \angle K) u(t) \]
\[ \ldots + \frac{k}{s+x-j\beta} + \frac{k^*}{s+x+j\beta} + \ldots \rightarrow 2|K| e^{\beta t} \cos (\beta t + \angle K) u(t) \]

For a stable circuit, natural response \( \rightarrow 0 \) as \( t \rightarrow \infty \)
\[ \lim_{t \to \infty} Y(t) = 2|K| \cos (\omega t + \angle K) = Y_{ss}(t) \]

To obtain \( K \) use the cover-up algorithm
\[ K = H(s) \frac{As}{(s-j\omega)(s+j\omega)} \bigg|_{s=j\omega} \]
\[ = H(s) \bigg|_{s=j\omega} \frac{A \cdot j\omega}{j\omega + j\omega} = \frac{A}{2} H(j\omega) \]
\[ |K| = \frac{A}{2} |H(j\omega)| \quad \angle K = \angle H(j\omega) \]
\[ Y_{ss}(t) = \frac{2}{A} \frac{A}{2} |H(j\omega)| \cos (\omega t + \angle H(j\omega)) \]