1) Laplace Transforms
2) Inverse Laplace Transforms
3) Applications to Circuits

The impulse function \( \delta(t) \) is an idealized representation of a very short duration, very high amplitude pulse. \( K \delta(t) \) is an impulse of strength \( K \). The area under the pulse:

\[
\delta(t) = 0 \quad \text{for} \quad t \neq 0 \quad ; \quad \int_{-\infty}^{\infty} \delta(t) \, dt = 1
\]

If \( f(t) \) is continuous at \( t = 0 \)

\[
f(t) \delta(t) = f(0) \delta(t)
\]

\[
f(t) \delta(t-3) = f(3) \delta(t-3)
\]

provided \( f(t) \) is continuous at \( t = 3 \)

Basic Laplace Transforms

\[
\mathcal{L} [ \delta(t) ] = 1
\]

\[
\mathcal{L} [ u(t) ] = \frac{1}{s}
\]

\[
\mathcal{L} [ r(t) ] = \mathcal{L} [ tu(t) ] = \frac{1}{s^2}
\]

\[
\mathcal{L} [ \sin \omega t \, u(t) ] = \frac{\omega}{s^2 + \omega^2}
\]

\[
\mathcal{L} [ \cos \omega t \, u(t) ] = \frac{s}{s^2 + \omega^2}
\]

\[
\mathcal{L} [ e^{-\alpha t} \, u(t) ] = \frac{1}{s + \alpha}
\]
Transform Properties

1) Linearity
\[ \mathcal{L} \left[ \int_0^t f(t) \, dt \right] = \frac{F(s)}{s} \]

2) Time integration
\[ \mathcal{L} \left[ \int_0^t f(t) \, dt \right] = \frac{F(s)}{s} \]

3) Time differentiation
\[ \mathcal{L} \left[ \frac{df(t)}{dt} \right] = sF(s) - f(0) \]

4) \text{s-domain translation}
\[ \mathcal{L} \left[ e^{-\alpha t} f(t) \right] = F(s+\alpha) \]

5) \text{time-domain translation}
\[ a > 0 \quad \mathcal{L} \left[ f(t-a)u(t-a) \right] = e^{-as} F(s) \]

6) Differentiation in the s-domain
\[ \mathcal{L} \left[ tf(t) \right] = -\frac{dF(s)}{ds} \]

Inverse Laplace Transforms

\[ F(s) = \frac{N(s)}{D(s)} = \frac{(s-z_1)(s-z_2) \ldots (s-z_m)}{(s-p_1)(s-p_2) \ldots (s-p_n)} \]

Pole-zero diagram

Partial Fraction Expansion

Proper rational functions: \# of zeros < \# of poles m < n

1. Real, distinct (no repeated) poles (simple poles)
\[ F(s) = \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \ldots + \frac{k_n}{s-p_n} \]

\( k_1, k_2, \ldots, k_n \) are determined by the cover-up algorithm
\[ F(s) = \frac{N(s)}{(s-p_1)(s-p_2) \ldots (s-p_j)(\ldots)(s-p_n)} \]

\[ k_j = \frac{N(s)}{(s-p_1)(s-p_2) \ldots (s-p_j)(\ldots)(s-p_n)} \bigg|_{s=p_j} \]

\[ f(t) = k_1 e^{p_1 t} u(t) + k_2 e^{p_2 t} u(t) + \ldots + k_n e^{p_n t} u(t) \]

2. Real repeated poles
- derivatives w.r.t. \( s \)

\[ F(s) = \frac{N(s)}{s(s-p_1)^2} \]

\[ = \frac{K_1}{s} + \frac{K_{21}}{s-p_1} + \frac{K_{22}}{(s-p_1)^2} \]

\( K_1, K_{21}, K_{22} \) determined from the Cover up algorithm

3. Distinct complex poles

\[ \text{PFE} \rightarrow \ldots + \frac{k}{(s+\alpha-j\beta)} + \frac{k^*}{(s+\alpha+j\beta)} + \ldots \]

determine \( k \) using Cover up algorithm = \( |k|/k \)

\[ f(t) = \ldots + 2 |k| e^{-\alpha t} \cos (\beta t + \angle k) u(t) \]

\[ L^{-1} \left[ \frac{1}{(s+\alpha)^2 + \beta^2} \right] = L^{-1} \left[ \frac{1}{\beta} \frac{\beta}{(s+\alpha)^2 + \beta^2} \right] \]

\[ \alpha, \beta > 0 \]

Complex poles: \( -\alpha \pm j\beta \)

\[ \frac{1}{\beta} e^{-\alpha t} \sin (\beta t) u(t) \]
What if $F(s)$ is not a proper rational function?

Use long division to get a quotient + remainder.

\[
\begin{align*}
\frac{s^2 + 2s + 2}{s + 1} &= \frac{s^2 + 2s + 2}{s + 1} \\
&= s + 1 + \frac{1}{s + 1}
\end{align*}
\]

Applications to circuits

- Draw a circuit in the transform domain (s-domain) with or without initial conditions
- KCL/KVL, mesh or nodal analysis

Voltage division
Series/parallel Combinations
Source transformations

Work is more important than the final answer

\[
\frac{64}{16} = 4 \quad \text{Not OK!}
\]