1) Laplace Transforms
2) Inverse Laplace transforms
3) Applications to circuits

The impulse function $\delta(t)$ is an idealized representation of a very short duration, very high amplitude pulse.

$K\delta(t)$ is an impulse of strength $K$.

**Unit impulse:**

$$\delta(t) = 0 \text{ for } t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t) \, dt = 1$$

$$\int_{0^{-}}^{0^{+}} \delta(t) \, dt = 1$$

If $f(t)$ is continuous at $t = 0$

$$f(t) \delta(t) = f(0) \delta(t)$$

$$f(t) \delta(t-3) = f(3) \delta(t-3)$$

provided $f(t)$ is continuous at $t = 3$

**Basic Laplace Transforms**

$$\mathcal{L}[\delta(t)] = 1$$

$$\mathcal{L}[u(t)] = \frac{1}{s}$$

$$\mathcal{L}[r(t)] = \frac{1}{s^2} = \mathcal{L}[tu(t)]$$

$$\mathcal{L}[\sin(\omega t) u(t)] = \frac{\omega}{s^2 + \omega^2}$$

$$\mathcal{L}[\cos(\omega t) u(t)] = \frac{s}{s^2 + \omega^2}$$

$$\mathcal{L}[e^{-\alpha t} u(t)] = \frac{1}{s + \alpha}$$
Transform Properties

1) Linearity: \[ \mathcal{L} \left[ a f_1(t) + b f_2(t) \right] = aF_1(s) + bF_2(s) \]

2) Time integration: \[ \mathcal{L} \left[ \int_0^t f(\tau) \, d\tau \right] = \frac{F(s)}{s} \]

3) Time differentiation: \[ \mathcal{L} \left[ \frac{df(t)}{dt} \right] = sF(s) - f(0^-) \]

4) s-domain translation
   \[ \mathcal{L} \left[ e^{-at} f(t) \right] = F(s+a) \]

5) Time-domain translation
   \[ a > 0 \quad \mathcal{L} \left[ f(t-a) u(t-a) \right] = e^{-as} F(s) \]

6) Differentiation in the s-domain
   \[ \mathcal{L} \left[ tf(t) \right] = -\frac{dF(s)}{ds} \]

Inverse Laplace Transforms

\[ F(s) = \frac{N(s)}{D(s)} = \frac{(s-z_1)(s-z_2) \ldots (s-z_m)}{(s-p_1)(s-p_2) \ldots (s-p_n)} \]

Pole-zero diagram

Partial Fraction Expansion

Proper rational functions: \( m < n \)
\# of zeros < \# of poles

1. Real, distinct (no repeated) poles (simple poles)

\[ F(s) = \frac{K_1}{s-p_1} + \frac{K_2}{s-p_2} + \ldots + \frac{K_n}{s-p_n} \]

\( K_1, K_2, \ldots, K_n \) determined by cover up algorithm
\[ F(s) = \frac{N(s)}{(s-p_1)(s-p_2) \cdots (s-p_j) \cdots (s-p_n)} \]

\[ K_j = \frac{N(s)}{(s-p_1)(s-p_2) \cdots (s-p_j) \cdots (s-p_n)} \bigg|_{s=p_j} \]

\[ f(t) = k_1 e^{p_1 t} u(t) + k_2 e^{p_2 t} u(t) + \cdots + k_n e^{p_n t} u(t) \]

2. Real repeated poles - derivatives w.r.t. s

\[ F(s) = \frac{N(s)}{s(s-p_1)^2} \]

\[ = \frac{k_1}{s} + \frac{k_{21}}{(s-p_1)} + \frac{k_{22}}{(s-p_1)^2} \]

\[ K_1 \text{ and } K_{22} \text{ determined by the cover up algorithm} \]

3. Distinct complex poles

\[ \text{PFE} \quad \ldots + \frac{k}{(s+\alpha+j\beta)} + \frac{k^*}{(s+\alpha-j\beta)} \]

\[ \text{determine } k \text{ using the cover up algorithm} \]

\[ k = |k| e^{-j\beta} \]

\[ f(t) = \cdots + 2|k| e^{-\beta t} \cos(\beta t + \angle k) u(t) \]

\[ \frac{1}{\beta} \mathcal{L}^{-1} \left[ \frac{\beta}{(s+\alpha)^2 + \beta^2} \right] = \frac{1}{\beta} e^{-\beta t} \sin(\beta t) u(t) \]

\[ \uparrow \text{complex poles} \quad -\alpha \pm j\beta \]
What if $F(s)$ is not a proper rational function?

Use long division to obtain a Quotient + Remainder

\[ \downarrow \]

\[ \text{impulse} \]

\[ \downarrow \]

\[ \text{PRF} \]

\[ \downarrow \]

\[ \text{under derivatives} \]

\[ \downarrow \]

\[ \text{PFE} \]

\[ \downarrow \]

\[ \text{of impulse} \]

Applications to Circuits

- Draw a circuit in the transform domain (s-domain) with or without initial conditions
- KCL/KVL, mesh, nodal analysis analysis
- Voltage division
- Series/parallel combination
- Source transformations

Look at the sample exam

1. Laplace transforms
2. Inverse Laplace transforms
3. Circuit analysis in s-domain
4. Pole-zero diagram
5. Transformed circuits + writing equations

Work is more important than the final answer

\[
\frac{\sqrt{4}}{1/6} = 4
\]

Not ok