OBJECTIVE: The objective of a communication system is to efficiently transmit an information-bearing signal (message) from one location to another through a communication channel (transmission medium).

WISH LIST FOR EFFICIENT COMMUNICATION:

• High reliability
• Minimum transmitted signal power
• Minimum transmission bandwidth
• Low implementation complexity (low cost)

EFFICIENT TRANSMISSION: Can be achieved by processing the signal through a technique called modulation.

MODULATION TYPES:

• Analog: Amplitude and angle modulation
• Digital: Amplitude, angle and hybrid (amplitude and angle together) modulation
MATHEMATICAL PRELIMINARIES

Let a complex signal in polar form be described by \( x(t) = Ae^{j(\omega_0 t + \theta)} \). Then,

\[
x(t) = A\cos(\omega_0 t + \theta) + jA\sin(\omega_0 t + \theta) = x_r(t) + jx_i(t).
\]

Thus, the real and imaginary parts of \( x(t) \) are

\[
x_r(t) = \text{Re}\{x(t)\} = A\cos(\omega_0 t + \theta) \quad \text{and} \quad x_i(t) = \text{Im}\{x(t)\} = A\sin(\omega_0 t + \theta).
\]

Clearly, \( x(t) = |x(t)|e^{j\phi(t)} \), where

\[
|x(t)| = \sqrt{x_r^2(t) + x_i^2(t)} \quad \text{and} \quad \phi(t) = \tan^{-1}\left(\frac{x_i(t)}{x_r(t)}\right).
\]

**Even and Odd signals**

\( x(t) \) is an even signal iff \( x(-t) = x(t) \) and \( x(t) \) is an odd signal iff \( x(-t) = -x(t) \).

In general,

\[
x(t) = x_e(t) + x_o(t),
\]
where

\[ x_e(t) = \frac{x(t) + x(-t)}{2} \quad \text{and} \quad x_o(t) = \frac{x(t) - x(-t)}{2}. \]

**Example:** Let \( x(t) \) be described by \( x(t) = A \cos(\omega_0 t + \theta) \). Then

\[ x(t) = A \cos \theta \cos(\omega_0 t) - A \sin \theta \sin(\omega_0 t). \]

Since \( \cos(-\omega_0 t) = \cos(\omega_0 t) \) and \( \sin(-\omega_0 t) = -\sin(\omega_0 t) \), we get

\[ x_e(t) = \frac{x(t) + x(-t)}{2} = A \cos \theta \cos(\omega_0 t) \quad \text{and} \quad x_o(t) = \frac{x(t) - x(-t)}{2} = A \sin \theta \sin(\omega_0 t). \]

**Energy and Power Signals**

A signal \( x(t) \) is an energy signal iff its energy

\[ E_x = \lim_{T \to \infty} \int_{-T/2}^{T/2} |x(t)|^2 \, dt < \infty. \]

A signal \( x(t) \) is a power signal iff its power \( P_x \) is such that

\[ 0 < P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt < \infty. \]

**Example:** If \( x(t) \) is bounded over a finite time interval and zero outside that interval, then \( x(t) \) is an energy signal.
**Example:** Let \( x(t) = A \cos(\omega_0 t + \theta) \). Then we can show that \( x(t) \) is a power signal, i.e.

\[
P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |A \cos(\omega_0 t + \theta)|^2 \, dt = \frac{A^2}{2} < \infty.
\]

**Fourier Series and Fourier Transforms:**

Let \( x(t) \) be a periodic signal, i.e. \( x(t) = x(t + nT) \), where \( T \) is the period of repetition. Then if \( x(t) \) satisfies the Dirichlet conditions, i.e.

a) \( x(t) \) is absolutely integrable over one period \( T \), i.e.

\[
\int_{t_0}^{t_0+T} |x(t)| \, dt < \infty, \text{ for arbitrary } t_0.
\]

b) The number of maxima and minima of \( x(t) \) in each period is finite.

c) The number and size of the discontinuities of \( x(t) \) in each period is finite.

the periodic signal \( x(t) \) has a Fourier series described by

\[
x(t) = \sum_{n=-\infty}^{\infty} x_n e^{j n \omega_0 t}, \quad \omega_0 = \frac{2\pi}{T},
\]

where \( x_n = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-j n \omega_0 t} \, dt \).
Let \( x(t) \) be an aperiodic signal. Then if \( x(t) \) satisfies the Dirichlet conditions, i.e.

\[
\text{a) } \int_{-\infty}^{\infty} |x(t)| \, dt < \infty.
\]

\[
\text{b) } \text{The number of maxima and minima of } x(t) \text{ in any finite interval over the real line is finite.}
\]

\[
\text{c) } \text{The number and size of the discontinuities of } x(t) \text{ in any finite interval over the real line is finite.}
\]

the signal \( x(t) \) has a Fourier transform in the Hz frequency domain described by

\[
X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} \, dt
\]

and the original signal can be recovered from

\[
x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} \, df
\]

Parseval’s Theorm:

Let \( x(t) \) be a periodic signal, i.e. \( x(t) = x(t + nT) \), where \( n \) is an integer and \( T \) is the period of repetition. Then the power content of \( x(t) \) is given by
\[ P_x = \frac{1}{T} \int_{t_0}^{t_0+T} |x(t)|^2 \, dt = \sum_{n=-\infty}^{\infty} |x_n|^2. \]

Let \( x(t) \) be an aperiodic signal which possesses a Fourier transform \( X(f) \). Then
\[
\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df.
\]

**Lowpass and Bandpass Signals:**

Let \( x(t) \) be described by \( x(t) = A(t) \cos(2\pi f_c t + \theta(t)) \), where \( A(t) \) and \( \theta(t) \) are slowly varying signals of time (they have small frequency content around zero frequency, i.e. they are low pass signals). Then,
\[
x(t) = A(t) \cos(\theta(t)) \cos(2\pi f_c t) - A(t) \sin(\theta(t)) \sin(2\pi f_c t)
= x_I(t) \cos(2\pi f_c t) - x_Q(t) \sin(2\pi f_c t),
\]
where the in-phase and quadrature components \( x_I(t) \) and \( x_Q(t) \) are given by
\[
x_I(t) = A(t) \cos(\theta(t)) \quad \text{and} \quad x_Q(t) = A(t) \sin(\theta(t)),
\]
respectively.
Hence, $x_l(t)$ and $x_Q(t)$ are lowpass signals and $x(t)$ is a bandpass signal with its spectrum around the center frequency $f_c$. The lowpass complex envelope of $x(t)$ is given by

$$x_l(t) = x_l(t) + jx_Q(t)$$

and

$$x(t) = \text{Re}\left\{ x_l(t)e^{j2\pi f_c t} \right\} = \text{Re}\left\{ (x_l(t) + jx_Q(t))\left(\cos(2\pi f_c t) + j \sin(2\pi f_c t)\right) \right\}$$

$$= \text{Re}\left\{ x_l(t)\cos(2\pi f_c t) + jx_l(t)\sin(2\pi f_c t) + jx_Q(t)\cos(2\pi f_c t) + j^2 x_Q(t)\sin(2\pi f_c t) \right\}$$

$$= \text{Re}\left\{ x_l(t)\cos(2\pi f_c t) - x_Q(t)\sin(2\pi f_c t) + j\left( x_l(t)\sin(2\pi f_c t) + x_Q(t)\cos(2\pi f_c t) \right) \right\} \cdot$$

$$= x_l(t)\cos(2\pi f_c t) - x_Q(t)\sin(2\pi f_c t)$$

$$= A(t)\cos\left(2\pi f_c t + \theta(t)\right)$$
Standard Amplitude Modulation (AM)

In this type of modulation the amplitude of a sinusoidal carrier is varied according to the transmitted message signal. Let \( m(t) \) be the message signal we would like to transmit, \( k_a \) be the amplitude sensitivity (modulation index), and \( c(t) = A_c \cos(2\pi f_c t) \) be the sinusoidal carrier signal, where \( A_c \) is the amplitude of the carrier and \( f_c \) is the carrier frequency. Then the transmitted standard AM signal waveform is described by

\[
s_{AM}(t) = A_c [1 + k_a m(t)] \cos(2\pi f_c t)
\]

Requirements:

1. \(|k_a m(t)| < 1, \forall t\), to avoid overmodulation and phase reversals
2. \( f_c \gg W_m \), where \( W_m \) is the message bandwidth
Taking the Fourier transform of the modulated waveform, we get

\[ S_{AM}(f) = \Im \{ s_{AM}(t) \} = \Im \{ A_c [1 + k_a m(t)] \cos(2\pi f_c t) \} \]

\[ = \Im \{ A_c \cos(2\pi f_c t) + A_c k_a m(t) \cos(2\pi f_c t) \} \]

\[ = \frac{A_c}{2} [\delta(f - f_c) + \delta(f + f_c)] + \frac{k_a A_c}{2} [M(f - f_c) + M(f + f_c)] \]

Let \(|M(f)|\) be described by

Magnitude spectrum of \(m(t)\)
Then the magnitude spectrum of $s_{AM}(t)$ is

Magnitude spectrum of $s_{AM}(t)$

**Observations:**

1. The transmission bandwidth is $B_{AM} = 2W_m$
2. Carrier signal is transmitted explicitly (delta functions are present in frequency spectrum)
Let the message signal be the frequency tone \( m(t) = \cos(0.2\pi t) \), then with \( s_{AM}(t) = A_c [1 + k_a \cos(2\pi f_m t)]\cos(2\pi f_c t) = 10[1 + 0.5\cos(0.2\pi t)]\cos(2\pi t) \)
AM waveform for a 0.1 Hz tone with $k_a = 1.2$
To conserve transmitted power, let us suppress the carrier, i.e., let the transmitted waveform be described by

\[ s_{DSB}(t) = A_c m(t) \cos(2\pi f_c t). \]

This is called double side-band suppressed carrier (DSB-SC) modulation.
In the frequency domain,

\[
S_{DSB}(f) = \mathcal{F}\{s(t)\} \\
= \mathcal{F}\{A_c m(t) \cos(2\pi f_c t)\} \\
= \frac{A_c}{2} M(f) \ast [\delta(f - f_c) + \delta(f + f_c)] = \frac{A_c}{2} \left[ M(f - f_c) + M(f + f_c) \right]
\]

The transmitted modulated waveform now has the following spectral characteristics:

Magnitude spectrum of \( s_{DSB}(t) \)
Observations:

1. Transmission bandwidth is $B_{DSB} = 2W_m$ (same as standard AM)
2. Transmitted power is less than that used by standard AM

**Example:** Let a sinc pulse be transmitted using DSB-SC modulation.
DSB modulated waveform when $f_c = 0.25$ Hz and $m(t) = \sin(0.1^\circ t)/0.1^\circ t$
Receivers for AM and DSB

Receivers can be classified into coherent and non-coherent categories.

**Definition**: If a receiver requires knowledge of the carrier frequency and phase to extract the message signal from the modulated waveform, then it is called coherent.

**Definition**: If a receiver does **not** require knowledge of the phase (only rough knowledge of the carrier frequency) to extract the message signal from the modulated waveform, then it is called non-coherent.

Non-coherent demodulator (receiver) for standard AM

![Diagram of a non-coherent demodulator](image)

Peak envelope detector (Standard AM demodulator)
Observations:

- The net effect of the diode is the multiplication (mixing) of the signals applied to its input. Therefore, its output will contain the original input frequencies, their harmonics and their cross products.

- The network is a lowpass filter with a single pole (lossy integrator) that removes most of the high frequencies. \( R \) and \( C \) have to be judiciously picked so that the time constant \( \tau = RC \) is neither too short (rectifier distortion) nor too long (diagonal clipping).

- A rule of thumb for choosing \( \tau \) is based on the highest modulating signal (message) frequency that can be demodulated by a peak envelope detector without attenuation, that is,

\[
\tau = \frac{\sqrt{1/k_a^2} - 1}{2\pi f_{m(\text{max})}} = \frac{\sqrt{1/k_a^2} - 1}{\omega_{m(\text{max})}}, \quad |k_a| < 1
\]

where \( f_{m(\text{max})} (\omega_{m(\text{max})}) \) is the maximum modulating signal (message) frequency in Hz (rad/s).
Let the message signal be described by the sinusoidal tone \( m(t) = \cos(2\pi f_m t) \). Then, the following graphs show the types of distortion that can occur when \( \tau \) is properly and improperly chosen.
Illustration of rectifier distortion and diagonal clipping (with improperly chosen $\tau$)

(a) Input waveform

(b) Output waveform

RC time constant too short

(c) Output waveform

RC time constant too long

Ideal waveform

Illustration of rectifier distortion and diagonal clipping (with improperly chosen $\tau$)
Coherent demodulator for DSB-SC: Consider the following demodulator which assumes $f_c$ has been estimated perfectly at the receiver, though $\phi$ is not known.

At the output of the mixer,

$$v(t) = \hat{A}_c \cos(2\pi f_c t + \phi) s_{DSB}(t)$$

$$= \hat{A}_c \cos(2\pi f_c t + \phi) \cdot A_c m(t) \cos(2\pi f_c t)$$

$$= \frac{\hat{A}_c A_c}{2} m(t) \left[ \cos(4\pi f_c t + \phi) + \cos\phi \right]$$

and

$$V(f) = F\{v(t)\}$$

$$= \frac{\hat{A}_c A_c}{4} \left[ M(f - 2f_c)e^{j\phi} + M(f + 2f_c)e^{-j\phi} \right] + \frac{A_c \hat{A}_c}{2} M(f) \cos\phi$$
Let the message signal have the following magnitude spectrum

Then, if \( f_c > W_m \),
Suppose $H_{lpf}(f)$ is such that

Then, if $W_m \leq B < 2f_c - W_m$, \[ \hat{M}(f) = \frac{A_c \hat{A}_c}{2} M(f) \cos \phi \]
Coherent Costas loop receiver for DSB-SC:
I-channel:

After downconversion,

\[ v_I(t) = A_c m(t) \cos(\omega_c t) \cdot \cos(\omega_c t + \phi) \]

\[ = \frac{A_c}{2} m(t)[\cos(2\omega_c t + \phi) + \cos\phi] \]

At the output of the lowpass filter, with \(|H(0)| = 1\),

\[ m_I(t) = \frac{A_c}{2} \cos\phi \cdot m(t) \]

Q-channel:

\[ v_Q(t) = \frac{A_c}{2} m(t)[\sin(2\omega_c t + \phi) + \sin\phi] \]

\[ m_Q(t) = \frac{A_c}{2} \sin\phi \cdot m(t) \]
Feedback path:

At the output of the multiplier,

\[ m_e(t) = \frac{A_c^2}{4} m^2(t) \sin \phi \cos \phi \]

\[ = \frac{A_c^2}{8} m^2(t) \sin 2\phi \]

At the output of the feedback lowpass filter,

\[ m_{ef}(t) = \int_{-\infty}^{\infty} m_e(\tau) h_f(t - \tau) d\tau \]

The purpose of \( h_f(t) \) is to smooth out fast time variations of \( m_e(t) \)

The output of the VCO is described by

\[ x_{VCO}(t) = \cos \left( \omega_c t + \phi(t) \right), \]
where $\omega_c$ is the VCO’s reference frequency and $\phi(t) = k_v \int_0^t m_{ef}(\tau)d\tau$, is the residual phase angle due to the tracking error. The constant $k_v$ is the frequency sensitivity of the VCO in rad/s/volt (it depends on the circuit implementation).

The instantaneous frequency in radians/sec of the VCO’s output is given by

$$\frac{d}{dt} \left[ \omega_c t + \phi(t) \right] = \frac{d}{dt} \left[ \omega_c t + k_v \int_0^t m_{ef}(\tau)d\tau \right] = \omega_c + k_v m_{ef}(t),$$

Clearly, if $\phi(t)$ were small and slowly varying, then the instantaneous frequency would be close to $\omega_c$ and the output of the I-path would also be proportional to the desired message signal $m(t)$, since $\cos(\phi) \approx 1$, for $\phi(t)$ small.
Angle Modulation

Let \( \theta_i(t) \) be the instantaneous angle of a modulated sinusoidal carrier, i.e.,

\[
s(t) = A_c \cos \theta_i(t),
\]

where \( A_c \) is the constant amplitude.

The instantaneous frequency is

\[
\omega_i(t) = \frac{d\theta_i(t)}{dt}.
\]

Observation: The signal \( s(t) \) can be thought of as a rotating phasor of length \( A_c \) and angle \( \theta_i(t) \), i.e.

\[
\vec{s}(t) = A_c \angle \phi(t)
\]
If \( s(t) \) were an unmodulated carrier signal, then the instantaneous angle would be

\[
\theta_i(t) = \omega_c t + \phi_c
\]

where \( \omega_c \equiv \text{Constant angular velocity in rad/s} \)

\( \phi_c \equiv \text{Constant but arbitrary phase angle in radians} \)

Let \( \theta_i(t) \) be varied linearly with the message signal \( m(t) \) and \( \phi_c = 0 \), then

\[
\theta_i(t) = \omega_c t + k_p m(t),
\]

where \( k_p \equiv \text{Phase sensitivity of the modulator in rad/volt (circuit dependent)} \)

In this case we say that the carrier has been **phase modulated**.

The phase modulated waveform is given by

\[
s_{PM}(t) = A_c \cos(\omega_c t + k_p m(t))
\]

Let the instantaneous frequency \( \omega_i(t) \) be varied linearly with the message signal \( m(t) \), i.e.,

\[
\omega_i(t) = \omega_c + k_\omega m(t)
\]
where \( k_\omega \equiv \) Frequency sensitivity of the modulator in rad/s/volt (circuit dependent)

In this case we say that the carrier has been **frequency modulated** and the instantaneous angle is obtained by integrating the instantaneous frequency, i.e.,

\[
\theta_i(t) = \int_0^t \omega_i(\tau) d\tau = \int_0^t [\omega_c + k_\omega m(\tau)] d\tau = \omega_c t + k_\omega \int_0^t m(\tau) d\tau
\]

The modulated waveform is therefore described by

\[
s_{FM}(t) = A_c \cos \left( \omega_c t + k_\omega \int_0^t m(\tau) d\tau \right)
\]

**Observation:** Both phase and frequency modulation are related to each other and one can be obtained from the other. Hence, we could deduce the properties of one of the two modulation schemes once we know the properties of the other. This is illustrated in the following 2 block diagrams:
FM modulation
PM modulation
The figure below shows a comparison between AM, FM and PM modulation of the same message waveform:

![Modulating signal with AM, FM, and PM examples]
Frequency Modulation

Consider the frequency modulation of a message signal (frequency tone)

\[ m(t) = A_m \cos(2\pi f_m t), \quad \Rightarrow s_{FM}(t) = A_c \cos \left( \omega_c t + k_\omega \int_0^t A_m \cos(2\pi f_m \tau) \, d\tau \right) \]

The instantaneous frequency (in Hz) of the FM signal is

\[ f_i(t) = f_c + k_f A_m \cos(2\pi f_m t), \quad \text{since} \quad \omega_c = 2\pi f_c \quad \text{and} \quad k_\omega = 2\pi k_f. \]

Define the maximum frequency deviation as \( \Delta f = \max \{ f_i(t) \} - f_c = k_f A_m \),

Since \( \max \{ f_i(t) \} = \max \{ f_c + k_f A_m \cos(2\pi f_m t) \} = f_c + k_f A_m = f_c + \Delta f \).

The instantaneous phase angle of the FM signal is

\[ \theta_i(t) = 2\pi \int_0^t f_i(\tau) \, d\tau = 2\pi \int_0^t \left( f_c + k_f A_m \cos(2\pi f_m \tau) \right) \, d\tau \]

\[ = 2\pi f_c t + \frac{k_f A_m}{f_m} \sin(2\pi f_m t) \]

\[ = 2\pi f_c t + \beta \sin(2\pi f_m t) \]
where $\beta = k_f A_m / f_m = \Delta f / f_m$, is known as the FM modulation index (for a tone) or the maximum phase deviation (in rad) produced by the tone in question

The FM modulated tone is therefore given by

$$s_{FM}(t) = A_c \cos \left(2\pi f_c t + \beta \sin(2\pi f_m t)\right)$$

$$= A_c \left[\cos(2\pi f_c t)\cos(\beta \sin(2\pi f_m t)) - \sin(2\pi f_c t)\sin(\beta \sin(2\pi f_m t))\right]$$

This $s_{FM}(t)$ signal is nonperiodic unless $f_c = nf_m$, where $n$ is a positive integer.

For the general case, $f_c$ is not necessarily a multiple of $f_m$. Now,

$$s_{FM}(t) = \Re \left\{ A_c e^{j[2\pi f_c t + \beta \sin(2\pi f_m t)]} \right\}$$

$$= \Re \left\{ A_c e^{j\beta \sin(2\pi f_m t)} \cdot e^{j2\pi f_c t} \right\} = \Re \left\{ s_e(t) e^{j2\pi f_c t} \right\},$$

where the complex envelope of the FM signal is described by

$$s_e(t) = A_c e^{j\beta \sin(2\pi f_m t)}$$

**Observation:** Unlike $s_{FM}(t)$, $s_e(t)$ is periodic with period $1/f_m$.  


Since \( s_e(t) \) meets the Dirichlet conditions, we can compute its Fourier series, i.e.,

\[
s_e(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_m t},
\]

where the Fourier series coefficients are given by

\[
c_n = \frac{1}{T} \int_{-T/2}^{T/2} s_e(t) e^{-j2\pi n f_m t} dt, \quad T = 1/ f_m
\]

\[
= f_m \int_{-1/2 f_m}^{1/2 f_m} s_e(t) e^{-j2\pi n f_m t} dt
\]

\[
= f_m \int_{-1/2 f_m}^{1/2 f_m} A_c e^{j\beta \sin(2\pi f_m t)} e^{-j2\pi n f_m t} dt
\]

\[
= A_c f_m \int_{-1/2 f_m}^{1/2 f_m} e^{j[\beta \sin(2\pi f_m t) - 2\pi n f_m t]} dt.
\]
Let \( x = 2\pi f_m t \).

Then,
\[
dt = \frac{1}{2\pi f_m} \, dx
\]

and
\[
c_n = \frac{A_c}{2\pi} \int_{-\pi}^{\pi} e^{j[\beta \sin x - nx]} \, dx = A_c J_n(\beta),
\]

where \( J_n(\beta) \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j[\beta \sin x - nx]} \, dx \) is the Bessel function of the first kind of order \( n \).

Alternatively,
\[
J_n(\beta) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+n+1)} \left( \frac{\beta}{2} \right)^{2m+n},
\]

where the Gamma function \( \Gamma(t) \) is defined by
\[
\Gamma(t) = \int_{0}^{\infty} x^{t-1} e^{-x} \, dx
\]
Therefore,

\[ s_e(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j2\pi n f_m t} \]

and the FM tone waveform is described by

\[ s_{FM}(t) = \Re \left\{ A_c \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j2\pi n f_m t} \cdot e^{j2\pi f_c t} \right\} = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos \left[ 2\pi (f_c + n f_m) t \right]. \]
In the frequency domain,

\[ S_{FM}(f) = \Im \{s_{FM}(t)\} \]

\[ = \Im \left\{ A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos[2\pi(f_c + nf_m)t] \right\} \]

\[ = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \Im \{\cos[2\pi(f_c + nf_m)t]\} \]

\[ = A_c \sum_{n=-\infty}^{\infty} \frac{J_n(\beta)}{2} \left[ \delta(f + f_c + nf_m) + \delta(f - f_c - nf_m) \right] \]

**Average power of the FM waveform:**

The average power delivered to a 1 ohm load resistor by the FM waveform is

\[ P = A_c^2 / 2 \]

But, \[ P = \int_{-\infty}^{\infty} \left| S_{FM}(f) \right|^2 df = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| s_{FM}(t) \right|^2 dt \]

is also the average power of the FM waveform delivered to a 1 ohm resistor.
Examples:

(a) $\beta = 1.0$

(b) $\beta = 2.0$

(c) $\beta = 5.0$
It can be shown, in the limit, that
\[ \int_{-\infty}^{\infty} |S_{FM}(f)|^2 df = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |s_{FM}(t)|^2 dt \to \frac{A_c^2}{2} \sum_{n=-\infty}^{\infty} J_n^2(\beta) \]

Let us now take a look at the properties of the Bessel function.

1. \[ J_n(\beta) = (-1)^n J_{-n}(\beta) \]
2. \[ \sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1 \]

Hence, the average power of an FM tone is, as expected, \( A_c^2 / 2 \).

Suppose \( \beta \) is small, i.e., \( 0 < \beta \leq 0.3 \), then

- \( J_0(\beta) \approx 1 \)
- \( J_1(\beta) \approx \beta / 2 \)
- \( J_n(\beta) \approx 0, \quad n \geq 2 \)

Under the assumption that \( \beta \) is small, the Fourier series representation of the FM waveform can be simplified to three terms.
Thus, for $\beta$ small, the FM tone may be described by

$$s_{FM}(t) \approx A_c \left\{ \cos(2\pi f_c t) + \frac{\beta}{2} \cos(2\pi (f_c + f_m)t) - \frac{\beta}{2} \cos(2\pi (f_c - f_m)t) \right\}$$

$$= A_c \cos(2\pi f_c t) + A_c \frac{\beta}{2} \cos(2\pi (f_c + f_m)t) - A_c \frac{\beta}{2} \cos(2\pi (f_c - f_m)t)$$

In the frequency domain,

$$S_{FM}(f) \approx \frac{A_c}{2} \left[ \delta(f + f_c) + \delta(f - f_c) \right] - \frac{A_c \beta}{4} \left[ \delta(f + f_c - f_m) + \delta(f - f_c + f_m) \right]$$

$$+ \frac{A_c \beta}{4} \left[ \delta(f + f_c + f_m) + \delta(f - f_c - f_m) \right]$$
A plot of the magnitude spectrum of the FM tone with $\beta$ small is shown below.

The time domain FM waveform can be represented in phasor form as follows:

$$\vec{S}_{FM} = A_c \angle 0^\circ + \frac{1}{2} A_c \beta \angle 2\pi f_m t + \frac{1}{2} A_c \beta \angle -2\pi f_m t + \pi$$

For arbitrary $t = t_0$, and small $\beta$, we can illustrate graphically the phasor representation and arrive at some conclusion.
The following figure shows an example of the phasor representation

\[ \tilde{S}_{FM} = A_c + \frac{1}{2} A_c \beta \left[ \cos(2\pi f_m t) + j \sin(2\pi f_m t) + \cos(-2\pi f_m t + \pi) + j \sin(-2\pi f_m t + \pi) \right] \]

Observation:

The resultant phasor \( \tilde{S}_{FM} \), has magnitude \( \left| \tilde{S}_{FM} \right| \approx A_c \), and is out of phase with respect to the carrier phasor \( A_c \angle 0^\circ \).
But, \[
\cos(-2\pi f_m t + \pi) = \cos(2\pi f_m t) \cos \pi + \sin \pi \sin(2\pi f_m t) \\
= -\cos(2\pi f_m t)
\]

and \[
\sin(-2\pi f_m t + \pi) = -\sin(2\pi f_m t) \cos \pi + \sin \pi \cos(2\pi f_m t) \\
= \sin(2\pi f_m t)
\]

Consequently, the resultant phasor (in rectangular form) is given by
\[
\bar{S}_{FM} = A_c + jA_c \beta \sin(2\pi f_m t)
\]
The magnitude of the resultant may be approximated by
\[
|\bar{S}_{FM}| = \sqrt{A_c^2 + A_c^2 \beta^2 \sin^2(2\pi f_m t)} = A_c \left[ 1 + \beta^2 \sin^2(2\pi f_m t) \right]^{1/2} \\
\cong A_c \left[ 1 + \frac{1}{2} \beta^2 \sin^2(2\pi f_m t) \right],
\]
since \((1+x)^n \cong 1 + nx, \quad |nx|<1, \quad n = \frac{1}{2}\) and \(x = \beta^2 \sin^2(2\pi f_m t)\) in our case.
Finally, the magnitude and phase of the resultant are found to be

\[ |\vec{S}_{FM}(t)| = A_c \left[ 1 + \frac{\beta^2}{4} - \frac{\beta^2}{4} \cos(4\pi f_m t) \right] \]

\[ \angle \vec{S}_{FM} = \varphi_{\vec{S}}(t) = \tan^{-1} \left[ \frac{A_c \beta \sin(2\pi f_m t)}{A_c} \right] = \tan^{-1} \left[ \beta \sin(2\pi f_m t) \right] \]

**Observation:**

- For an FM tone, the spectral lines sufficiently away from the carrier may be ignored because their contribution (amplitude) is very small when \( \beta \) is small.

**FM Transmission Bandwidth:**

For an FM **tone**, as \( \beta \) becomes large \( J_n(\beta) \) has significant components only for

\[ |n| \leq \beta = k_f A_m / f_m = \Delta f / f_m. \]

\[ f_c \pm \beta f_m = f_c \pm \Delta f, \]

where \( \Delta f \) is the peak frequency deviation.
Let $\beta$ be small, i.e., $0 < \beta \leq 0.3$, then

$$J_0(\beta) \gg J_n(\beta), \ n \neq 0$$

and only the first pair of spectral lines are significant, i.e., the significant lines are contained in the range $f_c \pm f_m$

**Observation:** The previous analysis of an FM tone suggests that

1. For large $\beta$ the FM bandwidth is $B_{FM} = 2 \Delta f$

2. For small $\beta$ the FM bandwidth is $B_{FM} = 2 f_m$.

In general, the FM transmission bandwidth may be approximated by

$$B_T \approx 2 \Delta f + 2 f_m$$

$$= 2 \Delta f \left(1 + f_m / \Delta f \right)$$

$$= 2 \Delta f \left(1 + 1 / \beta \right)$$

This is known as the Carson’s rule.

**Observation:** Carson’s rule underestimates the transmission bandwidth by about 10%.
Alternative definition of FM tone transmission bandwidth:

A band of frequencies that keeps all spectral lines whose magnitudes are greater than 1% of the unmodulated carrier amplitude $A_c$, i.e.,

$$B_T = 2n_{\text{max}}f_m,$$

where $n_{\text{max}} = \max\left\{ n : |J_n(\beta)| > 0.01 \right\}$. 

![Graph showing the relationship between $B_T/\Delta f$ and $\beta$]
General Case:
Let an arbitrary message signal $m(t)$ have bandwidth $W_m$.
Define the peak frequency deviation and the deviation ratio by

$$\Delta f \doteq k_f \max_t |m(t)|$$

and

$$D \doteq \Delta f / W_m.$$ 

Then Carson’s rule can be used to define the transmission bandwidth of an arbitrary FM signal, i.e., when $m(t)$ is arbitrary.

Specifically, the FM transmission bandwidth can be defined by

$$B_T \doteq 2\Delta f + 2W_m$$

$$= 2\Delta f \left(1 + \frac{W_m}{\Delta f}\right)$$

$$= 2\Delta f \left(1 + 1/D\right)$$
Example: In commercial FM in the US, $\Delta f = 75 \text{ kHz}$, $W_m = 15 \text{ kHz}$. Therefore, the deviation ratio is $D = 75 \text{ kHz} / 15 \text{ kHz} = 5$.

Using Carson’s rule, the transmission bandwidth is

$$B_T = 2\Delta f (1 + 1/D) = 180 \text{ kHz},$$

Using the Universal curve, the transmission bandwidth is

$$B_T = 3.2\Delta f = 240 \text{ kHz}.$$ 

In practice, FM radio in the US uses a transmission bandwidth of $B_T = 200 \text{ kHz}$. 

**Generation of FM**

The frequency of the carrier can be varied by the modulating signal $m(t)$ directly or indirectly.
Direct generation of FM
If a very high degree of stability of the carrier frequency is not a concern, then we can generate FM directly using circuits without external crystal oscillators. Examples of this method are VCO’s, varactor diode modulators, reactance modulators, Crosby modulators (modulators that use automatic frequency control), etc..

Reactance FM modulator
Indirect generation of FM

Commercial applications of FM (as established by the FCC and other spectrum governing bodies) require a high degree of stability of the carrier frequency. Such restrictions can be satisfied by using external crystal oscillators, a narrowband phase modulator, several stages of frequency multiplication and mixers.

Let us begin with the synthesis of narrow-band FM.

\[
\mathbf{s_{NB}(t) = A_c \cos \left[ 2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau \right]}
\]
with $k_f$ (and thus $\Delta f_{NB}$) small.

Let us now consider a nonlinear technique to increase the FM signal bandwidth. Let $s_{NB}(t)$ be input to a nonlinear device with transfer characteristic $y(t) = ax^n(t)$, where $x(t)$ is its input, namely,

$$s_{NB}(t) \rightarrow a(\cdot)^n \rightarrow y(t)$$

Nonlinear device.

Let $\theta_i(t) = 2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau$, then at the output of the nonlinear device, we observe

$$y(t) = aA_c^n \cos^n \theta_i(t)$$

Let us expand this last equation to infer the effect of this nonlinear device.
cos^n \theta_i(t) can be expanded as follows:

\[
\cos^n \theta_i(t) = \cos^2 \theta_i(t) \cos^{n-2} \theta_i(t)
\]
\[
= \frac{1}{2} \left[ 1 + \cos 2\theta_i(t) \right] \cos^{n-2} \theta_i(t)
\]
\[
= \frac{1}{2} \cos^{n-2} \theta_i(t) + \frac{1}{2} \cos 2\theta_i(t) \cos^{n-2} \theta_i(t)
\]

Likewise,
\[
\cos^{n-2} \theta_i(t) = \frac{1}{2} \cos^{n-4} \theta_i(t) + \frac{1}{2} \cos 2\theta_i(t) \cos^{n-4} \theta_i(t)
\]

Thus,
\[
\cos^n \theta_i(t) = \frac{1}{2} \cos^{n-2} \theta_i(t) + \frac{1}{4} \cos 2\theta_i(t) \cos^{n-4} \theta_i(t) + \frac{1}{4} \cos^2 2\theta_i(t) \cos^{n-4} \theta_i(t)
\]

Expanding the last term of the last equation, we get
\[
\cos^2 2\theta_i(t) \cos^{n-4} \theta_i(t) = \frac{1}{2} \left[ 1 + \cos 4\theta_i(t) \right] \cos^{n-4} \theta_i(t).
\]
Rewriting the equation before the last one, we get

\[
\cos^n \theta_i(t) = \frac{1}{2} \cos^{n-2} \theta_i(t) + \frac{1}{4} \cos 2\theta_i(t) \cos^{n-4} \theta_i(t) \\
+ \frac{1}{8} \cos^{n-4} \theta_i(t) + \frac{1}{8} \cos 4\theta_i(t) \cos^{n-4} \theta_i(t) \\
= \frac{1}{2} \cos^{n-2} \theta_i(t) + \frac{1}{8} \cos^{n-4} \theta_i(t) + \frac{1}{4} \cos 2\theta_i(t) \cos^{n-4} \theta_i(t) \\
+ \frac{1}{16} \cos 4\theta_i(t) \cos^{n-6} \theta_i(t) + \frac{1}{32} \cos 2\theta_i(t) \cos^{n-6} \theta_i(t) \\
+ \frac{1}{32} \cos 6\theta_i(t) \cos^{n-6} \theta_i(t)
\]

The last term in the expansion of \( \cos^n \theta_i(t) \) is given by

\[
\frac{1}{2^{k-1}} \cos k\theta_i(t) \cos^{n-k} \theta_i(t).
\]
Let $n$ be an even number, then, when $k = n$, the last term is

$$\frac{1}{2^{n-1}} \cos n \theta_i (t)$$

If, on the other hand, $n$ is an odd number, then when $k = n-1$, the last term is

$$\frac{1}{2^{n-2}} [\cos(n-1) \theta_i (t) \cos \theta_i (t)] = \frac{1}{2^{n-1}} \cos(n-2) \theta_i (t) + \frac{1}{2^{n-1}} \cos n \theta_i (t)$$

Therefore, the last term in the expansion of $\cos^n \theta_i (t)$ is always

$$\frac{1}{2^{n-1}} \cos n \theta_i (t)$$

So, $y(t)$ can be expanded as

$$y(t) = c_0 + c_1 \cos \theta_i (t) + c_2 \cos 2 \theta_i (t) + \ldots + a \frac{A^n}{2^{n-1}} \cos n \theta_i (t)$$
Example: Consider the cases when $n = 2$ and $n = 3$.

Let $n = 2$, then

$$y(t) = aA_c^2 \cos^2 \theta_i(t)$$

or

$$y(t) = aA_c^2 \left[ \frac{1 + \cos 2\theta_i(t)}{2} \right] = \frac{aA_c^2}{2} + \frac{aA_c^2}{2} \cos 2\theta_i(t)$$

Let $n = 3$, then

$$y(t) = aA_c^3 \left[ \frac{1}{2} \cos 2\theta_i(t) + \frac{1}{2} \cos 2\theta_i(t) \cos \theta_i(t) \right]$$

$$= aA_c^3 \left[ \frac{1}{2} \cos \theta_i(t) + \frac{1}{2} \left\{ \frac{1}{2} \cos 3\theta_i(t) + \frac{1}{2} \cos \theta_i(t) \right\} \right]$$

$$= \frac{3aA_c^3}{4} \cos \theta_i(t) + \frac{aA_c^3}{4} \cos 3\theta_i(t)$$
Finally,

\[
y(t) = c_0 + c_1 \cos \left( 2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau \right) + c_2 \cos \left( 4\pi f_c t + 4\pi k_f \int_0^t m(\tau) d\tau \right) \\
+ \ldots + a \frac{A^n_c}{2^{n-1}} \cos \left( 2\pi n f_c t + 2\pi nk_f \int_0^t m(\tau) d\tau \right)
\]

Let \( y(t) \) be input to an ideal bandpass filter with unity gain, bandwidth wide enough to accommodate spectrum of a wide band signal with center frequency \( n_f_c \), i.e.,

\[
y(t) \xrightarrow{H_{BP}(f)} s_{WB}(t)
\]

Then,

\[
s_{WB}(t) = \frac{aA^n_c}{2^{n-1}} \cos \left( 2\pi n f_c t + 2\pi nk_f \int_0^t m(\tau) d\tau \right)
\]
The instantaneous frequency in Hz of $s_{WB}(t)$ is

$$f_i(t) = n f_c + n k_f m(t)$$

Observations about $s_{WB}(t)$:

1. The carrier frequency is $n f_c$
2. The peak frequency deviation is $n \Delta f_{NB}$

These are the desired properties of the WB FM signal.

The overall frequency multiplier device is shown below:

$$A_c \cos(\theta_i t) \rightarrow a(\bullet)^n \rightarrow H_{BP}(f) \rightarrow \frac{a A_c^n}{2^{n-1}} \cos(n \theta_i t)$$

$$f_{center} = n f_c$$

Complete frequency multiplier
Example: Noncommercial FM broadcast in the US uses the 88-90 MHz band and commercial FM broadcast uses the 90-108 MHz band (divided into 200 kHz channels). In either case $\Delta f = 75$ kHz. Suppose we target a station with $f_c = 90.1$ MHz. Then the indirect FM generation method suggested by Armstrong enables us to achieve our goals.

Let us start with a 400 kHz crystal oscillator and a narrow band phase modulator with $\Delta f = 14.47$ Hz.
Armstrong indirect method of FM generation

\[ f_c = 1.408 \]
\[ f_c = 1.408 \text{MHz} \]
\[ \Delta f = 1172 \text{Hz} \]
\[ \Delta f = 75 \text{kHz} \]

33.808 MHz
Crystal osc.

33.81 MHz

\[ s_2(t) \]

RF power amplifier

\[ s_{WB}(t) \]
\[ f_c = 90.1 \text{MHz} \]
\[ \Delta f = 75 \text{kHz} \]
Demodulation of FM signals
Consider the following receiver architecture

Frequency discriminator implementation of an FM demodulator

The slope circuit is characterized by purely imaginary transfer functions

\[ H_i(s), \ i = 1,2. \]
Let $H_1(f)$ be described by

$$H_1(f) = \begin{cases} 
  j2\pi a(f - f_c + B_T/2), & f_c - B_T/2 \leq f \leq f_c + B_T/2 \\
  j2\pi a(f + f_c - B_T/2), & -f_c - B_T/2 \leq f \leq -f_c + B_T/2 \\
  0, & \text{elsewhere}
\end{cases}$$

Graphically,
where $a > 0$ is a constant that determines the slope of $H_1(s)$.

Define $G_1(f) \equiv H_1(f)/j$, then $g_1(t) = \mathcal{F}^{-1}\{G_1(f)\}$ is the impulse response of a real bandpass system described by $G_1(f)$.

In the time domain,

$$g_1(t) = g_{1,I}(t) \cos(2\pi f_c t) - g_{1,Q}(t) \sin(2\pi f_c t)$$

where $g_{1,I}(t)$ and $g_{1,Q}(t)$ are the in-phase and quadrature components of $g_1(t)$.

Therefore, the complex envelope of $g_1(t)$ is described by

$$\tilde{g}_1(t) = g_{1,I}(t) + jg_{1,Q}(t)$$

which implies that

$$g_1(t) = \mathcal{R}e\{\tilde{g}_1(t)e^{j2\pi f_c t}\}$$

Using this information, we get

$$\tilde{g}_1(t)e^{j2\pi f_c t} + \tilde{g}_1^*(t)e^{-j2\pi f_c t} = (g_{1,I}(t) + jg_{1,Q}(t))e^{j2\pi f_c t} + (g_{1,I}(t) - jg_{1,Q}(t))e^{-j2\pi f_c t}$$

$$= 2g_{1,I}(t) \cos(2\pi f_c t) - 2g_{1,Q}(t) \sin(2\pi f_c t)$$

$$= 2g_1(t)$$
But,
\[ \Im\{2g_1(t)\} = F\{\tilde{g}_1(t)e^{j2\pi f_c t} + \tilde{g}_1^*(t)e^{-2j2\pi f_c t}\} \]
or
\[ 2G_1(f) = \tilde{G}_1(f - f_c) + \tilde{G}_1^*(-(f + f_c)) \]
since \[ \Im\{\tilde{g}_1^*(t)\} = \tilde{G}_1^*(-f) \]
which implies that \( \tilde{G}_1(f) \) has a lowpass frequency response limited to \( |f| \leq B_T / 2 \)
and that
\[ \tilde{G}_1(f - f_c) = 2G_1(f), f > 0. \]

**Observations:** \( \tilde{G}_1(f) \) can be obtained by taking the part of \( G_1(f) \) that corresponds to positive frequencies, shifting it to the origin and then scaling it by a factor of 2.

In the next figure \( G_1(f) \) is replaced by \( H_1(f)/j \) and \( \tilde{G}_1(f) \) replaced by \( \tilde{H}_1(f)/j \).
Frequency responses of $H_1(f)$, $\tilde{H}_1(f)$, and $H_2(f)$.
From the previous derivation,

\[ \tilde{H}_1(f) = \begin{cases} j4\pi a(f + B_T / 2), & |f| \leq B_T / 2 \\ 0, & \text{elsewhere} \end{cases} \]

But, \( s_{FM}(t) = A_c \cos \left( 2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau \right) = \Re \left\{ A_c e^{j2\pi k_f \int_0^t m(\tau) d\tau} \times e^{j2\pi f_c t} \right\} \]

\[ = \Re \left\{ \tilde{s}_{FM}(t) \times e^{j2\pi f_c t} \right\}. \]

which implies that \( \tilde{s}_{FM}(t) = A_c e^{j2\pi k_f \int_0^t m(\tau) d\tau} \)

If \( s_1(t) \) is the output of the slope filter \( H_1(f) \) when the input is \( s_{FM}(t) \) then the complex envelope of the output is

\[ \tilde{s}_1(t) = \frac{1}{2} \tilde{h}_1(t) \ast \tilde{s}_{FM}(t) \]
In the frequency domain,

\[
\tilde{S}_1(f) = \frac{1}{2} \tilde{H}_1(f) \tilde{S}_{FM}(f)
\]

\[
= \begin{cases} 
  j2\pi a(f + B_T/2)\tilde{S}_{FM}(f) = a\left(j2\pi f \tilde{S}_{FM}(f)\right) + j\pi aB_T\tilde{S}_{FM}(f), & |f| \leq B_T/2 \\
  0, & \text{elsewhere}
\end{cases}
\]

But,

\[
\mathcal{F}\left\{\frac{dx(t)}{dt}\right\} = j2\pi f X(f)
\]

which implies that in the time domain we get

\[
\tilde{s}_1(t) = a \frac{d\tilde{S}_{FM}(t)}{dt} + ja\pi B_T\tilde{S}_{FM}(t)
\]

\[
= j2\pi aA_c k_f m(t) e^{j2\pi k_f \int_0^t m(\tau) d\tau} + j\pi aA_c B_T e^{j2\pi k_f \int_0^t m(\tau) d\tau}
\]

\[
= j\pi aA_c B_T \left[1 + \frac{2k_f}{B_T} m(t)\right] e^{j2\pi k_f \int_0^t m(\tau) d\tau}
\]
Therefore,

\[
s_1(t) = \Re \left\{ \tilde{s}_1(t)e^{j2\pi f_c t} \right\}
\]

\[
= \Re \left\{ j\pi A e B_T \left[ 1 + \frac{2k_f}{B_T} m(t) \right] e^{j2\pi f_c t + j2\pi k_f \int_0^t m(\tau) d\tau} \right\}
\]

\[
= -\pi A e B_T \left[ 1 + \frac{2k_f}{B_T} m(t) \right] \sin \left( 2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau \right)
\]

\[
= \pi A e B_T \left[ 1 + \frac{2k_f}{B_T} m(t) \right] \cos \left( 2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau + \frac{\pi}{2} \right)
\]

**Observation:** \( s_1(t) \) contains both AM and FM.

However, if

\[
\left| \frac{k_f}{B_T} m(t) \right| < 1, \ \forall t
\]
Then a distortionless envelope detector can extract \( m(t) \) plus a bias, i.e.,

\[
s_{1e}(t) = \pi a A_c B_T \left[ 1 + \frac{2k_f}{B_T} m(t) \right].
\]

Finally, if

\[
\tilde{H}_2(f) = \tilde{H}_1(-f)
\]

then

\[
s_{2e}(t) = \pi a A_c B_T \left[ 1 - \frac{2k_f}{B_T} m(t) \right].
\]

Moreover,

\[
\hat{m}(t) = s_{1e}(t) - s_{2e}(t) = 4\pi a A_c k_f m(t)
\]

The cascade of a slope circuit and an envelope detector is known as a frequency discriminator.
**Frequency discriminator**

**FM demodulation via phase-locked loops**

We consider the phase-locked loop (PLL) FM detector shown below.

**Phase-locked loop FM detector**
From the previous block diagram, assuming an ideal propagation channel, and

\[ x_r(t) = A_c \cos(\omega_c t + \phi(t)) \]
\[ e_0(t) = A_v \sin(\omega_c t + \theta(t)) \]

The phase detector can be modeled by

\[
\begin{align*}
    e_1(t) &= A_c \cos(\omega_c t + \phi(t)) \cdot A_v \sin(\omega_c t + \theta(t)) \cdot k_d \\
    &= \frac{k_d A_c A_v}{2} \left[ \sin[2\omega_c t + \phi(t) + \theta(t)] + \sin[\phi(t) - \theta(t)] \right]
\end{align*}
\]

\[ k_d \] depends on the phase detector implementation.
with the proper choice of lowpass filter, the output of the phase detector is

\[ e_d(t) = \frac{k_d A_c A_v}{2} \sin[\phi(t) - \theta(t)] \]

A VCO is an FM modulator with peak frequency deviation

\[ \Delta \omega = \max_t \left| \frac{d\theta(t)}{dt} \right| \]

where \( \frac{d\theta(t)}{dt} = k_{vco} \hat{m}(t) \) implies that \( \theta(t) = k_{vco} \int_0^t \hat{m}(\tau) d\tau. \)

An equivalent nonlinear model is now shown

Nonlinear model of PLL FM demodulator
Assuming the PLL is operating in the near lock condition, i.e., $\theta(t) \cong \phi(t)$, or that $|\theta(t) - \phi(t)|$ is small. Then,

$$\sin[\phi(t) - \theta(t)] \cong \phi(t) - \theta(t)$$

and the linear approximation of the PLL is given by

Let the loop filter have the transfer function $H_{LF}(s) = 1$, then

$$\hat{m}(t) \cong \frac{k_d A_c A_v \mu}{2} [\phi(t) - \theta(t)]$$

$$= k_t [\phi(t) - \theta(t)], \quad k_t = \frac{k_d A_c A_v \mu}{2}$$
Thus, the output of the VCO is given by

\[ \theta(t) = k_{vco} \int_{0}^{t} \hat{m}(\tau) d\tau = k_{t} k_{vco} \int_{0}^{t} \left[ \phi(\tau) - \theta(\tau) \right] d\tau \]

Let \( k_{0} = k_{t} k_{vco} \), then

\[ \frac{d\theta(t)}{dt} = k_{0} \left[ -\theta(t) + \phi(t) \right] \]

or

\[ \frac{d\theta(t)}{dt} + k_{0} \theta(t) = k_{0} \phi(t), \]

In the s-domain, assuming zero initial conditions,

\[ (s + k_{0}) \Theta(s) = k_{0} \Phi(s), \]

The closed-loop transfer function is therefore given by

\[ H_{cl}(s) = \frac{\Theta(s)}{\Phi(s)} = \frac{k_{0}}{s + k_{0}} \Rightarrow \Theta(s) = \frac{k_{0}}{s + k_{0}} \Phi(s) \]
The corresponding impulse response is

\[ h_{cl}(t) = k_0 e^{-k_0t} u(t) \]

Let us now find out what happens when the loop gain \( k_0 \) is increased, i.e.,

\[ \lim_{k_0 \to \infty} H_{cl}(s) = \lim_{k_0 \to \infty} \frac{k_0}{s + k_0} = 1 \]

Clearly, \( \Theta(s) \to \Phi(s) \)

or \( \theta(t) \to \phi(t) \), faster for large \( k_0 > 0 \).

**Example:** Let the message signal be a step function, i.e., \( m(t) = A u(t) \), then

\[ x_{FM}(t) = A_c \cos \left( \omega_c t + k_\omega \int_0^t A u(\tau) d\tau \right) \]

In this case,

\[ \phi(t) = k_\omega A \int_0^t u(\tau) d\tau = k_\omega A t \]

In the s-domain, \( \Phi(s) = \frac{A k_\omega}{s^2} \) and \( \Theta(s) = \frac{A k_\omega k_0}{s^2 (s + k_0)} \).
The Laplace transform of \( \hat{m}(t) \) is then given by

\[
\hat{M}(s) = k_t \left[ \Phi(s) - \Theta(s) \right] = \frac{A k_\omega k_t}{s(s + k_0)} = A k_\omega k_t \left[ \frac{1}{k_0} \right] \left[ \frac{1}{s} - \frac{1}{s + k_0} \right] = \frac{A k_\omega}{k_{vco}} \left[ \frac{1}{s} - \frac{1}{s + k_0} \right].
\]

Let \( k_1 = \frac{k_\omega}{k_{vco}} \), then in the time domain,

\[
\hat{m}(t) = A k_1 u(t) - A k_1 e^{-k_0 t} u(t) = k_1 m(t) - A k_1 e^{-k_0 t} u(t)
\]

Clearly, as \( t \to \infty \), the estimate \( \hat{m}(t) \to k_1 m(t) \).

**Observation:** This result is valid when the initial phase error is small.

**Remark:** A large loop gain \( k_0 \) results in practical difficulties, hence, a different loop filter has to be used.

Consider the loop filter described by

\[
H_{LF}(s) = \left( s + a \right)/s, \quad a > 0
\]

Then the output of the VCO is given by

\[
\Theta(s) = k_{vco} \frac{\hat{M}(s)}{s} = k_{vco} \frac{k_t H_{LF}(s)}{s} \left[ \Phi(s) - \Theta(s) \right] = \frac{k_0 H_{LF}(s)}{s} \left[ \Phi(s) - \Theta(s) \right]
\]
Let $|\phi(t) - \theta(t)|$ be small, then the closed-loop transfer function is

$$H_{cl}(s) = \frac{\Theta(s)}{\Phi(s)} = \frac{k_0 H_{LF}(s)}{s + k_0 H_{LF}(s)} = \frac{k_0 (s + a)}{s^2 + k_0 s + k_0 a}$$

Define the phase angle error by $\Psi(s) \equiv \Phi(s) - \Theta(s)$, then

$$\Psi(s) = \Phi(s) - \frac{k_0 H_{LF}(s)}{s + k_0 H_{LF}(s)} \Phi(s) = \frac{s^2}{s^2 + k_0 s + k_0 a} \Phi(s) = \frac{s^2}{s^2 + 2\xi \omega_n s + \omega_n^2} \Phi(s)$$

where

$$\omega_n = \sqrt{k_0 a}$$

$$2\xi \omega_n = k_0$$

$$\xi = \frac{k_0}{2\omega_n} = \frac{k_0}{2\sqrt{k_0 a}} = \frac{1}{2} \sqrt{\frac{k_0}{a}}$$

Consider again the step function message $m(t) = Au(t)$. Then

$$\phi(t) = k_\omega \int_0^t Au(\tau) d\tau = Ak_\omega \int_0^t d\tau = Ak_\omega t$$
In the complex frequency domain,

\[ \Phi(s) = \frac{k_\omega A}{s^2} \]

Let \( \Delta \omega \equiv k_\omega A \), then \( \Phi(s) = \Delta \omega / s^2 \)

and

\[ \Psi(s) = \frac{\Delta \omega}{s^2 + 2 \xi \omega_n s + \omega_n^2} \]

If \( 0 < \xi < 1 \), then

\[ \psi(t) = \frac{\Delta \omega}{\omega_n \sqrt{1 - \xi^2}} e^{-\xi \omega_n t} \sin\left(\omega_n \sqrt{1 - \xi^2} t\right) \]

and

\[ \lim_{t \to \infty} \psi(t) = 0. \]

Hence the steady-state phase difference error is zero and \( \theta(t) \to \phi(t) \) as \( t \to \infty \).

A typical value of \( \xi \) is 0.707.
Superheterodyne Receiver

Definition: To heterodyne means to combine a radio frequency wave with a locally generated wave of different frequency, in order to produce a new frequency equal to the sum or difference of the two.

Specifically, a superheterodyne receiver is one that performs the operations of carrier frequency tuning of the desired signal, filtering it to separate it from unwanted signals, in most instances, amplifying it to compensate for signal power loss due to propagation medium.

Generic superheterodyne receiver
Example: Let $m(t)$ modulate a sinusoidal carrier with frequency $f_c = 10$ MHz. Let the bandwidth of the modulated carrier be $B_T = 200$ kHz and let $f_{IF} = 1$ MHz. Let the local oscillator frequency be $f_{LO} = 11$ MHz and let an interferer have its spectrum located at $11.95 \leq f \leq 12.05$ MHz. Then, if no bandpass filter is used in the RF section, at the output of the RF block and of the mixer we have (for $f \geq 0$).

Spectra when no RF filter is used