Coherent Phase-Shift Keying

Consider first Binary Phase-Shift Keying (BPSK). In this case, the transmitted signal is described by

\[ s_i(t) = \sqrt{\frac{2E_b}{T_b}} \cos \left( \omega_c t + (i-1)\pi \right), \quad i = 1, 2, \ 0 \leq t \leq T_b, \quad \omega_c = \frac{2n\pi}{T_b}, \]

where \( n \) is a positive integer.

That is, when \( i = 1 \), a “1” is transmitted, or the pulse

\[ s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos (\omega_c t), \quad 0 \leq t \leq T_b, \]

is transmitted and when \( i = 2 \), a “0” is transmitted, or the pulse

\[ s_2(t) = \sqrt{\frac{2E_b}{T_b}} \cos (\omega_c t + \pi) = -\sqrt{\frac{2E_b}{T_b}} \cos (\omega_c t), \quad 0 \leq t \leq T_b, \]

is transmitted.
Let $\phi_1(t) = \sqrt{2/T_b} \cos(\omega_c t)$, $0 \leq t \leq T_b$. Then $s_1(t) = \sqrt{E_b} \phi_1(t)$ and

$s_2(t) = -\sqrt{E_b} \phi_1(t), 0 \leq t \leq T_b$. So, the signal space $S$ is one-dimensional or $K = 1$. Also, $M = 2$ and the coordinates of the message signals are

$$s_{11} = \int_0^{T_b} s_1(t) \phi_1^*(t) dt = \sqrt{E_b} \int_0^{T_b} \phi_1(t)\phi_1^*(t) dt = \sqrt{E_b} \int_0^{T_b} \left| \phi_1(t) \right|^2 dt$$

$$= \sqrt{E_b} \int_0^{T_b} 2 \cos^2(\omega_c t) dt = \frac{2}{T_b} \sqrt{E_b} \int_0^{T_b} \left[ \frac{1 + \cos(2\omega_c t)}{2} \right] dt$$

$$= \sqrt{E_b}.$$ 

and $s_{21} = \int_0^{T_b} s_2(t) \phi_1^*(t) dt = -\sqrt{E_b}$

The signal constellation for BPSK is shown in the next figure. Note that this signal constellation is the same as that of bi-level ASK, i.e., when $m_1 = \sqrt{E_b}$ and $m_2 = -\sqrt{E_b}$. 

131
BPSK Performance Evaluation.

Now, consider the following coherent receiver in AWGN

BPSK coherent correlator receiver
The following figure shows the effect of additive white Gaussian noise on the transmitted BPSK waveform.

Assuming symbol \( m_1 \) was transmitted, the probability of a correct decision is

\[
P\{X \text{ lies in } R_1 \mid m_1 \text{ sent}\} = \int_0^\infty f_{X|m_1}(x \mid m_1) \, dx = \int_0^\infty \frac{1}{\sqrt{\pi N_0}} e^{-\frac{1}{N_0}(x-\sqrt{E_b})^2} \, dx.
\]

Let \( u = \frac{x - \sqrt{E_b}}{\frac{\sqrt{N_0}}{2}} \), then \( du = \frac{1}{\sqrt{N_0}} dx \) and
\[ P\{ X \text{ lies in } R_1 \mid m_1 \text{ sent} \} = \frac{1}{\sqrt{\pi N_0}} \int_{\frac{-2E_b}{\sqrt{N_0}}}^{\infty} e^{-u^2/2} \sqrt{\frac{N_0}{2}} \, du \]

\[ = \frac{1}{\sqrt{2\pi}} \left[ \int_{\frac{-2E_b}{\sqrt{N_0}}}^{\infty} e^{-u^2/2} \, du - \int_{-\infty}^{1} e^{-u^2/2} \, du - \int_{-\infty}^{\frac{-2E_b}{\sqrt{N_0}}} e^{-u^2/2} \, du \right] \]

\[ = 1 - Q\left( \frac{2E_b}{\sqrt{N_0}} \right). \]

Likewise, the probability of a correct decision, given that \( m_2 \) was transmitted is given by

\[ P\{ X \text{ lies in } R_2 \mid m_2 \text{ sent} \} = \int_{-\infty}^{0} f_{X \mid m_2}(x \mid m_2) \, dx = \int_{-\infty}^{0} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{1}{N_0}(x+\sqrt{E_b})^2} \, dx \]

\[ = 1 - Q\left( \frac{2E_b}{\sqrt{N_0}} \right). \]
Finally, the average bit error probability (BER) $P\{\text{error occurs}\}$, assuming equal probability of occurrence, i.e., $P\{m_1 \text{ sent}\} = P\{m_2 \text{ sent}\} = \frac{1}{2}$, is equal to

$$P\{\text{error occurs}\} = 1 - P\{\text{correct decision}\}$$

$$= 1 - P\{\{\text{choose } m_1 \text{ and } m_1 \text{ sent}\} \cup \{\text{choose } m_2 \text{ and } m_2 \text{ sent}\}\}$$

$$= 1 - \left[ P\{\text{choose } m_1 \text{ and } m_1 \text{ sent}\} + P\{\text{choose } m_2 \text{ and } m_2 \text{ sent}\} \right]$$

$$= 1 - \left[ P\{\text{choose } m_1 | m_1 \text{ sent}\} P\{m_1 \text{ sent}\} + P\{\text{choose } m_2 | m_2 \text{ sent}\} P\{m_2 \text{ sent}\} \right]$$

$$= 1 - \left[ P\{X \text{ lies in } R_1 | m_1 \text{ sent}\} \cdot \frac{1}{2} + P\{X \text{ lies in } R_2 | m_2 \text{ sent}\} \cdot \frac{1}{2} \right]$$

$$= 1 - \frac{1}{2} \left[ 2 \left( 1 - Q\left( \sqrt{\frac{2E_b}{N_0}} \right) \right) \right] = Q\left( \sqrt{\frac{2E_b}{N_0}} \right).$$

The following plots show the effect of AWGN on the detector. Such effects will lead to erroneous decisions.
The following plot shows the average BER performance of BPSK in AWGN.


**Power Spectral Density of BPSK**

BPSK is generated by

\[
\{b_n\} \xrightarrow{\text{Polar non-return to zero encoder}} \{g_n(t)\} \xrightarrow{\text{Product modulator}} s_i(t) \quad (BPSK)
\]

\[
cos(\omega_c t)
\]

**Generation of BPSK**

Under the assumption that the bits occur on a purely random basis and that they are statistically independent of each other, the baseband power spectrum is given by

\[
S_g(f) = \frac{1}{T_b} |\Im\{g_n(t)\}|^2 = \frac{1}{T_b} \left| 2 \sqrt{\frac{2E_b}{T_b}} \frac{\sin(\omega T_b/2)}{\omega} e^{-j \omega T_b/2} \right|^2
\]

\[
= \frac{1}{T_b} \cdot 4 \left( \frac{2E_b}{T_b} \right) \frac{\sin^2(\pi f T_b)}{(2\pi f)^2}
\]

\[
= 2E_b \frac{\sin^2(\pi f T_b)}{(\pi f T_b)^2} = 2E_b \text{sinc}^2(f T_b).
\]
Hence, the passband power spectrum is described by

\[
S_{s_i}(f) = \frac{1}{4} \left[ S_g(f - f_c) + S_g(f + f_c) \right] = \frac{E_b}{2} \left[ \text{sinc}^2 \left( (f - f_c)T_b \right) + \text{sinc}^2 \left( (f + f_c)T_b \right) \right]
\]

The power spectra shown in the next two figures assume \( E_b = 1 \) and \( T_b = 10 / f_c \).

PSD of BPSK
PSD of BPSK in dB
Quadriphase – Shift Keying (QPSK)

Let \( s(t) \triangleq A_1 m_1(t) \cos(2\pi f_c t + \theta) - A_2 m_2(t) \sin(2\pi f_c t + \theta) \),

where \( \theta \) = phase shift

\[
m_i(t) = \text{data signal} = \pm 1, \quad 0 \leq t \leq T_s, \quad i = 1, 2, \quad T_s = 2T_b
\]

\( A_i, \quad i = 1, 2 \) are constants

\( f_c = \text{carrier frequency} = \frac{n_c}{T_s} \)

If both \( m_1(t) \) and \( m_2(t) \) change signs at the same time instants and \( A_1 = A_2 = \frac{A}{\sqrt{2}} \), then \( s(t) \) is a QPSK signal, i.e.,

\[
s(t) = \frac{A}{\sqrt{2}} m_1(t) \cos(2\pi f_c t + \theta) - \frac{A}{\sqrt{2}} m_2(t) \sin(2\pi f_c t + \theta)
\]

On the other hand, if the time when \( m_2(t) \) changes sign is offset by \( T_s / 2 = T_b \) from the time \( m_1(t) \) changes sign and \( A_1 = A_2 = A \), then \( s(t) \) is an Offset Quadrature Phase-Shift Keyed (OQPSK) signal, i.e.,

\[
s(t) = A m_1(t) \cos(2\pi f_c t + \theta) - A m_2(t - T_b) \sin(2\pi f_c t + \theta).
\]
(a) Data input

(b) QPSK I and Q data

(c) OQPSK I and Q data
Let $\theta = 0$, then for QPSK with $A = \sqrt{\frac{2E_s}{T_s}}$,

$$s_i(t) = \begin{cases} \sqrt{\frac{2E_s}{T_s}} \cos \left[ 2\pi f_c t + (2i - 1)\frac{\pi}{4} \right], & 0 \leq t \leq T_s \\ 0, & \text{otherwise} \end{cases}$$

where $i = 1, 2, 3, 4$, $E_s$ is the symbol energy and $T_s$ is the symbol duration. The carrier frequency $f_c$ is chosen so that $f_c = n_c / T_s$, $n_c$ is a positive integer. Note that there are four symbols because four distinct combinations occur when $m_1(t) = \pm 1$ and $m_2(t) = \pm 1$ over $T_s = 2T_b$ seconds. Explicitly,

<table>
<thead>
<tr>
<th>$m_1(t)$</th>
<th>$m_2(t)$</th>
<th>Symbol (dibit)</th>
<th>$\theta(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$11 - M_1$</td>
<td>$\pi / 4$</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>$01 - M_2$</td>
<td>$3\pi / 4$</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>$00 - M_3$</td>
<td>$5\pi / 4$</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>$10 - M_4$</td>
<td>$7\pi / 4$</td>
</tr>
</tbody>
</table>
Also, in the interval $0 \leq t \leq T_s$,

$$s_i(t) = \sqrt{\frac{2E_s}{T_s}} \cos \left[ (2i-1)\frac{\pi}{4} \right] \cos(2\pi f_c t) - \sqrt{\frac{2E_s}{T_s}} \sin \left[ (2i-1)\frac{\pi}{4} \right] \sin(2\pi f_c t), \quad i = 1, 2, 3, 4,$$

or

$$s_i(t) = \sqrt{E_s} \cos \left[ (2i-1)\frac{\pi}{4} \right] \phi_1(t) - \sqrt{E_s} \sin \left[ (2i-1)\frac{\pi}{4} \right] \phi_2(t), \quad i = 1, 2, 3, 4,$$

where $\phi_1(t) = \sqrt{\frac{2}{T_s}} \cos(2\pi f_c t)$ and $\phi_2(t) = \sqrt{\frac{2}{T_s}} \sin(2\pi f_c t)$,

and

$$\int_0^{T_s} \phi_j(t) \phi_k^* dt = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}, \quad j, k = 1, 2.$$
Now,

\[ s_i(t) = \begin{cases} 
\sqrt{\frac{E_s}{2}} \phi_1(t) - \sqrt{\frac{E_s}{2}} \phi_2(t), & i = 1 \\
-\sqrt{\frac{E_s}{2}} \phi_1(t) - \sqrt{\frac{E_s}{2}} \phi_2(t), & i = 2 \\
-\sqrt{\frac{E_s}{2}} \phi_1(t) + \sqrt{\frac{E_s}{2}} \phi_2(t), & i = 3 \\
\sqrt{\frac{E_s}{2}} \phi_1(t) + \sqrt{\frac{E_s}{2}} \phi_2(t), & i = 4 
\end{cases} \]

Clearly, \( s_i = \begin{bmatrix} \sqrt{E_s} \cos \left( (2i-1)\frac{\pi}{4} \right) & -\sqrt{E_s} \sin \left( (2i-1)\frac{\pi}{4} \right) \end{bmatrix}^T \)
The signal constellation diagrams for both QPSK and OQPSK are shown in the next two figures.

QPSK signal constellation and allowed phase shifts
OQPSK signal constellation and allowed phase shifts
Phase shifts in QPSK and OQPSK
QPSK Performance Evaluation: Consider the following coherent receiver:

$$ \Sigma \quad s_i(t) \quad x(t) \quad W(t) \quad \phi_1^*(t) \quad y_1(t) \quad X_1 $$

$$ t = T_s $$

$$ \phi_2^*(t) \quad y_2(t) \quad X_2 $$

with $W(t)$ WGN with zero mean and p.s.d. $S_W(f) = \frac{N_0}{2} \frac{\text{Watts}}{\text{Hz}}$. 

$$ \int_0^{T_s} (\cdot) dt $$
Assuming all 4 symbols are transmitted with equal probability, the observation space at the output of the QPSK demodulator and the decision regions are shown in the next figure.

Observation space at the output of coherent QPSK demodulator and its decision regions
\[ y_1(t) = \int_0^{T_s} x(t)\phi_i^*(t)dt = \int_0^{T_s} \left[ s_i(t) + W(t) \right] \phi_i^*(t)dt \]

\[ = \int_0^{T_s} s_i(t)\phi_i^*(t)dt + \int_0^{T_s} W(t)\phi_i^*(t)dt = \sqrt{E_s} \cos \left( (2i - 1) \frac{\pi}{4} \right) + W_1 = X_1 \]

Now, \[ E \{ y_1(t) | M_i \} = \sqrt{E_s} \cos \left( (2i - 1) \frac{\pi}{4} \right) + E \{ W_1 \} \]

\[ E \{ W_1 \} = E \left\{ \int_0^{T_s} W(t)\phi_i^*(t)dt \right\} = \int_0^{T_s} E \{ W(t) \} \phi_i^*(t)dt = 0 \]

\[ \Rightarrow E \{ y_1(t) | M_i \} = \sqrt{E_s} \cos \left( (2i - 1) \frac{\pi}{4} \right) = E \{ X_1 | M_i \} \]

\[ Var(X_1 | M_i) = E \left\{ |X_1 - E \{ X_1 | M_i \}|^2 | M_i \right\} = E \{ |W_1|^2 \} = E \left\{ \left( \int_0^{T_s} W(t)\phi_i^*(t)dt \right) \left( \int_0^{T_s} W(\tau)\phi_i^*(\tau)d\tau \right)^* \right\} \]

\[ = E \left\{ \int_0^{T_s} \int_0^{T_s} W(t)W^*(\tau)\phi_i^*(t)\phi_i(\tau)dtd\tau \right\} = \int_0^{T_s} \int_0^{T_s} E \{ W(t)W^*(\tau) \} \phi_i^*(t)\phi_i(\tau)dtd\tau \]

\[ = \int_0^{T_s} \int_0^{T_s} \frac{N_0}{2} \delta(t-\tau)\phi_i^*(t)\phi_i(\tau)dtd\tau = \frac{N_0}{2} \int_0^{T_s} \phi_i^*(\tau)\phi_i(\tau)d\tau = \frac{N_0}{2} \int_0^{T_s} |\phi_i(\tau)|^2 d\tau = \frac{N_0}{2} \]
\[
\Rightarrow X_1 | M_i \sim G\left(\sqrt{E_s} \cos\left(\frac{(2i-1)\pi}{4}\right), \frac{N_0}{2}\right), i = 1, 2, 3, 4.
\]

Likewise,
\[
X_2 | M_i \sim G\left(-\sqrt{E_s} \sin\left(\frac{(2i-1)\pi}{4}\right), \frac{N_0}{2}\right), i = 1, 2, 3, 4.
\]

**Decision Rule:**

Using the ML criterion, if the observed measurement \( X = \bar{x} = [x_1 \quad x_2]^T \) falls in region \( R_i, i = 1, 2, 3, 4 \), message \( M_i \) is selected. A correct decision is made if symbol \( M_i \) is selected when symbol \( M_i \) was transmitted. An erroneous decision is made if the sampled observation vector \( \bar{x} \) falls outside region \( R_i \) when message \( M_i \) was transmitted.

Since the regions are symmetrical because of equal probability of symbol occurrence at the transmitter, the probability of interpreting the received signal vectors correctly is the same regardless of which particular signals were actually transmitted.
Suppose message $M_4$ was transmitted, then the probability of making the correct decision $P_c$ is the conditional probability that $X_1 > 0$ and $X_2 > 0$, given that $M_4$ was sent. The r.v.’s $X_1$ and $X_2$ are independent, thus

$$P_{c|M_4} = P\{X_1 > 0, \; X_2 > 0 \mid M_4\} = P\{X_1 > 0, \; X_2 > 0 \mid m_1 = 1, m_2 = -1\}$$

$$= P\{X_1 > 0 \mid m_1 = 1\} \cdot P\{X_2 > 0 \mid m_2 = -1\}.$$

But, $f_{X_1|m_1}(x|m_1 = 1) = \frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{1}{N_0} \left(x - \sqrt{\frac{E_s}{2}}\right)^2\right\}$

So,

$$P\{X_1 > 0 \mid m_1 = 1\} = \int_0^\infty f_{X_1|m_1}(x \mid m_1) dx = \int_0^\infty \frac{1}{\sqrt{\pi N_0}} e^{-\frac{1}{N_0} \left(x - \sqrt{\frac{E_s}{2}}\right)^2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{\frac{E_s}{N_0}}}^{\infty} e^{-\frac{u^2}{2}} du$$

$$= 1 - Q\left(\sqrt{\frac{E_s}{N_0}}\right) = P\{X_2 > 0 \mid m_2 = -1\}$$

Hence, $P_{c|M_4} = \left[1 - Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right]^2$ and $P(c \cap M_4) = P_{c|M_4} P(M_4) = \left[1 - Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right]^2 \cdot \left(\frac{1}{4}\right)$. 

153
Since all four symbols are equiprobable, then

\[
P_c = \sum_{i=1}^{4} P(c \cap M_i) = \sum_{i=1}^{4} P_{c|M_i} P(M_i) = \sum_{i=1}^{4} P_{c|M_i} \left( \frac{1}{4} \right) = \left( \frac{1}{4} \right) \sum_{i=1}^{4} \left[ 1 - Q\left( \frac{E_s}{\sqrt{N_0}} \right) \right]^2 = \left[ 1 - Q\left( \frac{E_s}{\sqrt{N_0}} \right) \right]^2.
\]

The symbol error probability (SER) is therefore given by

\[
SER = P_e = 1 - P_c = 1 - \left[ 1 - Q\left( \frac{E_s}{\sqrt{N_0}} \right) \right]^2
\]

\[
= 1 - \left[ 1 - 2Q\left( \frac{E_s}{\sqrt{N_0}} \right) + Q^2\left( \frac{E_s}{\sqrt{N_0}} \right) \right] = 2Q\left( \frac{E_s}{\sqrt{N_0}} \right) - Q^2\left( \frac{E_s}{\sqrt{N_0}} \right).
\]

If \( \frac{E_s}{N_0} \gg 1 \), \( Q^2\left( \frac{E_s}{\sqrt{N_0}} \right) \approx 0 \) and \( SER = P_e \approx 2Q\left( \frac{E_s}{\sqrt{N_0}} \right) \).

Furthermore, in QPSK, \( E_s = 2E_b \Rightarrow SER = P_e \approx 2Q\left( \sqrt{\frac{2E_b}{N_0}} \right) \).
The following plot shows QPSK system performance in the presence of AWGN (as well as that of BPSK for comparison purposes).
If Gray encoding were used for the dibits (bit pairs), then $P_e$ would be halved (find this out!) and its performance would be the same as BPSK, even though QPSK only uses half the bandwidth.

In principle, QPSK can be generated as follows:

![Generation of QPSK](image)
Over a signaling period $T_s$, $a_j(t) = \pm \sqrt{E_b} = \pm \sqrt{\frac{E_s}{2}}$, $j = 1, 2$.

Now, $s_i(t) = i(t) \cos(2\pi f_c t) - q(t) \sin(2\pi f_c t)$,

where $i(t) = \begin{cases} \pm \sqrt{\frac{E_s}{T_s}}, & 0 \leq t \leq T_s \\ 0, & \text{otherwise} \end{cases}$ and $q(t) = \begin{cases} \pm \sqrt{\frac{E_s}{T_s}}, & 0 \leq t \leq T_s \\ 0, & \text{otherwise} \end{cases}$

$i(t)$ and $q(t)$ are independent of each other. Therefore, the p.s.d. of $s_i(t)$ is given by

$$S_{s_i}(f) = \frac{1}{4} S_i(f + f_c) + \frac{1}{4} S_q(f + f_c) + \frac{1}{4} S_i(f - f_c) + \frac{1}{4} S_q(f - f_c)$$

$$= \frac{1}{2} [S_i(f + f_c) + S_i(f - f_c)] = \frac{1}{2} [S_q(f + f_c) + S_i(f - f_c)],$$

since $S_i(f) = S_q(f) = 2E_s \text{sinc}^2(T_s f) = 4E_b \text{sinc}^2(2T_b f)$

$$\Rightarrow S_{s_i}(f) = 2E_b \left[ \text{sinc}^2(2T_b (f + f_c)) + \text{sinc}^2(2T_b (f - f_c)) \right].$$
The previous PSD assumes $E_b = 1$ and $T_b = 10 / f_c$.
Coherent M-ary PSK:

Here the signal set is represented by

\[
s_i(t) = \sqrt{\frac{2E_s}{T_s}} \cos \left[ 2\pi f_c t + \frac{2(i-1)}{M} \pi \right], \quad 0 \leq t \leq T_s, \quad i = 1, \ldots, M
\]

= 0, otherwise

As in QPSK, \( E_s \) is the symbol energy, \( T_s \) is the symbol duration, and the carrier frequency is given by \( f_c = n_c/T_s \). Over a signaling period,

\[
s_i(t) = \sqrt{\frac{2E_s}{T_s}} \cos \left[ 2(i-1) \frac{\pi}{M} \right] \cos(2\pi f_c t) - \sqrt{\frac{2E_s}{T_s}} \sin \left[ 2(i-1) \frac{\pi}{M} \right] \sin(2\pi f_c t).
\]

Let \( \phi_1(t) = \sqrt{2} \cos \omega_t \) and \( \phi_2(t) = \sqrt{2} \sin \omega_t \), \( 0 \leq t \leq T_s \). Then

\[
s_i(t) = \sqrt{E_s} \cos \left[ 2(i-1) \frac{\pi}{M} \right] \phi_1(t) - \sqrt{E_s} \sin \left[ 2(i-1) \frac{\pi}{M} \right] \phi_2(t), \quad i = 1, \ldots, M.
\]
Consider 8-PSK, then the signal constellation is as shown in the next figure.

8-PSK signal constellation
Let the receiver have the same structure as that used for QPSK signals, then for $M = 8$ we have

$$X_1 = \int_0^{T_s} [s_i(t) + W(t)] \phi_i^*(t) \, dt = \sqrt{E_s} \cos \left( \frac{(i-1)\pi}{4} \right) + W_i, \ i = 1, \ldots, 8,$$

$$X_2 = \int_0^{T_s} [s_i(t) + W(t)] \phi_2^*(t) \, dt = -\sqrt{E_s} \sin \left( \frac{(i-1)\pi}{4} \right) + W_i, \ i = 1, \ldots, 8,$$

where

$$W_k = \int_0^{T_s} W(t) \phi_k^*(t) \, dt, \ k = 1, 2$$

$$\therefore X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} s_{i1} + W_1 \\ s_{i2} + W_2 \end{bmatrix} = \begin{bmatrix} \sqrt{E_s} \cos \left( \frac{(i-1)\pi}{4} \right) + W_1 \\ -\sqrt{E_s} \sin \left( \frac{(i-1)\pi}{4} \right) + W_2 \end{bmatrix}.$$
If all symbols are transmitted with equal probability, the decision regions are symmetric (see next figure).

Observation space and decision regions for 8-PSK
Let’s now compute the symbol error rate $P_e$.

$$P\{\text{correct decision} \mid m_i \text{ sent}\} = P\{X \text{ lies in } R_i \mid m_i \text{ sent}\}$$

$$= \int \int_{R_i} f_{X|m_i}(x|m_i) dx_1dx_2,$$

where

$$f_{X|m_i}(x|m_i) = \frac{1}{\pi N_0} \exp\left\{-\frac{1}{N_0} \sum_{j=1}^{2}(x_j - s_{ij})^2\right\}, i = 1, \ldots, 8.$$

As in the QPSK case,

$$E\{X \mid m_i\} = \begin{bmatrix} \sqrt{E_s} \cos \left[\frac{(i-1)\pi}{4}\right] \\ -\sqrt{E_s} \sin \left[\frac{(i-1)\pi}{4}\right] \end{bmatrix}$$

and

$$E\{(X - S_i)(X - S_i)^T\} = \begin{bmatrix} N_0/2 & 0 \\ 0 & N_0/2 \end{bmatrix}.$$
Since the message points exhibit circular symmetry, $P_{c|m_i}$ is independent of $m_i$. So, let’s consider $m_1$ only.

$$P_{c|m_1} = P\left\{ X \text{ lies in } R_1 | m_1 \text{ sent} \right\} = \int_{0}^{\infty} \int_{-\tan(\pi/8) x_1}^{\tan(\pi/8) x_1} \frac{1}{\pi N_0} e^{-\frac{1}{N_0} \sum_{j=1}^{2} (x_j - s_{ij})^2} \, dx_2 \, dx_1$$

$$= \int_{0}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{1}{N_0} (x_1 - s_{11})^2} \tan(\pi/8) x_1 \int_{-\tan(\pi/8) x_1}^{\tan(\pi/8) x_1} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{1}{N_0} (x_2 - s_{12})^2} \, dx_2 \, dx_1$$

$$= \frac{2}{\pi N_0} \int_{0}^{\infty} e^{-\frac{1}{N_0} (x_1 - \sqrt{E_0})^2 \tan(\pi/8) x_1} \int_{0}^{\infty} e^{-x_2^2/N_0} \, dx_2 \, dx_1, \text{ since } s_{12} = 0.$$

The last two integrals cannot be solved analytically and an alternate representation has to be found.

Define $\Theta \equiv \tan^{-1} (X_2 / X_1)$, then $\Theta \in (-\pi/8, \pi/8)$ implies that $m_1$ was sent. Thus, if

if we let $P_{c|m_1} \triangleq P\left\{ X \text{ lies in } R_1 | m_1 \text{ sent} \right\}$, then

$$P_{c|m_1} = P\left\{ \Theta \text{ lies in } \left( -\frac{\pi}{8}, \frac{\pi}{8} \right) | m_1 \right\} = \int_{-\pi/8}^{\pi/8} f_{\Theta|m_1}(\theta | m_1) \, d\theta.$$
If $m_1$ is sent, then $X_1 = \sqrt{E_s} + W_1$ and $X_2 = W_2$. Hence, the conditional probability of $\Theta$ given $m_1$ was transmitted can be obtained as follows:

Let $D = \sqrt{X_1^2 + X_2^2}$, then $X_1 = D \cos \Theta$ and $X_2 = D \sin \Theta$.

Also, $f_{\Theta,D|m_1}(\theta,d|m_1) = f_{X|m_1}(x|m_1) \left| J \begin{pmatrix} x_1 & x_2 \\ d & \theta \end{pmatrix} \right|_{x_1 = d \cos \theta, x_2 = d \sin \theta}$

where $J \begin{pmatrix} x_1 & x_2 \\ d & \theta \end{pmatrix} = \begin{vmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{vmatrix} = d \cos^2 \theta + d \sin^2 \theta = d$

so,

$f_{\Theta,D|m_1}(\theta,d|m_1) = \frac{d}{\pi N_0} e^{-\frac{1}{N_0} \left( (d \cos \theta - \sqrt{E_s})^2 + d^2 \sin^2 \theta \right)} = \frac{d}{\pi N_0} e^{-\frac{1}{N_0} \left( E_s + d^2 - 2 \sqrt{E_s} d \cos \theta \right)}$

and

$f_{\Theta|m_1}(\theta|m_1) = \int_0^\infty f_{\Theta,D|m_1}(\theta,d|m_1) dd = \frac{1}{\pi N_0} e^{-\frac{E_s}{N_0}} \int_0^\infty \frac{d}{\pi N_0} e^{-\frac{1}{N_0} \left( d^2 - 2 \sqrt{E_s} d \cos \theta \right)} dd$. 

165
Completing the square of the exponent of the integrand, yields

\[ f_{\Theta|m_1}(\theta|m_1) = \frac{1}{\pi N_0} e^{-\frac{E_s}{N_0} \int_0^\infty de \left[ -\frac{1}{N_0} \left( (d - \sqrt{E_s} \cos \theta)^2 - E_s \cos^2 \theta \right) \right] dd} \]

\[ = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{E_s}{N_0} \left( 1 - \cos^2 \theta \right)} \left[ \frac{1}{\sqrt{\pi N_0}} \int_0^\infty de \left[ -\frac{1}{N_0} \left( (d - \sqrt{E_s} \cos \theta)^2 \right) \right] dd \right]. \]

Let \( \frac{r}{\sqrt{2}} = \frac{d - \sqrt{E_s} \cos \theta}{\sqrt{N_0}} \), then \( dr = \frac{2}{\sqrt{N_0}} dd \) and

\[ f_{\Theta|m_1}(\theta|m_1) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{E_s}{N_0} \frac{\sin^2 \theta}{2}} \left[ \frac{1}{\sqrt{\pi N_0}} \int_0^\infty \left\{ \sqrt{\frac{N_0}{2}} r + \sqrt{E_s} \cos \theta \right\} e^{-\frac{r^2}{2} \sqrt{\frac{N_0}{2}}} dr \right] \]

\[ = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{E_s}{N_0} \frac{\sin^2 \theta}{2}} \left[ \frac{1}{2 \sqrt{\pi}} \int_0^\infty re^{-\frac{r^2}{2}} dr + \sqrt{E_s} \cos \theta \cdot \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{r^2}{2}} dr \right] \]

\[ = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{E_s}{N_0} \frac{\sin^2 \theta}{2}} \left[ \frac{1}{2 \sqrt{\pi}} \int_0^\infty re^{-\frac{r^2}{2}} dr + \sqrt{E_s} \cos \theta \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{r^2}{2}} dr \right] \]
For large $\frac{E_s}{N_0}$ and small $\theta$, \[ \sqrt{\frac{N_0}{4\pi}} \int_{-\frac{2E_s}{N_0}\cos\theta}^{\infty} re^{-\frac{r^2}{2}} dr \approx 0 \] and

\[ f_{\Theta|m_1}(\theta|m_1) \approx \sqrt{\frac{E_s}{\pi N_0}} \cos\theta e^{-\frac{E_s\sin^2\theta}{N_0}} \cdot \left[ 1 - Q\left(\sqrt{\frac{2E_s}{N_0}} \cos\theta\right) \right]. \]

Again, for $\frac{E_s}{N_0}$ large and $\theta$ small,

\[ 1 - Q\left(\sqrt{\frac{2E_s}{N_0}} \cos\theta\right) \approx 1. \]

Hence,

\[ f_{\Theta|m_1}(\theta|m_1) \approx \sqrt{\frac{E_s}{\pi N_0}} \cos\theta e^{-\frac{E_s\sin^2\theta}{N_0}} \cdot \left[ 1 - Q\left(\sqrt{\frac{2E_s}{N_0}} \cos\theta\right) \right]. \]

Therefore, the probability of a correct decision, given that $m_1$ was sent is

\[ P_c|m_1 = \int_{-\pi/8}^{\pi/8} f_{\Theta|m_1}(\theta|m_1) d\theta \approx \int_{-\pi/8}^{\pi/8} \sqrt{\frac{E_s}{\pi N_0}} \cos\theta e^{-\frac{E_s\sin^2\theta}{N_0}} d\theta. \]
Let \( u = \sqrt{\frac{2E_s}{N_0}} \sin \theta \), then \( du = \sqrt{\frac{2E_s}{N_0}} \cos \theta \, d\theta \) and

\[
P_c \mid m_1 \approx \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{\frac{2E_s}{N_0}} \sin \left( \frac{\pi}{8} \right)}^{\sqrt{\frac{2E_s}{N_0}} \sin \left( \frac{\pi}{8} \right)} e^{-\frac{u^2}{2}} \, du = 2 \left[ \frac{1}{\sqrt{2\pi}} \int_{0}^{\sqrt{\frac{2E_s}{N_0}} \sin \left( \frac{\pi}{8} \right)} e^{-\frac{u^2}{2}} \, du \right] = 2 \left[ \frac{1}{2} - Q \left( \sqrt{\frac{2E_s}{N_0}} \sin \left( \frac{\pi}{8} \right) \right) \right] = 2 \left[ \frac{1}{2} - Q \left( \sqrt{\frac{2E_s}{N_0}} \sin \left( \frac{\pi}{8} \right) \right) \right]
\]

or

\[
P_c \mid m_1 = P_c \mid m_i \approx 1 - 2Q \left( \sqrt{\frac{2E_s}{N_0}} \sin \left( \frac{\pi}{8} \right) \right), \ i = 1, 2, \ldots, 8,
\]

because all symbols occur with equal probability, which implies, in this case, symmetric decision regions.
Now,

\[ P_c = \sum_{i=1}^{8} P\{\text{choose } m_i \text{ and } m_i \text{ sent}\} = \sum_{i=1}^{8} P\{\text{choose } m_i|m_i \text{ sent}\}P\{m_i \text{ sent}\} \]

\[ = \frac{1}{8} \sum_{i=1}^{8} P\{\text{choose } m_i|m_i \text{ sent}\} = \frac{1}{8} \sum_{i=1}^{8} P_c|m_i \]

\[ = \frac{1}{8} \sum_{i=1}^{8} P_c|m_i \approx \frac{1}{8} \sum_{i=1}^{8} \left\{ 1 - 2Q\left[\sqrt{\frac{2E_s}{N_0}} \sin\left(\frac{\pi}{8}\right)\right]\right\} = 1 - 2Q\left[\sqrt{\frac{2E_s}{N_0}} \sin\left(\frac{\pi}{8}\right)\right] \]

Hence, the average probability of symbol error is given by

\[ \text{SER} = P_e = 1 - P_c \approx 2Q\left[\sqrt{\frac{2E_s}{N_0}} \sin\left(\frac{\pi}{8}\right)\right]. \]

For arbitrary \( M, M \) a power of 2, \( \text{SER} = P_e \approx 2Q\left[\sqrt{\frac{2E_s}{N_0}} \sin\left(\frac{\pi}{M}\right)\right]. \)
Effect of AWGN on an 8PSK detector output
Power Spectrum of M-ary PSK: We know that \( T_s = T_b \log_2 M \), and that the I and Q outputs are independent of each other. Moreover, even though each symbol contains \( \log_2 M \) bits and energy \( E_s \), the symbol sequence baseband spectrum is equal to the sum of the I and Q baseband spectra, i.e.,

\[
S_B(f) = E_s \sinc^2(T_s f) + E_s \sinc^2(T_s f) = 2E_s \sinc^2(T_s f) = 2E_b \log_2(M) \sinc^2[T_b \log_2(M) \cdot f].
\]

Therefore, the passband spectrum of M-ary PSK is given by

\[
S_{s_i}(f) = \frac{1}{4} S_B(f + f_c) + \frac{1}{4} S_B(f - f_c) = \frac{1}{2} E_s \sinc^2(T_s(f + f_c)) + \frac{1}{2} E_s \sinc^2(T_s(f - f_c)).
\]
Generation of M-ary PSK: For the case of $M = 8$, $s_i(t)$ is generated as follows:

![Diagram of 8-PSK modulator with sequence $\{b_k\}$, 2-to-4 level converters, PAMs, and modulating signals $\cos(\omega_c t)$ and $\sin(\omega_c t)$]
Example: For symbol energy equal to 2 Joules, 

**I-channel**

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<th>I</th>
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<th>Output</th>
</tr>
</thead>
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<tr>
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<td>1.307 V</td>
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**Q-channel**

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<th>$\bar{C}$</th>
<th>Output</th>
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</thead>
<tbody>
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<td>-1.307 V</td>
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<tr>
<td>0</td>
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<tr>
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</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0.541 V</td>
</tr>
</tbody>
</table>

where $C$ is the control bit (component) and the radius of the circle is equal to $\sqrt{2}$.
Coherent Binary FSK (Frequency Shift-Keying)

\[
s_i(t) = \begin{cases} 
\sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_i t), & 0 \leq t \leq T_b, \ i = 1, 2, \\
0, & \text{otherwise}
\end{cases}
\]

where symbol “1” is transmitted when \( i = 1 \) and symbol “0” is transmitted when \( i = 2 \). Moreover,

\[f_i = \frac{n_c + i}{T_b}, \ n_c \text{ is a fixed positive integer, } i = 1, 2.\]

For \( t \in [0, T_b] \), let \( \phi_i(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_i t), i = 1, 2. \) Then both basis functions are orthogonal to each other, i.e.

\[
\int_{0}^{T_b} \phi_i(t)\phi_j^*(t)dt = \begin{cases} 
1, & i = j \\
0, & i \neq j
\end{cases}, \ i, j = 1, 2.
\]

and the two symbols are expressed by \( s_i(t) = \sqrt{E_b}\phi_i(t), i = 1, 2. \)
Clearly, $\mathbf{s}_1 = \begin{bmatrix} \sqrt{E_b} \\ 0 \end{bmatrix}$ and $\mathbf{s}_2 = \begin{bmatrix} 0 \\ \sqrt{E_b} \end{bmatrix}$.

The Euclidean distance between $\mathbf{s}_1$ and $\mathbf{s}_2$ is $d = \sqrt{(\sqrt{E_b} - 0)^2 + (0 - \sqrt{E_b})^2} = \sqrt{2E_b}$.