Consider the following receiver architecture:

Suppose \( s_1(t) \) is sent, then

\[
X_1 = \int_0^{T_b} \left[ s_1(t) + w(t) \right] \phi_1^*(t) \, dt = \int_0^{T_b} \sqrt{E_b} \phi_1(t) \phi_1^*(t) \, dt + \int_0^{T_b} w(t) \phi_1^*(t) \, dt
= \sqrt{E_b} + W_1
\]

\[
X_2 = \int_0^{T_b} \left[ s_1(t) + w(t) \right] \phi_2^*(t) \, dt = \int_0^{T_b} \sqrt{E_b} \phi_1(t) \phi_2^*(t) \, dt + \int_0^{T_b} w(t) \phi_2^*(t) \, dt
= W_2
\]
Likewise, if $s_2(t)$ is sent, then $X_1 = W_1$ and $X_2 = \sqrt{E_b} + W_2$.

Suppose all symbols are transmitted with equal probability, then the decision regions in the observation 2-dimensional space are symmetric (see next figure).
**Decision strategy**: choose $M_1$ if the random observation vector $X = [X_1 \ X_2]^T$ falls in $R_1$ and choose $M_2$ if $X$ falls in $R_2$. Equivalently, choose $M_1$ if $X_1 > X_2$ or choose $M_2$ if $X_2 > X_1$.

Alternatively, if we choose $Y = X_1 - X_2$ as the decision variable, then a decision strategy based on a single variable can be implemented, namely, select $M_1$ if $Y > 0$, otherwise, select $M_2$. Since both $X_1$ and $X_2$ are Gaussian, then $Y$ is also Gaussian.

Now,

$$E\{Y|M_1\} = E\{X_1 - X_2|M_1\} = E\{X_1|M_1\} - E\{X_2|M_1\} = E\{\sqrt{E_b} + W_1|M_1\} - E\{W_2|M_1\}$$

$$= E\{\sqrt{E_b} + W_1\} - E\{W_2\} = E\{\sqrt{E_b}\} + E\{W_1\} - 0 = \sqrt{E_b} + 0 = \sqrt{E_b}$$

$$E\{Y|M_2\} = E\{X_1|M_2\} - E\{X_2|M_2\} = E\{W_1|M_2\} - E\{\sqrt{E_b} + W_2|M_2\}$$

$$= E\{W_1\} - E\{\sqrt{E_b} + W_2\} = 0 - (\sqrt{E_b} + 0) = -\sqrt{E_b}$$
\[
E\left\{ \left| Y - E\left\{ Y \mid M_1 \right\} \right|^2 \mid M_1 \right\} = E\left\{ \left| \sqrt{E_b} + W_1 - W_2 - \sqrt{E_b} \right|^2 \mid M_1 \right\} = E\left\{ \left| W_1 - W_2 \right|^2 \mid M_1 \right\}
\]
\[
= E\left\{ \left| W_1 - W_2 \right|^2 \right\} = E\left\{ \left| W_1 \right|^2 \right\} - E\left\{ W_1 W_2^* \right\} - E\left\{ W_1^* W_2 \right\} + E\left\{ \left| W_2 \right|^2 \right\}
\]
\[
= \frac{N_o}{2} - 0 - 0 + \frac{N_o}{2} = N_o
\]

and \( E\left\{ \left| Y - E\left\{ Y \mid M_2 \right\} \right|^2 \mid M_2 \right\} = N_o \), because \( E\left\{ W_1 W_2^* \right\} = E\left\{ W_1^* W_2 \right\} = 0 \).

\[\begin{align*}
\therefore \quad Y & \sim G\left( \pm \sqrt{E_b}, N_o \right) \\
f_{Y|M_1}(y \mid m_1) & = \frac{1}{\sqrt{2\pi N_o}} e^{-\frac{1}{2N_o}(y-\sqrt{E_b})^2} \\
and \quad f_{Y|M_2}(y \mid m_2) & = \frac{1}{\sqrt{2\pi N_o}} e^{-\frac{1}{2N_o}(y+\sqrt{E_b})^2}
\end{align*}\]

Thus, the average BER is given by
\[
P_e = P\{Y < 0 \text{ and } M_1 \text{ sent}\} + P\{Y > 0 \text{ and } M_2 \text{ sent}\}
\]
\[
= P\{Y < 0 \mid M_1 \text{ sent}\} P\{M_1 \text{ sent}\} + P\{Y > 0 \mid M_2 \text{ sent}\} P\{M_2 \text{ sent}\}.
\]
If $P\{M_1 \text{ sent}\} = P\{M_2 \text{ sent}\} = \frac{1}{2}$, then

$$P_e = \frac{1}{2} \left[ P\{Y < 0|M_1 \text{ sent}\} + P\{Y > 0|M_2 \text{ sent}\} \right]$$

$$P\{Y < 0|M_1 \text{ sent}\} = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi N_o}} \ e^{-\frac{(y-\sqrt{E_b})^2}{2N_o}} \ dy = \int_{-\infty}^{\sqrt{\frac{E_b}{N_o}}} \frac{1}{\sqrt{2\pi}} \ e^{-\frac{u^2}{2}} \ du$$

$$= \int_{\sqrt{\frac{E_b}{N_o}}}^{\infty} \frac{1}{\sqrt{2\pi}} \ e^{-\frac{v^2}{2}} \ dv = Q\left(\sqrt{\frac{E_b}{N_o}}\right),$$

$$P\{Y > 0|M_2 \text{ sent}\} = \int_{0}^{\infty} \frac{1}{\sqrt{2\pi N_o}} \ e^{-\frac{(y+\sqrt{E_b})^2}{2N_o}} \ dy = \int_{\sqrt{\frac{E_b}{N_o}}}^{\infty} \frac{1}{\sqrt{2\pi}} \ e^{-\frac{u^2}{2}} \ du = Q\left(\sqrt{\frac{E_b}{N_o}}\right)$$

and $P_e = \frac{1}{2} \left[ 2Q\left(\sqrt{\frac{E_b}{N_o}}\right) \right] = Q\left(\sqrt{\frac{E_b}{N_o}}\right)$. 

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Note that coherent FSK has the same performance as OOK.

Generation of coherent binary FSK

An implementation of binary FSK
The baseband power spectrum of coherent binary FSK is given by

$$S_{FSK,B}(f) = \frac{E_b}{2T_b} \left[ \delta \left( f + \frac{1}{2T_b} \right) + \delta \left( f - \frac{1}{2T_b} \right) \right] + \frac{8E_b \cos^2(\pi T_b f)}{\pi^2 \left( 4T_b^2 f^2 - 1 \right)^2}$$

Linear and dB plots of the PSD of binary FSK, $T_b = 1\text{s}$, are shown in the next two figures.
PSD of binary FSK in dB/Hz
Differential Phase-Shift Keying (DPSK)

It is the noncoherent version of PSK. Thus, it can be noncoherently demodulated at the receiver. Briefly, to send “0”, we phase advance the current signal waveform by $\pi$ radians, and to send a “1” we leave the phase of the current signal waveform unchanged. To achieve this, we must process the input bit stream at the logical level by performing a simple pre-coding operation.

Suppose a “1” is transmitted in the interval $[0, T_b]$, then the symbol is either described by

$$s_1(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t), & 0 \leq t \leq T_b \\ \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t), & T_b \leq t \leq 2T_b \end{cases}$$

when a “1” is transmitted in $t \in [T_b, 2T_b]$
or

\[ s_2(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t), & 0 \leq t \leq T_b \\ \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t + \pi) = -\sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t), & T_b \leq t \leq 2T_b \end{cases} \]

when a “0” is transmitted in \( t \in [T_b, 2T_b] \).

Let us now find out how \( s_1(t) \) and \( s_2(t) \) are related to each other.

\[
\int_0^{2T_b} s_1(t) s_2(t) \, dt = \int_0^{T_b} \frac{2E_b}{T_b} \cos^2(\omega_c t) \, dt + \int_{T_b}^{2T_b} \frac{2E_b}{T_b} \cos(\omega_c t)\cos(\omega_c t + \pi) \, dt \\
= \frac{2E_b}{T_b} \left[ \int_0^{T_b} \left( \frac{1 + \cos(2\omega_c t)}{2} \right) \, dt - \int_{T_b}^{2T_b} \left( \frac{1 + \cos(2\omega_c t)}{2} \right) \, dt \right] = 0
\]

\( \Rightarrow s_1(t) \perp s_2(t) \) in the interval \( [0, 2T_b] = [0, T_s] \) and \( E_s = 2E_b \).
Generation of DPSK: The input bit sequence is modified by an encoder as follows:

\[ d_k = b_k \oplus d_{k-1} \]

In block diagram form, the DPSK modulator is implemented as shown in the next figure.

Example of DPSK encoding for the sequence 10010011.

In order for the pre-coding to work properly, the encoder has to be initialized with a desired initial state (since the pre-coder contains a memory element).
The following plot shows the DPSK modulated waveform that is transmitted when the bit rate is $R_b$ bps and the carrier frequency is $2R_b$ Hz, assuming an initial condition of $d_{k-1} = "1"$. 

DPSK waveform representation of bit sequence 110010011
Consider the following noncoherent receiver:

\[ X_dl \]
\[ X(I) \]
\[ X(I) \]
\[ X(Q) \]
\[ X_dQ \]
\[ \cos(\omega_c t + \theta) \]
\[ \sin(\omega_c t + \theta) \]
\[ \int_0^{T_b} (\cdot) \, dt \]
\[ Y \]
\[ \Sigma \]
\[ \text{Decision} \]
\[ \hat{b}_k \]

A DPSK noncoherent demodulator

threshold = 0
Suppose there is a phase error $\theta$ at the local oscillator, then at the output of the down converter, we have (assuming no noise and $A = \sqrt{\frac{2E_b}{T_b}}$).

When a “1” is transmitted in $[0, T_b]$, 

$$x_{dI} = A \cos(\omega_c t + \theta) \cos \omega_c t = \frac{A}{2} \left[ \cos(2\omega_c t + \theta) + \cos \theta \right]$$

$$x_{dQ} = A \sin(\omega_c t + \theta) \cos \omega_c t = \frac{A}{2} \left[ \sin(2\omega_c t + \theta) + \sin \theta \right],$$

Likewise, when a “0” is transmitted in $[0, T_b]$, 

$$x_{dI} = -A \cos(\omega_c t + \theta) \cos \omega_c t = -\frac{A}{2} \left[ \cos(2\omega_c t + \theta) + \cos \theta \right]$$

$$x_{dQ} = -A \sin(\omega_c t + \theta) \cos \omega_c t = -\frac{A}{2} \left[ \sin(2\omega_c t + \theta) + \sin \theta \right]$$
At the output of the integrator, assuming \( \theta \) is constant for \( t \in [nT_b, (n+1)T_b] \), we get

\[
X_1 = \pm \frac{\Delta T_b}{2} \cos \theta, \quad X_Q = \pm \frac{\Delta T_b}{2} \sin \theta, 
\]

Thus, \((X_1, X_Q)\) is either \(\left(\frac{\Delta T_b}{2} \cos \theta, \frac{\Delta T_b}{2} \sin \theta\right)\) or \(\left(-\frac{\Delta T_b}{2} \cos \theta, -\frac{\Delta T_b}{2} \sin \theta\right)\).

Let \(\tilde{X}_0 = \begin{bmatrix} X_{I0} \\ X_{Q0} \end{bmatrix}\) be the observation at \(t = T_b\) and \(\tilde{X}_1 = \begin{bmatrix} X_{I1} \\ X_{Q1} \end{bmatrix}\) be the observation at \(t = 2T_b\). Then \(\tilde{X}_0\) and \(\tilde{X}_1\) point in the same direction if their inner product is positive, otherwise, they point in different directions and the transmitted bit in interval \([T_b, 2T_b]\) is different than that transmitted in interval \([0, T_b]\).

Mathematically,

\[
\tilde{X}_0^T \tilde{X}_1 = X_{I0} X_{I1} + X_{Q0} X_{Q1} = Y > 0 \quad \text{"1"} \\
< \quad \text{"0"}
\]
Let's now obtain the performance of the receiver in the presence of AWGN. Suppose \( s_1(t) \) is transmitted, \( t \in [0, 2T_b] \), then for a given \( \theta \) the conditional error probability is described by

\[
P_e | \theta, s_1 = P\left\{ X_0^T X_1 < 0 \mid s_1(t) \text{ sent, } \theta \right\}
\]

\[
X_{I0} = \int_0^{T_b} x(t) \cos(\omega_c t + \theta) \, dt = \int_0^{T_b} \left[ A \cos(\omega_c t) + W(t) \right] \cos(\omega_c t + \theta) \, dt
\]

\[
= \frac{AT_b}{2} \cos \theta + W_{I0}, \quad W_{I0} \triangleq \int_0^{T_b} W(t) \cos(\omega_c t + \theta) \, dt
\]

\[
X_{Q0} = \int_0^{T_b} x(t) \sin(\omega_c t + \theta) \, dt = \int_0^{T_b} \left[ A \cos(\omega_c t) + W(t) \right] \sin(\omega_c t + \theta) \, dt
\]

\[
= \frac{AT_b}{2} \sin \theta + W_{Q0}, \quad W_{Q0} \triangleq \int_0^{T_b} W(t) \sin(\omega_c t + \theta) \, dt
\]

\[
X_{I1} = \int_{T_b}^{2T_b} x(t) \cos(\omega_c t + \theta) \, dt = \frac{AT_b}{2} \cos \theta + W_{I1}, \quad W_{I1} \triangleq \int_{T_b}^{2T_b} W(t) \cos(\omega_c t + \theta) \, dt
\]

\[
X_{Q1} = \int_{T_b}^{2T_b} x(t) \sin(\omega_c t + \theta) \, dt = \frac{AT_b}{2} \sin \theta + W_{Q1}, \quad W_{Q1} \triangleq \int_{T_b}^{2T_b} W(t) \sin(\omega_c t + \theta) \, dt
\]
where \( W_{I0}, W_{Q0}, W_{I1}, \) and \( W_{Q1} \) are uncorrelated, zero-mean Gaussian r.v.’s with variances \( N_0 T_b / 4 \Rightarrow \) they are independent. Hence,

\[
P_e | \theta, s_1 = P \left\{ \left( \frac{A T_b}{2} \cos \theta + W_{I0} \right) \left( \frac{A T_b}{2} \cos \theta + W_{I1} \right) + \left( \frac{A T_b}{2} \sin \theta + W_{Q0} \right) \left( \frac{A T_b}{2} \sin \theta + W_{Q1} \right) \right\} < 0 | s_1(t), \theta
\]

Let us simplify things a bit by introducing a new variable that is equal to the dot product of the two vectors, i.e.

\[
Y \triangleq X_0^T X_1 = \frac{1}{4} \left[ (X_{I0} + X_{I1})^2 + (X_{Q0} + X_{Q1})^2 - \left( (X_{I0} - X_{I1})^2 + (X_{Q0} - X_{Q1})^2 \right) \right]
\]

\[
= \frac{1}{4} \left\{ (A T_b \cos \theta + W_{I0} + W_{I1})^2 + (A T_b \sin \theta + W_{Q0} + W_{Q1})^2 - \left[ (W_{I0} - W_{I1})^2 + (W_{Q0} - W_{Q1})^2 \right] \right\}
\]

With this definition, the event \( \{Y < 0\} \) means

\[
\left( \frac{A T_b}{2} \cos \theta + \frac{W_{I0}}{2} + \frac{W_{I1}}{2} \right)^2 + \left( \frac{A T_b}{2} \sin \theta + \frac{W_{Q0}}{2} + \frac{W_{Q1}}{2} \right)^2 < \left( \frac{W_{I0}}{2} - \frac{W_{I1}}{2} \right)^2 + \left( \frac{W_{Q0}}{2} - \frac{W_{Q1}}{2} \right)^2
\]
Let $W_1 \triangleq \frac{W_{10}}{2} + \frac{W_{11}}{2}$, $W_2 \triangleq \frac{W_{10}}{2} - \frac{W_{11}}{2}$, $W_3 \triangleq \frac{W_{Q0}}{2} + \frac{W_{Q1}}{2}$, $W_4 \triangleq \frac{W_{Q0}}{2} - \frac{W_{Q1}}{2}$, then $W_i \sim G\left(0, \frac{N_0 T_b}{8}\right)$, $i = 1, 2, 3, 4$.

Let $R_1^2 \triangleq \left(\frac{AT_b}{2} \cos \theta + W_1\right)^2 + \left(\frac{AT_b}{2} \sin \theta + W_3\right)^2$, $R_2^2 \triangleq W_2^2 + W_4^2$, then

$R_1$ is a Ricean r.v. and $R_2$ is a Rayleigh r.v. (this will be proved when we consider non-coherent binary FSK detection), and the conditional probability of a binary error, given that we know both $\theta$ and $s_1$ is given by

$$P_e | \theta, s_1 = P\left\{R_2^2 > R_1^2 \mid s_1(t) \text{ sent}, \theta\right\} = P\left\{R_2 > R_1 \mid s_1(t) \text{ sent}, \theta\right\} = \iint_{\mathbb{R}} f_{R_1, R_2 | S_1, \Theta}(r_1, r_2 | s_1, \theta) \, dr_2 \, dr_1 = \iint_{\mathbb{R}} f_{R_1 | S_1, \Theta}(r_1 | s_1, \theta)f_{R_2 | S_1, \Theta}(r_2 | s_1, \theta) \, dr_2 \, dr_1$$

$$= \iint_{\mathbb{R}} f_{R_2}(r_2)f_{R_1 | S_1, \Theta}(r_1 | s_1, \theta) \, dr_2 \, dr_1 = \iint_{\mathbb{R}} f_{R_2}(r_2)f_{R_1 | S_1, \Theta}(r_1 | s_1, \theta) \, dr_2 \, dr_1$$
Explicitly, \( f_{R_2}(r_2) = \frac{r_2}{\sigma^2} e^{-\frac{r_2^2}{2\sigma^2}} \) and \( f_{R_1|S_1,\theta}(r_1|s_1,\theta) = \frac{r_1}{\sigma^2} e^{-\frac{1}{2\sigma^2}[r_1^2+B^2]} I_0\left(\frac{B}{\sigma^2 r_1}\right) \),

where \( \sigma^2 = \frac{N_0T_b}{8} \) and \( B = \frac{AT_b}{2} \). Clearly, \( f_{R_1|S_1,\theta}(r_1|s_1,\theta) = f_{R_1|S_1}(r_1|s_1) \), independent of \( \theta \), which implies that \( P_e|\theta,s_1 = P_e|s_1 = P_b|s_1 = \text{BER}|s_1 \), or

\[
\text{BER}|s_1 = \int_0^\infty \left[ \int_{r_1}^\infty f_{R_2}(r_2) \, dr_2 \right] f_{R_1|S_1}(r_1|s_1) \, dr_1
\]

\[
= \int_0^\infty \left[ \int_{r_1}^\infty \frac{r_2}{\sigma^2} e^{-\frac{r_2^2}{2\sigma^2}} \, dr_2 \right] \frac{r_1}{\sigma^2} e^{-\frac{1}{2\sigma^2}[r_1^2+B^2]} I_0\left(\frac{B}{\sigma^2 r_1}\right) \, dr_1
\]

\[
= \int_0^\infty \left[ -e^{-\frac{x^2}{2}} \right]_{r_1/\sigma}^\infty \frac{r_1}{\sigma^2} e^{-\frac{1}{2\sigma^2}[r_1^2+B^2]} I_0\left(\frac{B}{\sigma^2 r_1}\right) \, dr_1, x = \frac{r_2}{\sigma}
\]

\[
= -e^{-\frac{B^2}{2\sigma^2}} \int_0^\infty r_1 e^{-\frac{r_1^2}{2\sigma^2}} I_0\left(\frac{B}{\sigma^2 r_1}\right) \, dr_1 = -e^{-\frac{B^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{\pi}} e^{-\frac{B^2}{4\sigma^2}} = \frac{e^{-\frac{B^2}{4\sigma^2}}}{\sqrt{\pi}} = \frac{e^{\frac{\sigma^2}{4}}}{\sqrt{\pi}} = e^{-\frac{A^2T_b}{N_0}} = e^{-\frac{E_b}{N_0}}
\]

If \( P\{S_i\} = \frac{1}{4}, i = 1, 2, 3, 4 \), then if we average out over the 4 symbols, \( \text{BER} = \frac{1}{2} e^{-\frac{E_b}{N_0}} \).
The following plot shows the BER performance of DPSK in AWGN.
Noncoherent Detection of Binary FSK

Let the transmitted signal be described by

\[
s_i(t) = \begin{cases} 
\sqrt{\frac{2E_b}{T_b}} \cos(\omega_i t), & 0 \leq t \leq T_b, \ i = 1, 2 \\
0, & \text{elsewhere}
\end{cases}
\]

where \(\omega_1\) and \(\omega_2\) are such that \(s_1(t) \perp s_2(t)\).

Let \(\phi_i(t) = \begin{cases} 
\sqrt{\frac{2}{T_b}} \cos(\omega_i t), & 0 \leq t \leq T_b, \ i = 1, 2 \\
0, & \text{elsewhere}
\end{cases}\). Then \(s_i(t) = \sqrt{E_b} \phi_i(t), \ i = 1, 2\).
Consider the following noncoherent receiver:

Noncoherent receiver for orthogonal FSK
Where $\hat{\phi}_i(t) = \sqrt{\frac{2}{T_b}} \cos \left( \omega_i t - \frac{\pi}{2} \right) = \sqrt{\frac{2}{T_b}} \sin(\omega_i t), \ 0 \leq t \leq T_b, \ i = 1, 2.$

Assuming an AWGN channel, the received signal be described by

$$x(t) = \sqrt{\frac{2E_b}{T_b}} \cos(\omega_i t + \theta) + W(t), \ 0 \leq t \leq T_b, \ \theta \sim U[-\pi, \pi],$$

then

$$x(t) = \sqrt{\frac{2E_b}{T_b}} \cos \theta \cdot \cos \omega_i t - \sqrt{\frac{2E_b}{T_b}} \sin \theta \cdot \sin \omega_i t + W(t), \ i = 1, 2$$

$$= \sqrt{E_b} \cos \theta \cdot \hat{\phi}_i(t) - \sqrt{E_b} \sin \theta \cdot \hat{\phi}_i(t) + W(t), \ i = 1, 2$$
Consider the $L_1$ path:

$$X_1 = \int_0^{T_b} x(t) \phi_1^*(t) \, dt$$

$$= \int_0^{T_b} \left[ \sqrt{E_b} \cos \theta \cdot \phi_i(t) - \sqrt{E_b} \sin \theta \cdot \phi_i(t) + W(t) \right] \phi_1^*(t) \, dt$$

$$= \begin{cases} 
\sqrt{E_b} \cos \theta + W_1, & \text{if } i = 1, \text{ } \theta \text{ is constant in } t \in [0, T_b] \\
W_1, & \text{if } i = 2
\end{cases}$$

$$Y_1 = \int_0^{T_b} x(t) \hat{\phi}_1^*(t) \, dt$$

$$= \int_0^{T_b} \left[ \sqrt{E_b} \cos \theta \cdot \hat{\phi}_i(t) - \sqrt{E_b} \sin \theta \cdot \hat{\phi}_i(t) + W(t) \right] \hat{\phi}_1^*(t) \, dt$$

$$= \begin{cases} 
-\sqrt{E_b} \sin \theta + \hat{W}_1, & \text{if } i = 1, \text{ } \theta \text{ is constant in } t \in [0, T_b] \\
\hat{W}_1, & \text{if } i = 2
\end{cases}$$
Likewise, for the $L_2$ path

$$X_2 = \begin{cases} \sqrt{E_b} \cos \theta + W_2, & i = 2 \\ W_2, & i = 1 \end{cases}$$

$$Y_2 = \begin{cases} -\sqrt{E_b} \sin \theta + \hat{W}_2, & i = 2 \\ \hat{W}_2, & i = 1 \end{cases}$$

Now, $L_i = \sqrt{X_i^2 + Y_i^2}$, $i = 1, 2$. Both $W_i$ and $\hat{W}_i \sim G\left(0, \frac{N_0}{2}\right)$, $i = 1, 2$. Since the receiver is symmetric, $P\{\text{choosing } M_2 \mid M_1 \text{ sent}\} = P\{\text{choosing } M_1 \mid M_2 \text{ sent}\}$. Hence, we need to compute only one of them.

Suppose $M_1$ was sent in $0 \leq t \leq T_b$, then an error is made if $W(t)$ is such that $L_2 > L_1$. In this case, $L_1 \mid M_1 = \sqrt{\left(\sqrt{E_b} \cos \theta + W_1\right)^2 + \left(-\sqrt{E_b} \sin \theta + \hat{W}_1^2\right)}$

$L_2 \mid M_1 = \sqrt{W_2^2 + \hat{W}_2^2}$. 200
But, $L_2 | M_1$ sent has a Rayleigh distribution, i.e. its pdf is described by

$$f_{L_2|M_1}(\ell_2 | m_1) = \begin{cases} \frac{2\ell_2}{N_0} e^{-\frac{\ell_2^2}{N_0}}, & \ell_2 \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

and $P\{\text{error} | M_1 \text{ sent}\} = \int_0^\infty P\{L_2 > L_1 | L_1 = \ell_1, M_1 = m_1\} f_{L_1|M_1}(\ell_1 | m_1) d\ell_1$. Now,

$$P\{L_2 > L_1 | L_1 = \ell_1, M_1\} = \int_{\ell_1}^\infty f_{L_2|M_1}(\ell_2 | m_1) d\ell_2 = \int_{\ell_1}^\infty \frac{2\ell_2}{N_0} e^{-\frac{\ell_2^2}{N_0}} d\ell_2 = -e^{-\frac{\ell_1^2}{N_0}} = e^{-\frac{\ell_1^2}{N_0}}$$

and $L_1 | m_1$ sent $= \sqrt{X_1^2 + Y_1^2}$.

For a given $\theta$, $X_1 \sim G\left(\sqrt{E_b} \cos \theta, \frac{N_0}{2}\right)$ and $Y_1 \sim G\left(-\sqrt{E_b} \sin \theta, \frac{N_0}{2}\right)$.

Thus, $f_{X_1,Y_1|M_1,\theta}(x_1, y_1 | m_1, \theta) = \frac{1}{\pi N_0} e^{-\frac{1}{N_0}\left[\left(x_1 - \sqrt{E_b} \cos \theta\right)^2 + \left(y_1 + \sqrt{E_b} \sin \theta\right)^2\right]}$, because $X_1$ and $Y_1$ are independent of each other.
Let \( X_1 = \sqrt{\frac{N_0}{2}} \) \( R \cos \Phi \) and \( Y_1 = \sqrt{\frac{N_0}{2}} \) \( R \sin \Phi \), \( \Phi = \tan^{-1} \left( \frac{Y_1}{X_1} \right) \). Then

\[
f_{R,\Phi|M_1}(r, \phi | m_1) = f_{x_1,y_1|M_1}(x_1, y_1 | m_1) \left| J \left( \begin{array}{c} x_1, y_1 \\ r, \phi \end{array} \right) \right| \bigg|_{x_1 = \sqrt{\frac{N_0}{2}} r \cos \phi, x_2 = \sqrt{\frac{N_0}{2}} r \sin \phi}
\]

where

\[
J \left( \begin{array}{c} x_1, y_1 \\ r, \phi \end{array} \right) = \left| \begin{array}{cc} \frac{\partial x_1}{\partial r} & \frac{\partial y_1}{\partial r} \\
\frac{\partial x_1}{\partial \phi} & \frac{\partial y_1}{\partial \phi} \end{array} \right| = \left| \begin{array}{cc} \sqrt{\frac{N_0}{2}} \cos \phi & \sqrt{\frac{N_0}{2}} \sin \phi \\
-\sqrt{\frac{N_0}{2}} r \sin \phi & \sqrt{\frac{N_0}{2}} r \cos \phi \end{array} \right| = \frac{N_0}{2} r
\]

Hence,

\[
f_{R,\Phi|M_1}(r, \phi | m_1) = \frac{1}{\pi N_0} e^{\frac{-1}{N_0} \left( \sqrt{\frac{N_0}{2}} r \cos \phi - \sqrt{E_b} \cos \theta \right)^2 + \left( \sqrt{\frac{N_0}{2}} r \sin \phi + \sqrt{E_b} \sin \theta \right)^2} \cdot \frac{N_0}{2} r
\]

\[
= \frac{r}{2\pi} e^{\frac{-1}{N_0} \left[ \frac{N_0}{2} r^2 + E_b - \sqrt{2N_0E_b} \ r \cos(\phi+\theta) \right]} = \frac{r}{2\pi} e^{\frac{-r^2 + E_b - \sqrt{2E_bN_0}}{2N_0} \sqrt{\frac{2E_b}{N_0}} \ r \cos(\phi+\theta)}
\]
and

\[ f_{R|M_1}(r | m_1) = \int_{-\pi}^{\pi} f_{R,\Phi|M_1}(r, \phi | m_1) \, d\phi = \frac{r}{2\pi} e^{-\left(\frac{r^2 + E_b}{2N_0}\right)} \int_{-\pi}^{\pi} e^{\sqrt{\frac{2E_b}{N_0}} r \cos(\phi+\theta)} \, d\phi. \]

But, the last integral is independent of \( \theta \), because it is periodic regardless of the value of \( \theta \). Hence,

\[ \int_{-\pi}^{\pi} e^{\sqrt{\frac{2E_b}{N_0}} r \cos(\phi+\theta)} \, d\phi = \int_{-\pi}^{\pi} e^{\sqrt{\frac{2E_b}{N_0}} r \cos \phi} \, d\phi. \]

Now,

\[ I_0(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x \cos \phi} \, d\phi \] is the modified Bessel function of the first kind of order zero.

\[ \Rightarrow f_{R|M_1}(r | m_1) = re^{-\left(\frac{r^2 + E_b}{2N_0}\right)} I_0\left(\sqrt{\frac{2E_b}{N_0}} r\right), \quad r \geq 0 \Rightarrow R \text{ is a Ricean distributed r.v.} \]
Clearly, \( \ell_1 = \sqrt{\frac{N_0}{2}} \) \( r = \sqrt{\left( \sqrt{\frac{N_0}{2}} \cos \phi \right)^2 + \left( \sqrt{\frac{N_0}{2}} \sin \phi \right)^2} \). Thus,

\[
P\{\text{error } | \text{ M}_1 \text{ sent}\} = \int_0^\infty f_{L_1|\text{M}_1}(\ell_1 | \text{m}_1) e^{-\frac{\ell_1^2}{N_0}} \ d\ell_1 = \int_0^\infty f_{R|\text{M}_1}(r | \text{m}_1) e^{-\frac{r^2}{2}} \ dr
\]

\[
= \int_0^\infty r e^{-r^2 - \frac{E_b}{N_0}} I_0 \left( \sqrt{\frac{2E_b}{N_0}} \ r \right) \ dr
\]

\[
= e^{-\frac{E_b}{N_0}} \int_0^\infty r e^{-r^2} I_0 \left( \sqrt{\frac{2E_b}{N_0}} \ r \right) \ dr.
\]

But, \( \int_0^\infty x e^{-ax^2} I_0(bx) \ dx = \frac{1}{2a} e^{\frac{b^2}{4a}} \). Hence,

\[
P\{\text{error } | \text{ M}_1\} = e^{-\frac{E_b}{N_0}} \cdot \frac{1}{2} e^{-\frac{\left(\frac{2E_b}{N_0}\right)}{4}} = \frac{1}{2} e^{-\frac{E_b}{2N_0}}.
\]
If both symbols are transmitted with equal probability, i.e. $P\{M_1\} = P\{M_2\} = \frac{1}{2}$, then $\text{BER} = P_b = \frac{1}{2}e^{-\frac{E_b}{2N_0}}$. Clearly, the BER performance of noncoherent FSK in AWGN is the same as that of DPSK.