Sensitivity and Reliability Analysis of Nonlinear Frame Structures

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Applied Mathematics and Computation Seminar
April 8, 2011

Outline

1. Structural Response Sensitivity Analysis
   - Governing Equilibrium Equations
   - Computing Response Sensitivity
   - Sensitivity Verification

2. Structural Reliability Analysis
   - Problem Statement
   - Computing Reliability
   - Sensitivity at Design Point
   - Cantilever Example
   - Reinforced Concrete Frame Example

Structural Equilibrium

Balance of external loads and internal resisting forces

$$\mathbf{P}_f(t) = \mathbf{P}_r(\mathbf{U}_f(t)) \tag{1}$$

- $\mathbf{P}_f(t)$ – time-varying external loads (given)
- $\mathbf{U}_f(t)$ – time-varying nodal displacements (unknown)
- $\mathbf{P}_r$ – resisting forces (by selection and assembly)

Residual Equilibrium Equations

Solve residual equilibrium using Newton-Raphson algorithm

$$\mathbf{R}_f(\mathbf{U}_f(t)) = \mathbf{P}_f(t) - \mathbf{P}_r(\mathbf{U}_f(t)) = \mathbf{0} \tag{2}$$

$$\Delta\mathbf{U}_f = \left(\mathbf{K}_T^j\right)^{-1} \mathbf{R}_f(\mathbf{U}_f^j) \tag{3}$$

where

$$\mathbf{K}_T^j = \frac{\partial \mathbf{P}_r}{\partial \mathbf{U}_f} \tag{4}$$

or any other suitable stiffness matrix according to other root finding algorithms (modified, quasi, accelerated Newton).
Regardless, use increment to update displacement

$$\mathbf{U}_f^{j+1} = \mathbf{U}_f^j + \Delta\mathbf{U}_f \tag{5}$$

Parameterized Equilibrium Equations

Let θ be an uncertain parameter of the external load or the finite element model

$$\mathbf{P}_f(t,\theta) = \mathbf{P}_r(\mathbf{U}_f(t,\theta),\theta) \tag{6}$$

- Nodal displacements, $\mathbf{U}_f$, depend on all θ, whether of the load or FE model
- Resisting forces depend explicitly on a θ of the FE model and implicitly on all θ via $\mathbf{U}_f$
- Load vector, $\mathbf{P}_f$, depends explicitly on a θ of the external load

Typical properties θ represents: steel yield stress, concrete strength, member cross-section dimensions, nodal coordinates, etc.

Sensitivity to Uncertain Parameters

Two approaches to determine $\partial \mathbf{U}_f / \partial \theta$, the sensitivity of structural response to changes in θ

1. Finite Differences – re-run analysis with perturbed parameter value, e.g., forward finite difference

$$\frac{\partial \mathbf{U}_f}{\partial \theta} \approx \frac{\mathbf{U}_f(\theta + \Delta\theta) - \mathbf{U}_f(\theta)}{\Delta\theta} \tag{7}$$

2. Direct Differentiation – develop analytic expressions for sensitivity by differentiating governing equations of the finite element analysis

Direct Differentiation Method (DDM)

Differentiate equilibrium equations wrt parameter, θ

$$\frac{\partial \mathbf{P}_f}{\partial \theta} = \frac{\partial \mathbf{P}_r}{\partial \mathbf{U}_f}\frac{\partial \mathbf{U}_f}{\partial \theta} + \left.\frac{\partial \mathbf{P}_r}{\partial \theta}\right|_{\mathbf{U}_f} \tag{8}$$

Solve for nodal displacement sensitivity, $\partial \mathbf{U}_f / \partial \theta$

$$\frac{\partial \mathbf{U}_f}{\partial \theta} = \mathbf{K}_T^{-1}\left(\frac{\partial \mathbf{P}_f}{\partial \theta} - \left.\frac{\partial \mathbf{P}_r}{\partial \theta}\right|_{\mathbf{U}_f}\right) \tag{9}$$

At end of a simulation time step:

- Conditional derivative, $\partial \mathbf{P}_r / \partial \theta|_{\mathbf{U}_f}$, assembled from element contributions in same manner as $\mathbf{P}_r$
- Factorized $\mathbf{K}_T$ from last Newton-Raphson iteration.
- Or reform and factorize $\mathbf{K}_T$ if using Newton-like iteration.

Verification

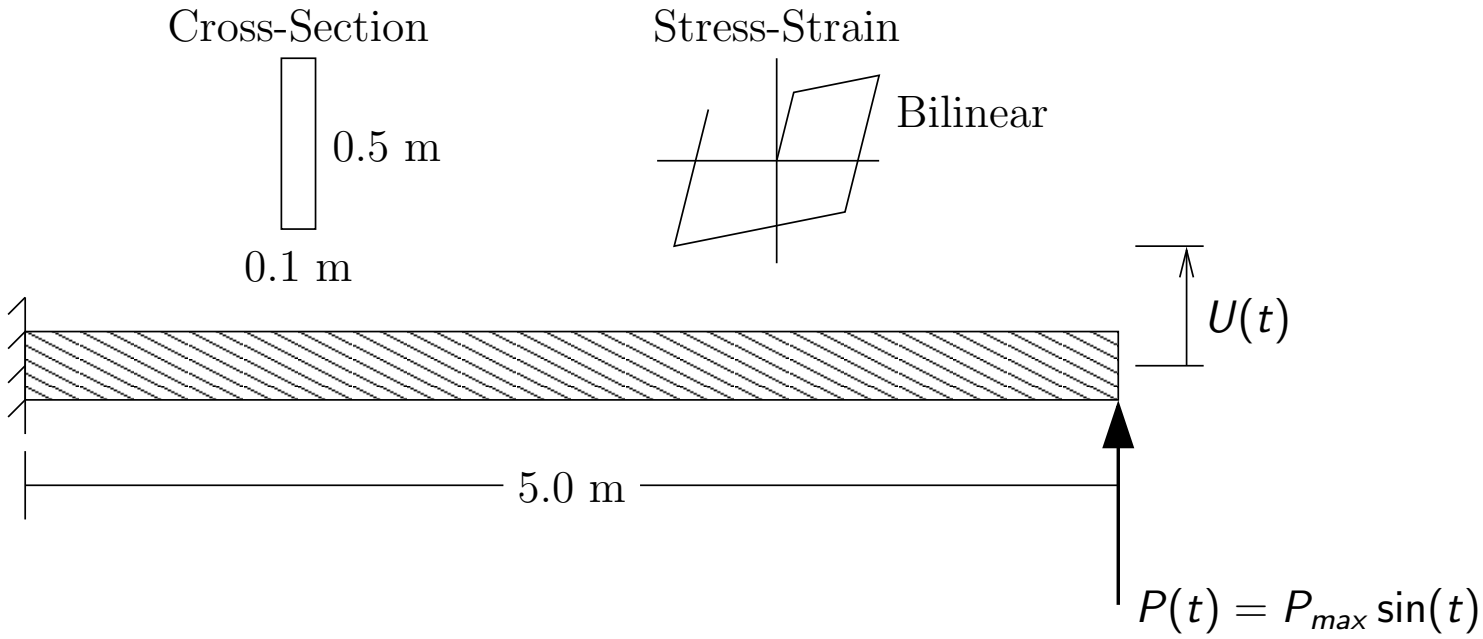
Compare with forward finite difference

$$\lim_{\Delta\theta \to 0} \frac{\mathbf{U}_f(\theta + \Delta\theta) - \mathbf{U}_f(\theta)}{\Delta\theta} = \frac{\partial \mathbf{U}_f}{\partial \theta} \tag{10}$$

- Repeat analysis with perturbed parameter
- If finite difference approximation converges to analytic sensitivity as $\Delta\theta$ decreases, then DDM is "correct"

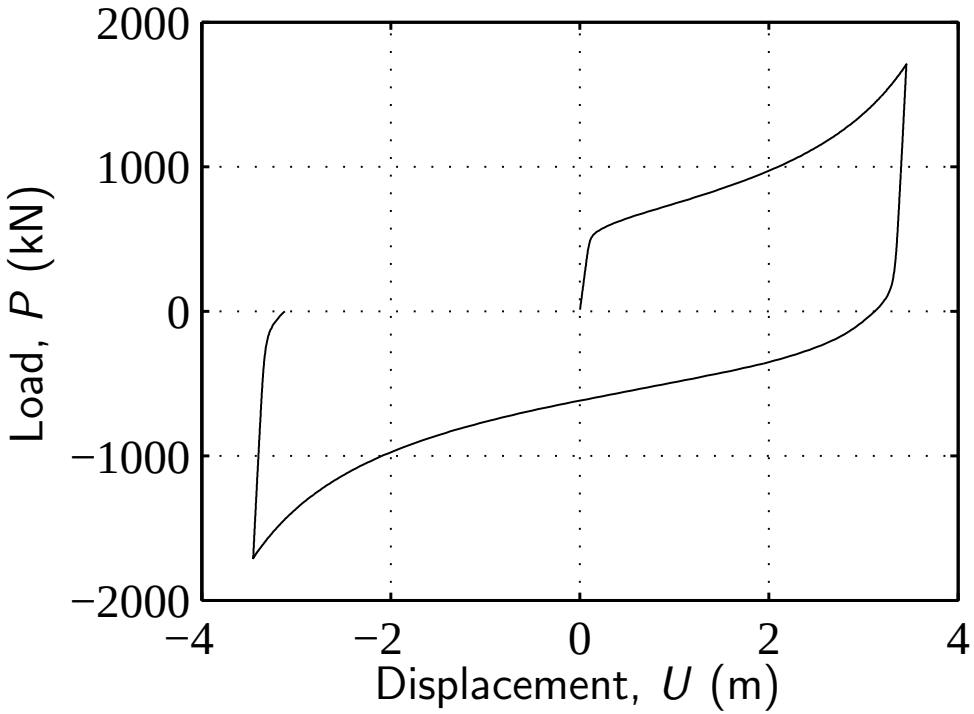
Verification Example

Slender cantilever with material and geometric nonlinearity



- Discretize with five frame finite elements (linear curvature, constant axial deformation approximation)
- Corotational formulation for large displacements
- Numerically integrate stress-strain response over cross-section
- E = 2.0e8 kPa, σ_y = 4.1e5 kPa, 2% kinematic strain-hardening
- One load cycle with P_{max} = 1710 kN (5 times yield load)

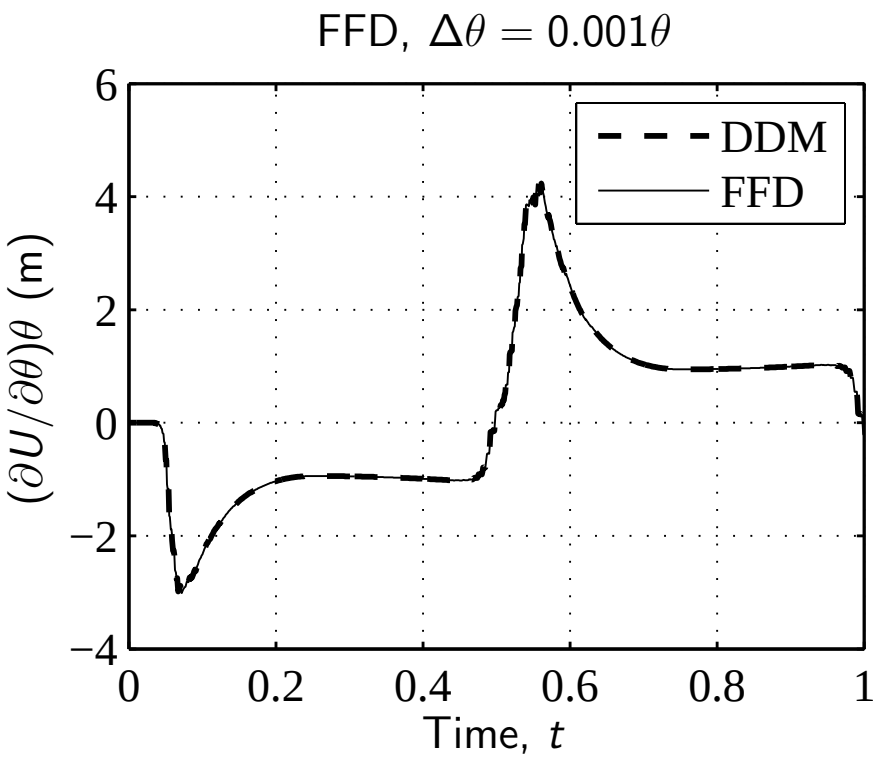
Load-Displacement Response

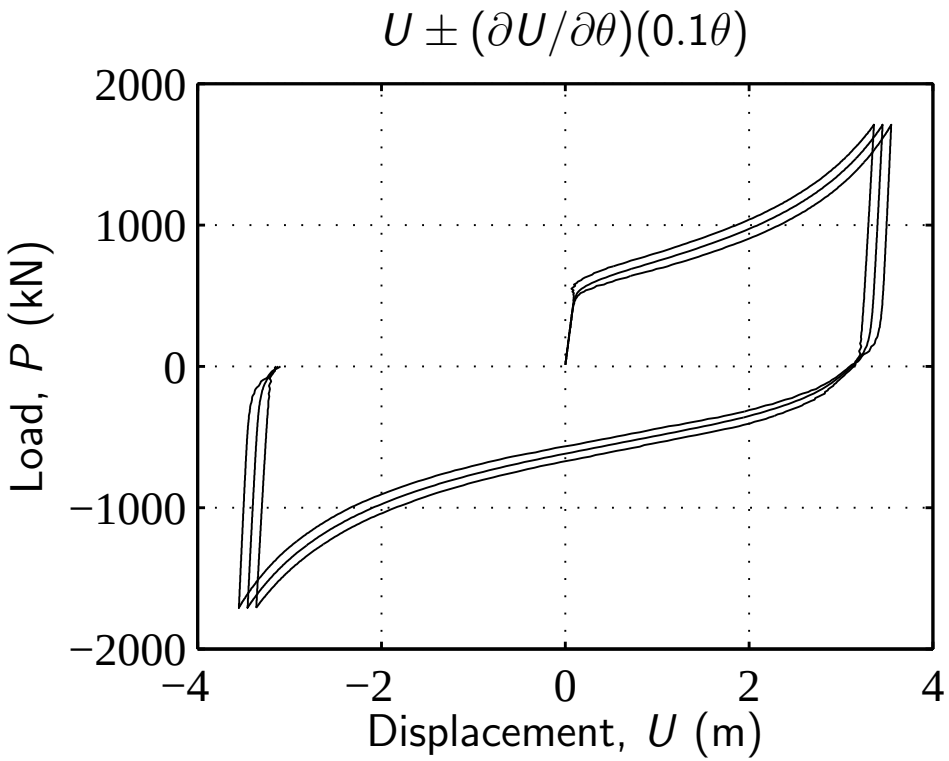


- Material yield at about 400 kN load
- Tension stiffening at about 1 m displacement
- Elastic unloading and reverse cyclic yielding

Sensitivity to Yield Stress ($\theta \equiv \sigma_y$)

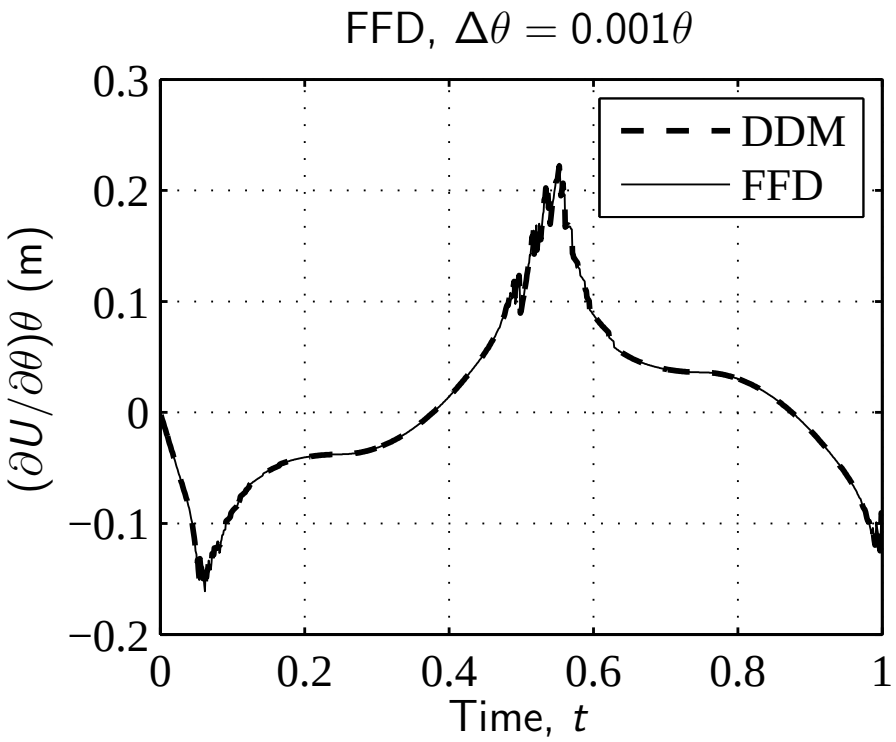
DDM verification and response envelope for strength parameter

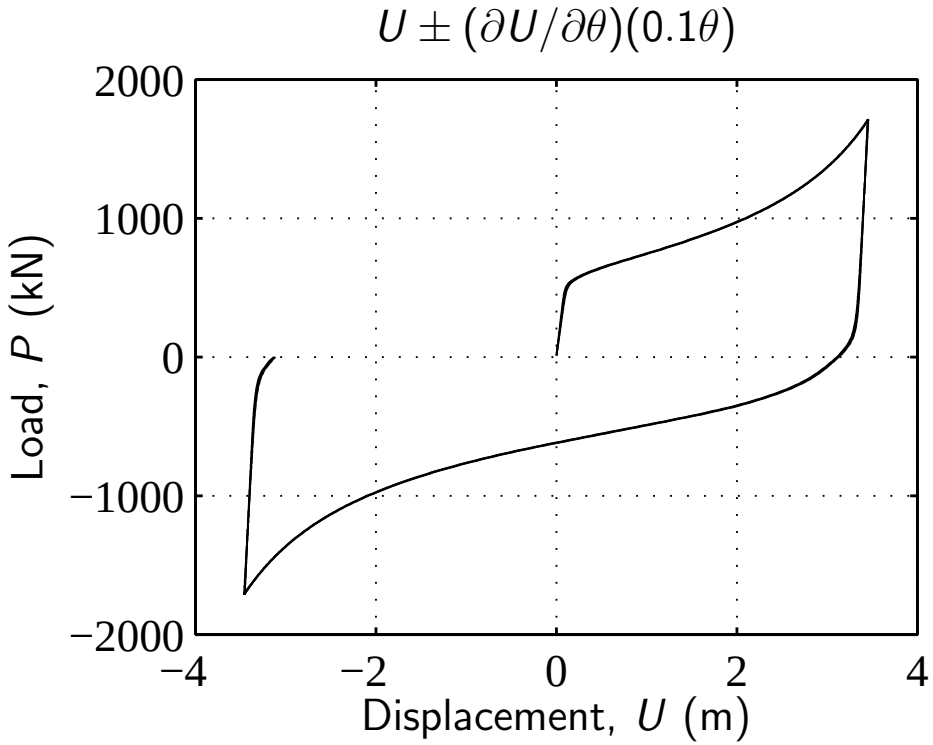




Sensitivity to Elastic Modulus ($\theta \equiv E$)

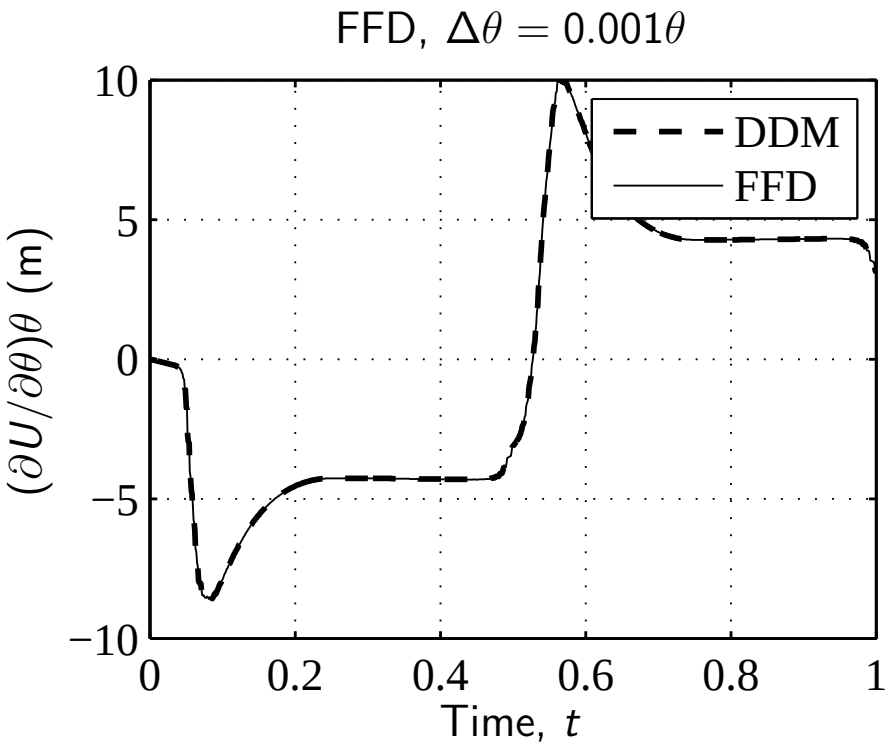
DDM verification and response envelope for stiffness parameter

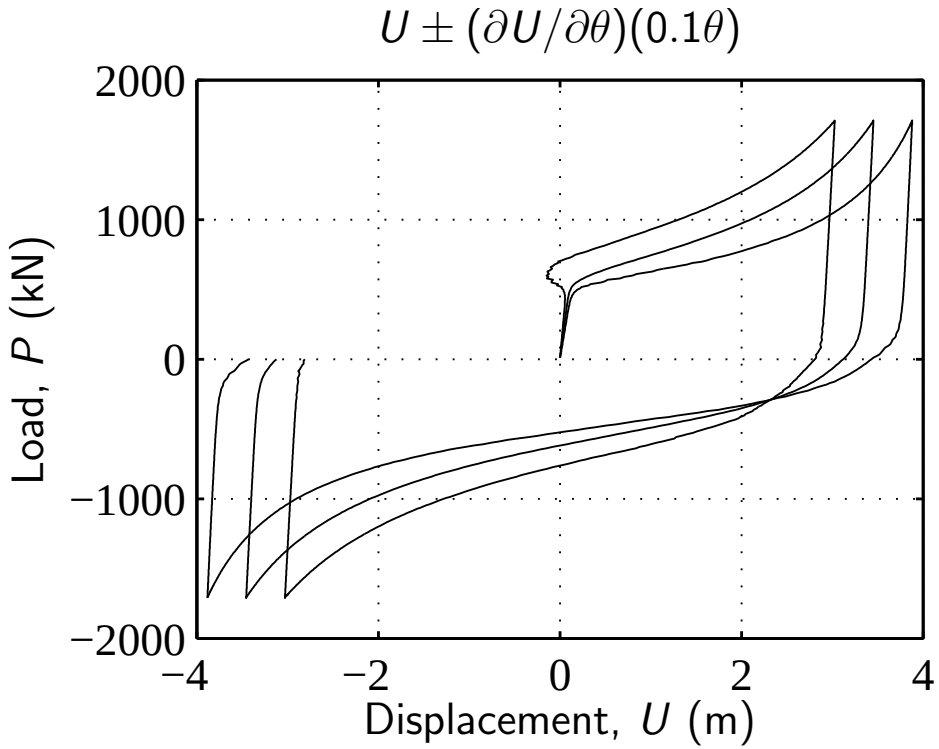




Sensitivity to Cross-Section Depth ($\theta \equiv d$)

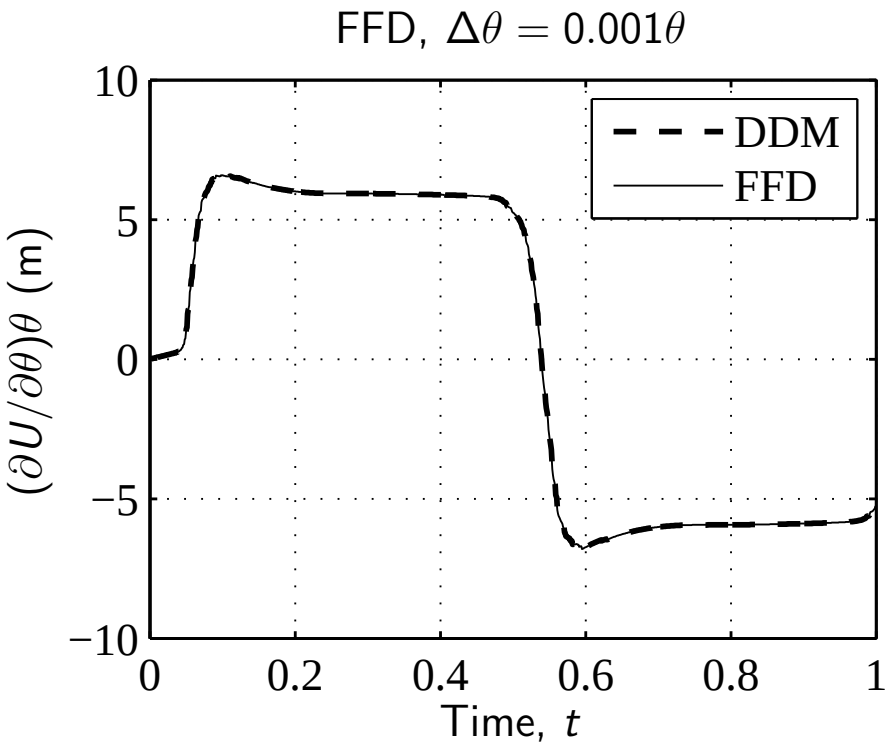
DDM verification and response envelope for local geometric parameter

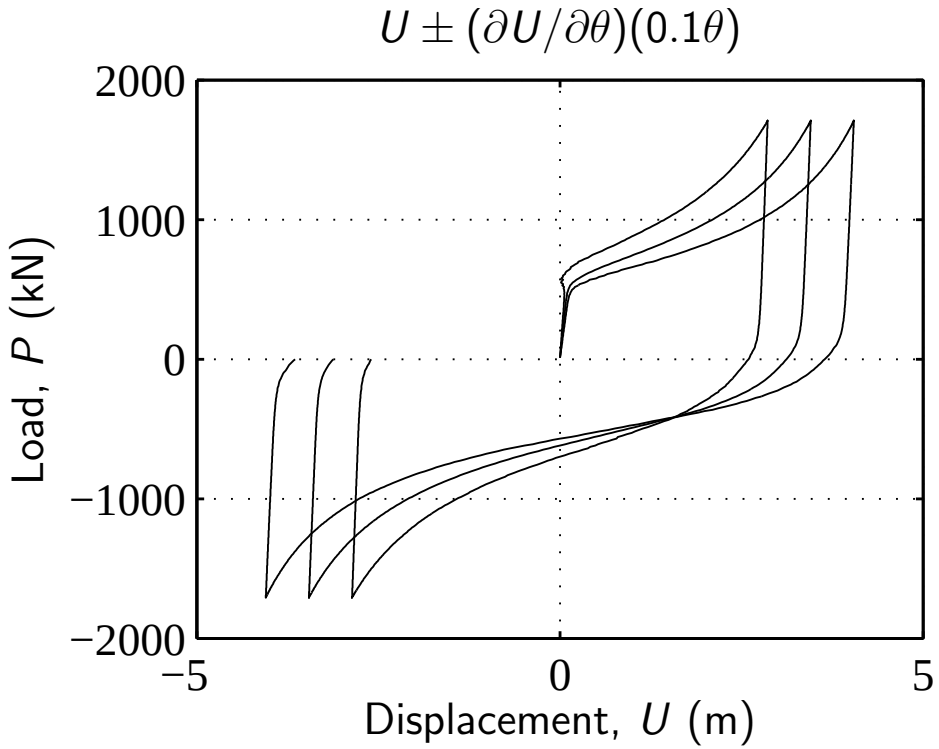




Sensitivity to Cantilever Length ($\theta \equiv L$)

DDM verification and response envelope for global geometric parameter





Structural Reliability Analysis

Assess uncertain structural response in terms of probability of failure under prescribed events, or limit states

Define scalar valued limit state function in terms of a vector of random variables, $\mathbf{x}$

$$g(\mathbf{x}) = \begin{cases} \leq 0 & \textit{fail} \\ > 0 & \textit{safe} \end{cases} \tag{11}$$

Each random variable in $\mathbf{x}$

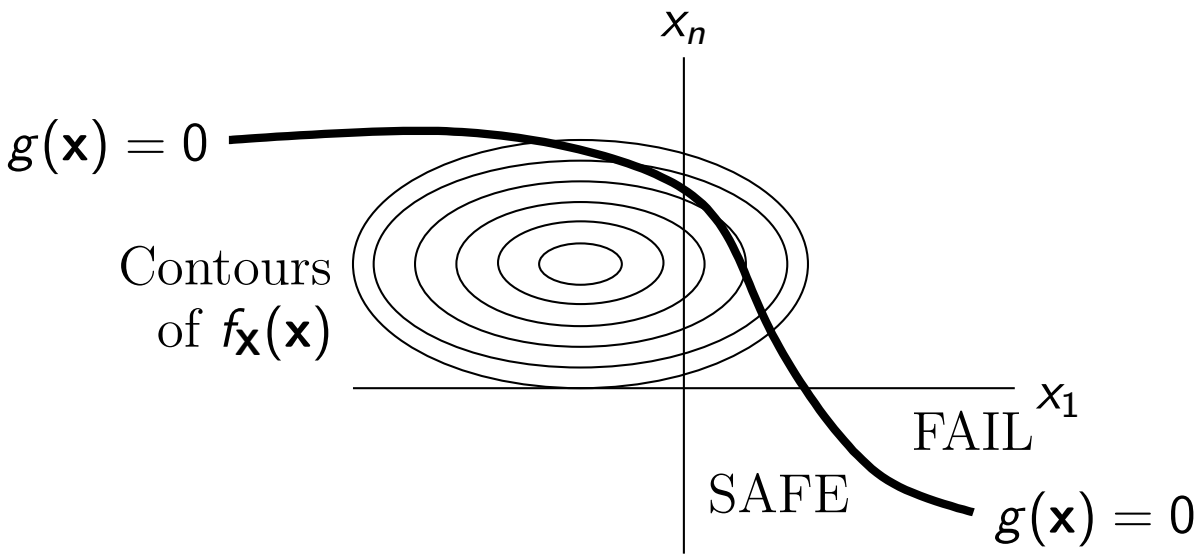
- Has a prescribed PDF, CDF, mean, st.dev, etc.;
- Generally, correlated and non-normal; and
- Maps to one or more parameters, θ, of the load or FE model

Probability of Failure

The probability that the limit state function evaluates to *fail* over all possible realizations of the random variables in $\mathbf{x}$

$$p_f = \int_{g(\mathbf{x})\leq 0} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \tag{12}$$

- $f_{\mathbf{X}}(\mathbf{x})$ is the joint PDF of all random variables
- Integrate over failure domain

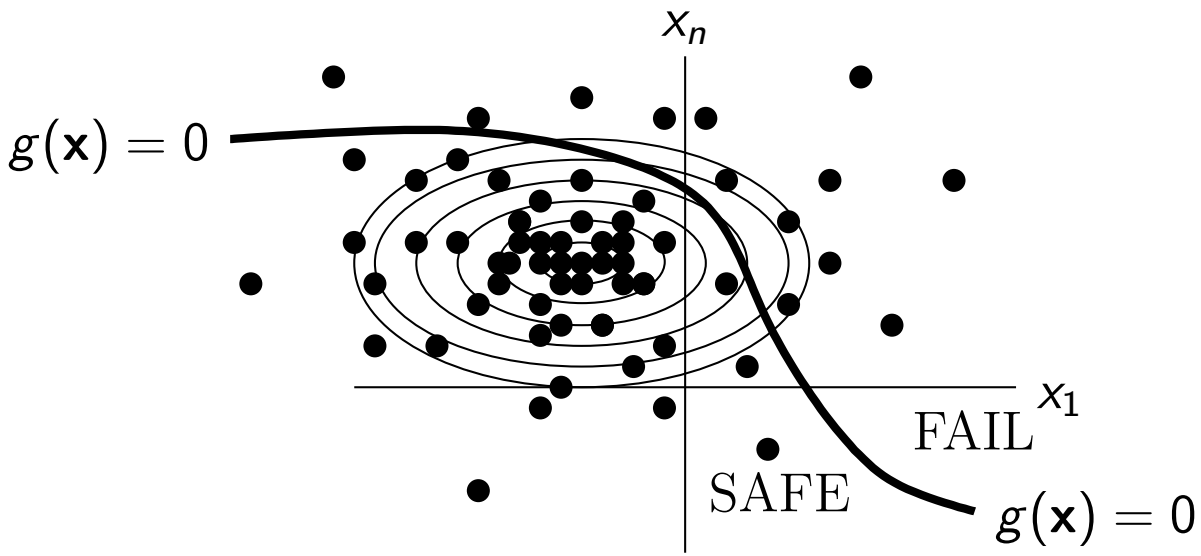


Monte Carlo Simulation

Generate N random realizations of $\mathbf{x}$ using joint inverse CDF

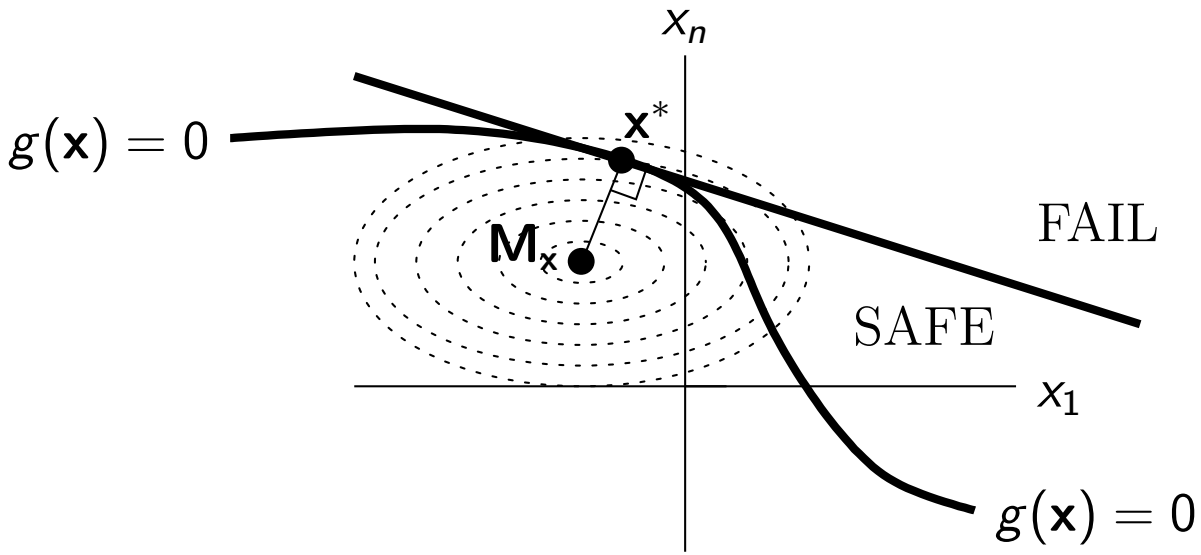
$$p_f \approx \frac{\sum_{i=1}^{N}(g(\mathbf{x}^{(i)}) \leq 0)}{N} \tag{13}$$

- Captures nonlinearities of limit state function
- Computationally intense: $N >= 1e5$ for accurate p_f
- Difficult to obtain sensitivity of failure to random variables



First Order Approximation

Linearize at design point, $\mathbf{x}^*$, on failure surface closest to $\mathbf{M_x}$, mean value of $\mathbf{x}$



$\mathbf{x}^*$ is solution to constrained optimization problem

$$\min_{g(\mathbf{x})=0} \|\mathbf{x} - \mathbf{M_x}\|_2 \tag{14}$$

Distance from mean to failure surface is measure of reliability

Higher order approximations of failure surface also possible

Standard Normal Space

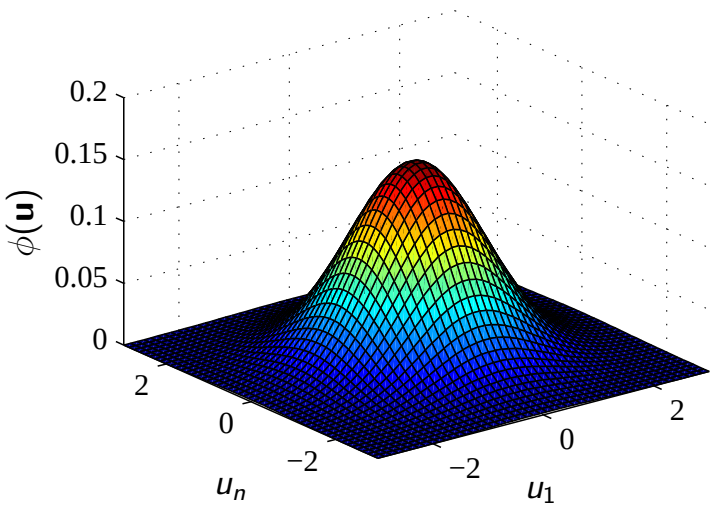
To find design point, transform all random variables to space of uncorrelated, standard normals, $\mathbf{u}$, with joint PDF

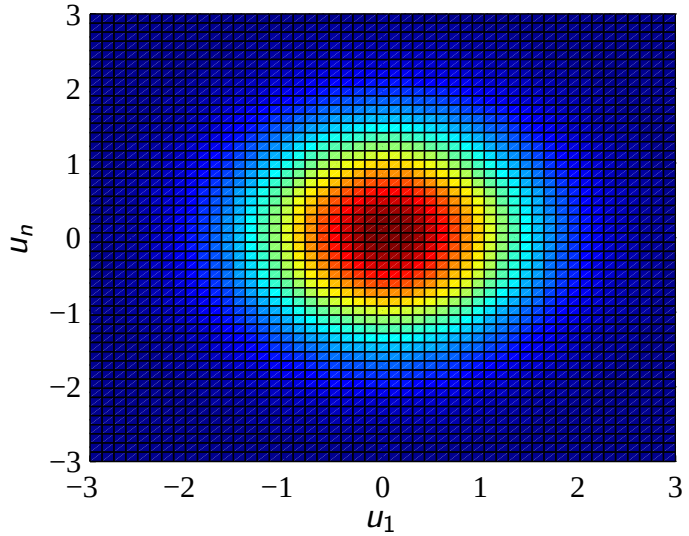
$$\phi(\mathbf{u}) = \frac{1}{(2\pi)^{-n/2}} \exp\left(-\frac{1}{2}\mathbf{u}^T\mathbf{u}\right) \tag{15}$$

Find design point, $\mathbf{u}^*$, in $\mathbf{u}$-space

$$\min_{g(\mathbf{u})=0} \|\mathbf{u}\|_2 \tag{16}$$

Rotational symmetry of $\mathbf{u}$-space: p_f from inverse normal CDF





HLRF Iteration

Find design point, $\mathbf{u}^*$, by Hasofer-Lind-Rackwitz-Fiessler (HLRF) iteration

Start at $\mathbf{u}^0$ (usually at mean point $\mathbf{0}$), then

$$\mathbf{u}^{m+1} = \left(\boldsymbol{\alpha}^T \mathbf{u}^m + \frac{g(\mathbf{x})}{\|\nabla_{\mathbf{u}} g\|} \right) \boldsymbol{\alpha} \tag{17}$$

where

$$\boldsymbol{\alpha} = -\frac{\nabla_{\mathbf{u}} g}{\|\nabla_{\mathbf{u}} g\|} \tag{18}$$

Converges to a point where

$$\mathbf{u}^* = \beta \boldsymbol{\alpha} \tag{19}$$

with the reliability index

$$\beta = \boldsymbol{\alpha}^T \mathbf{u}^* \tag{20}$$

Probability of failure from normal CDF

$$p_f = \Phi(-\beta) \tag{21}$$

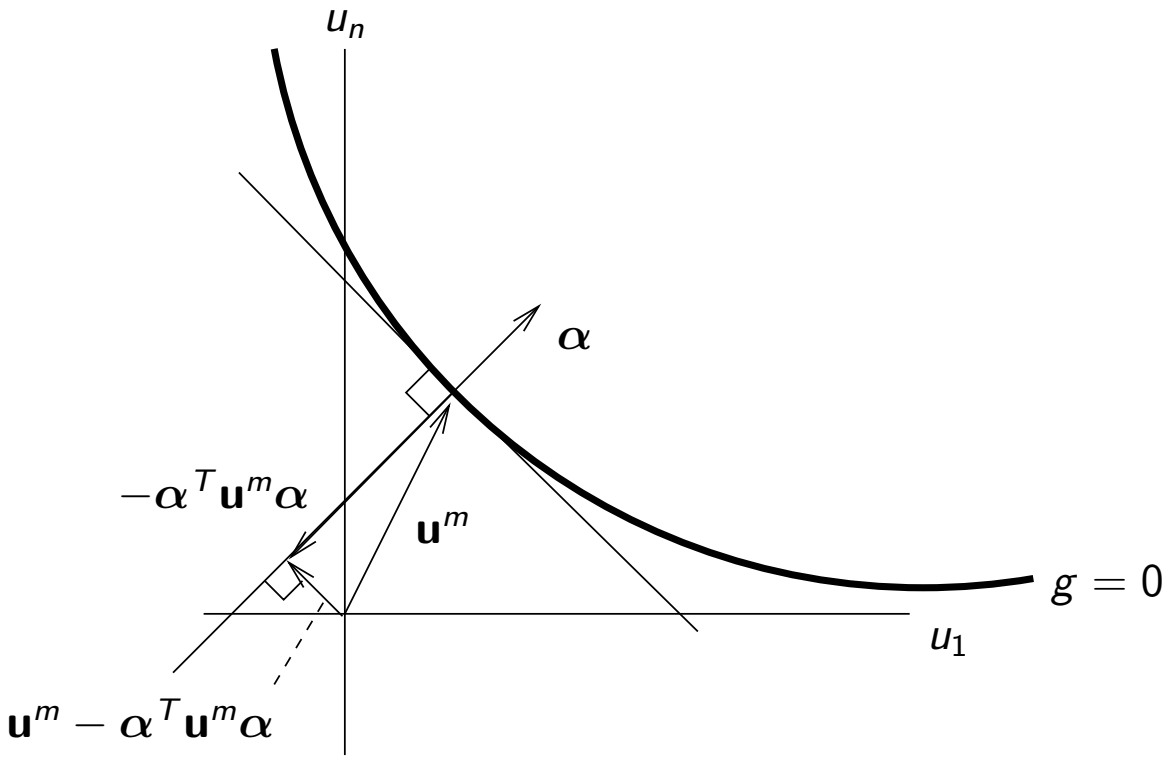
Convergence of HLRF Iteration

Design point, $\mathbf{u}^*$, where

$$\mathbf{u}^* - \boldsymbol{\alpha}^T \mathbf{u}^* \boldsymbol{\alpha} = \mathbf{0} \tag{22}$$

or magnitude is less than a specified tolerance

$$\left\| \mathbf{u}^* - \boldsymbol{\alpha}^T \mathbf{u}^* \boldsymbol{\alpha} \right\|_2 \leq tol \tag{23}$$



Sensitivity Vector

The vector $\boldsymbol{\alpha}$ is unit normal to failure surface

$$\boldsymbol{\alpha} = -\frac{\nabla_{\mathbf{u}} g}{\|\nabla_{\mathbf{u}} g\|}$$

In addition to providing search directions for HLRF iteration, component α_i indicates

- The nature of random variable *i* in **u**-space ($\alpha_i > 0 \rightarrow$ load, $\alpha_i < 0 \rightarrow$ resistance)
- The importance (via magnitude) of random variable *i* relative to other random variables in **u**-space

Transform to original **x**-space in order to determine nature and importance of random variables in problem domain

- Depends on $\boldsymbol{\alpha}$ and Jacobian of **x**-**u** transformation

Computation of Sensitivity Vector

All measures of sensitivity and importance depend on the gradient of the limit state function

$$\nabla_{\mathbf{u}} g = \nabla_{\mathbf{x}} g \frac{\partial \mathbf{x}}{\partial \mathbf{u}} \tag{24}$$

- $\partial \mathbf{x}/\partial \mathbf{u}$ – Jacobian of **x**-**u** transformation

For a limit state function defined in **x**-space in terms of finite element response, $\mathbf{U}_f$, which depends on an uncertain parameter θ, the chain rule gives

$$\nabla_{\mathbf{x}} g = \frac{\partial g}{\partial \mathbf{U}_f} \frac{\partial \mathbf{U}_f}{\partial \theta} \frac{\partial \theta}{\partial \mathbf{x}} \tag{25}$$

- $\partial g/\partial \mathbf{U}_f$ – analytic derivative of limit state function
- $\partial \mathbf{U}_f/\partial \theta$ – nodal response sensitivity of FE model
- $\partial \theta/\partial \mathbf{x}$ – mapping of random variable to FE model parameter

Direct Differentiation

Analytical $\partial \mathbf{U}_f / \partial \theta$ in gradient of limit state function

Pros:

- Requires as many limit state function evaluations as HLRF iterations
- Compute sensitivity as analysis proceeds
- Use factorized stiffness matrix, $\mathbf{K}_T$, for each parameter of FE model
- Not subject to round off error (beyond that of FE analysis)

Cons:

- Difficult to implement
- Must be implemented for all objects present in FE analysis when path-dependency is present

Finite Differences

Approximate $\partial \mathbf{U}_f / \partial \theta$ in gradient of limit state function

Pros:

- Easy to implement
- Applicable "for free" to all objects in FE analysis

Cons:

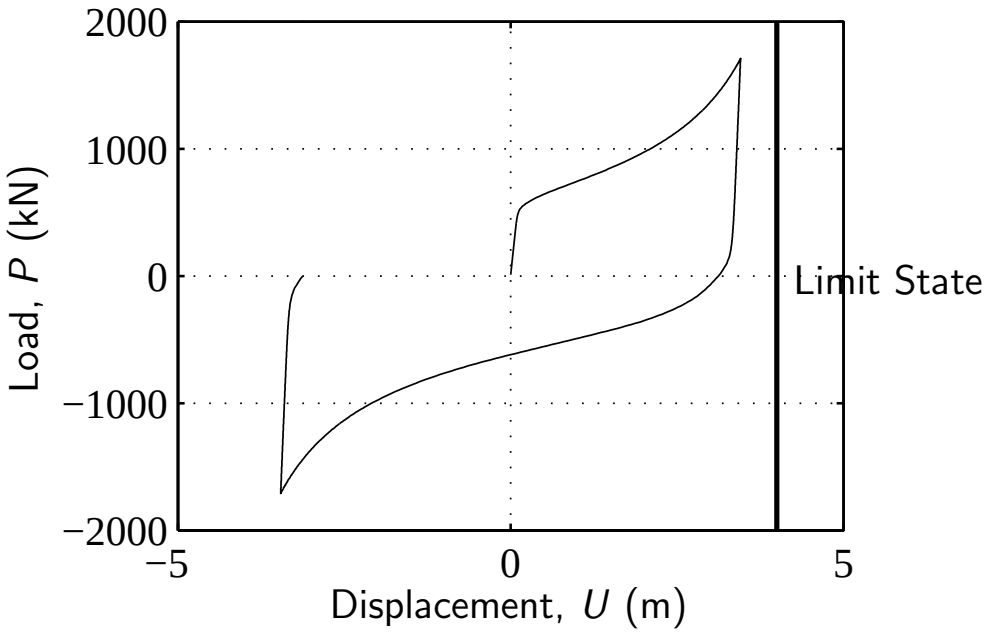
- Requires more limit state function evaulations than HLRF iterations
- Re-run analysis for each parameter perturbation
- Reform and factorize tangent stiffness matrix, $\mathbf{K}_T$, for each parameter perturbation at each iteration
- Subject to round off error based on parameter perturbation

Cantilever Reliability

Determine probability that peak displacement exceeds 4.0 m

$$g(\mathbf{x}) = 4.0 - U$$

$x_1 \sim LN(4.1e5, 0.05)$ MPa maps to yield stress, σ_y
$x_2 \sim LN(2.0e8, 0.1)$ MPa maps to elastic modulus, E
$x_3 \sim N(1710, 0.15)$ kN maps to applied load, P_{max}
$x_4 \sim N(5.0, 0.02)$ m maps to cantilever length, L
$x_5 \sim N(0.5, 0.04)$ m maps to cross-section depth, d



Reliability Results

HLRF converges in 7 iterations with both approaches to $\partial \mathbf{U}_f / \partial \theta$:

- DDM – 7 evaluations of limit state function
- Finite Difference – 42 evaluations of limit state function

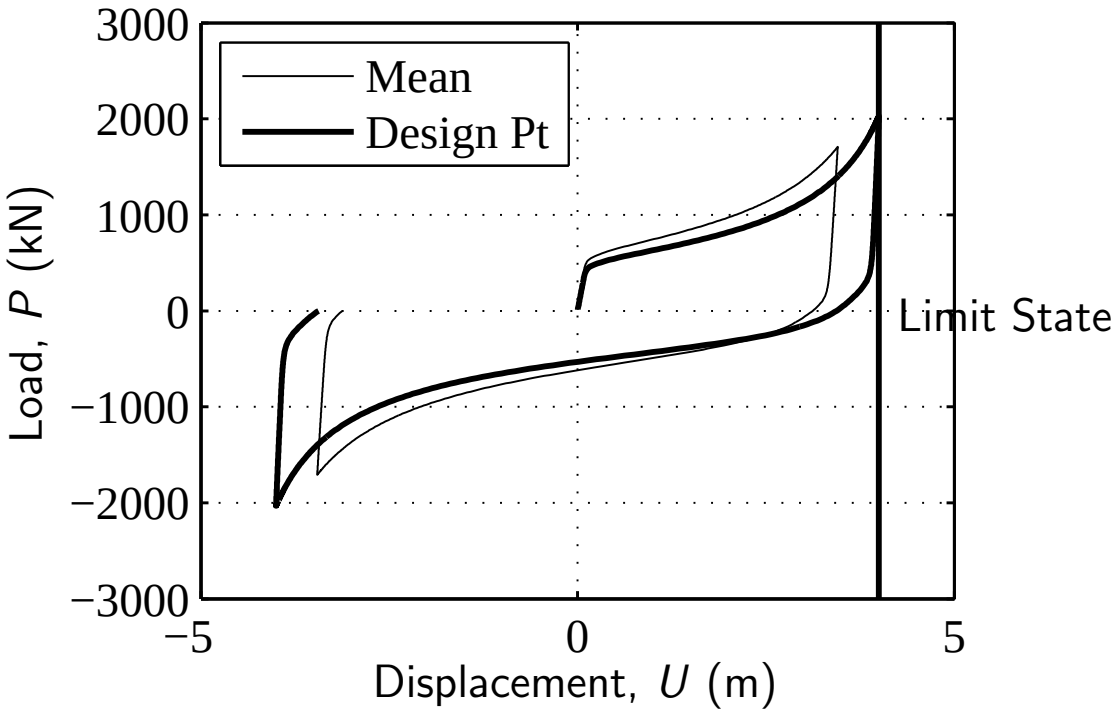
Reliability index, $\beta = 1.983$

Probability of failure, $p_f = \Phi(-1.983) = 2.368\%$

Random Variable, i		μ_i	x_i^*	α_i
1	σ_y	4.1e5 kPa	3.897e5 kPa	-0.2317
2	E	2.0e8 kPa	1.996e8 kPa	-0.005413
3	P_{max}	1710 kN	2032 kN	0.6393
4	L	5.0 m	5.100 m	0.5008
5	d	0.5 m	0.4787 m	-0.5356

Mean and Design Point Response

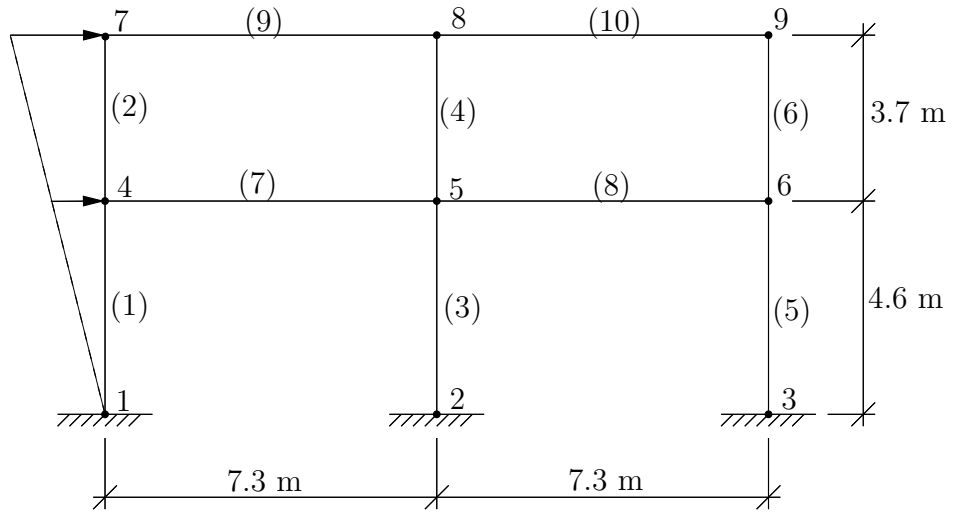
Cantilever load-displacement response at realization of random variables in $\mathbf{x}^*$

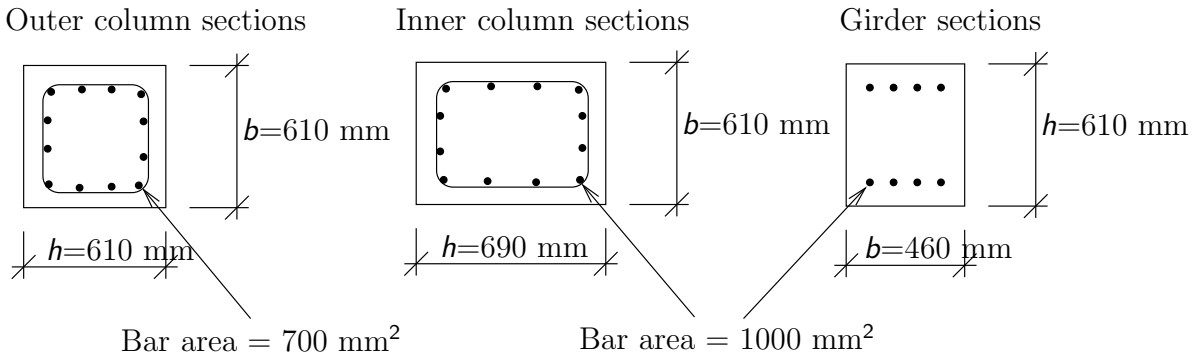


Reinforced Concrete Frame

Limit state – 2% roof drift

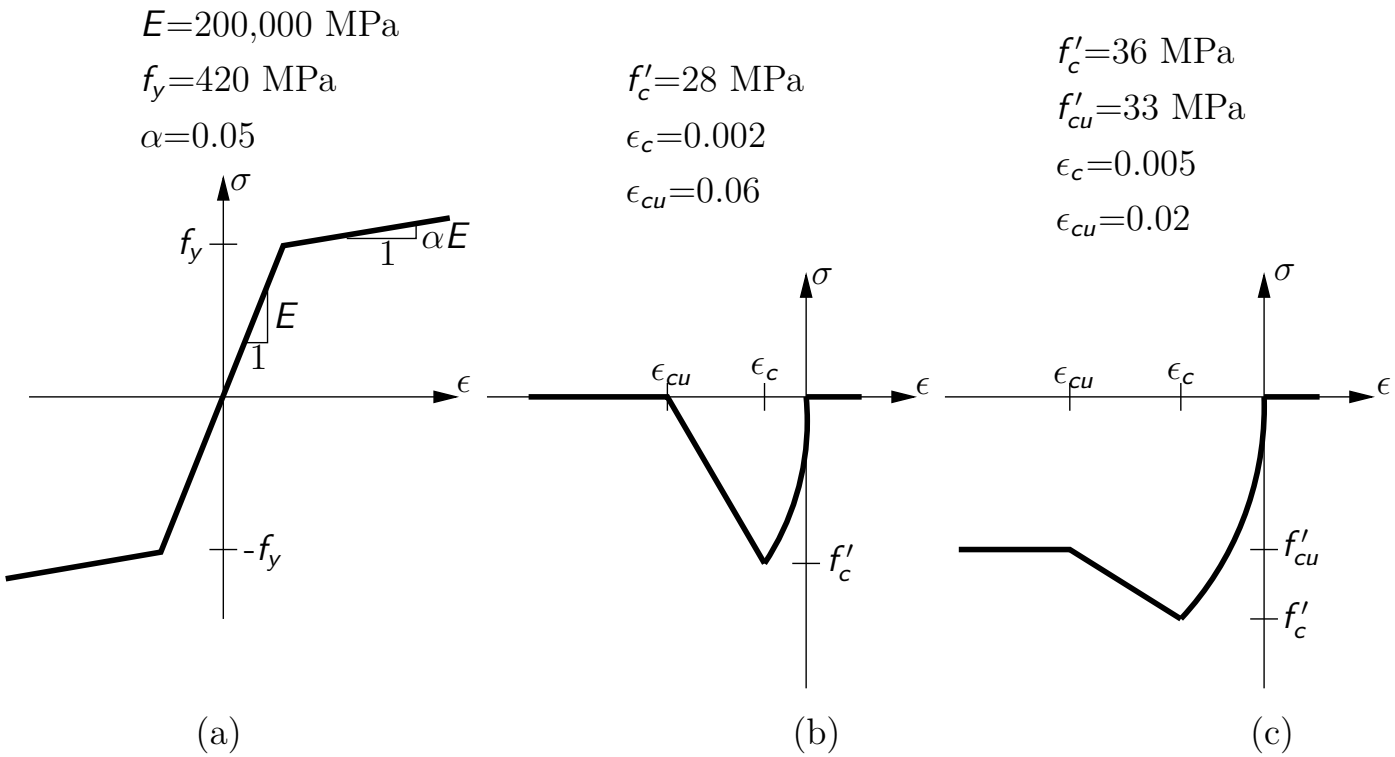
$$g(\mathbf{x}) = 0.02(8.3) - U_7 \tag{26}$$





Material Properties

Capture axial-moment interaction of reinforced concrete using numerical integration of stress-strain response over cross-sections



Uncertainty Modeling

Total of 142 random variables

For each member:

- Reinforcing steel material properties (LN and 0.6 group correlation)
- Concrete material properties (LN and 0.6 group correlation)
- Cross-section width and height (N and no group correlation)
- Reinforcing bar area (N and no group correlation)

For column members only:

- Cover concrete thickness (N and no group correlation)

For all joints:

- X and Y coordinates (N and no group correlation)

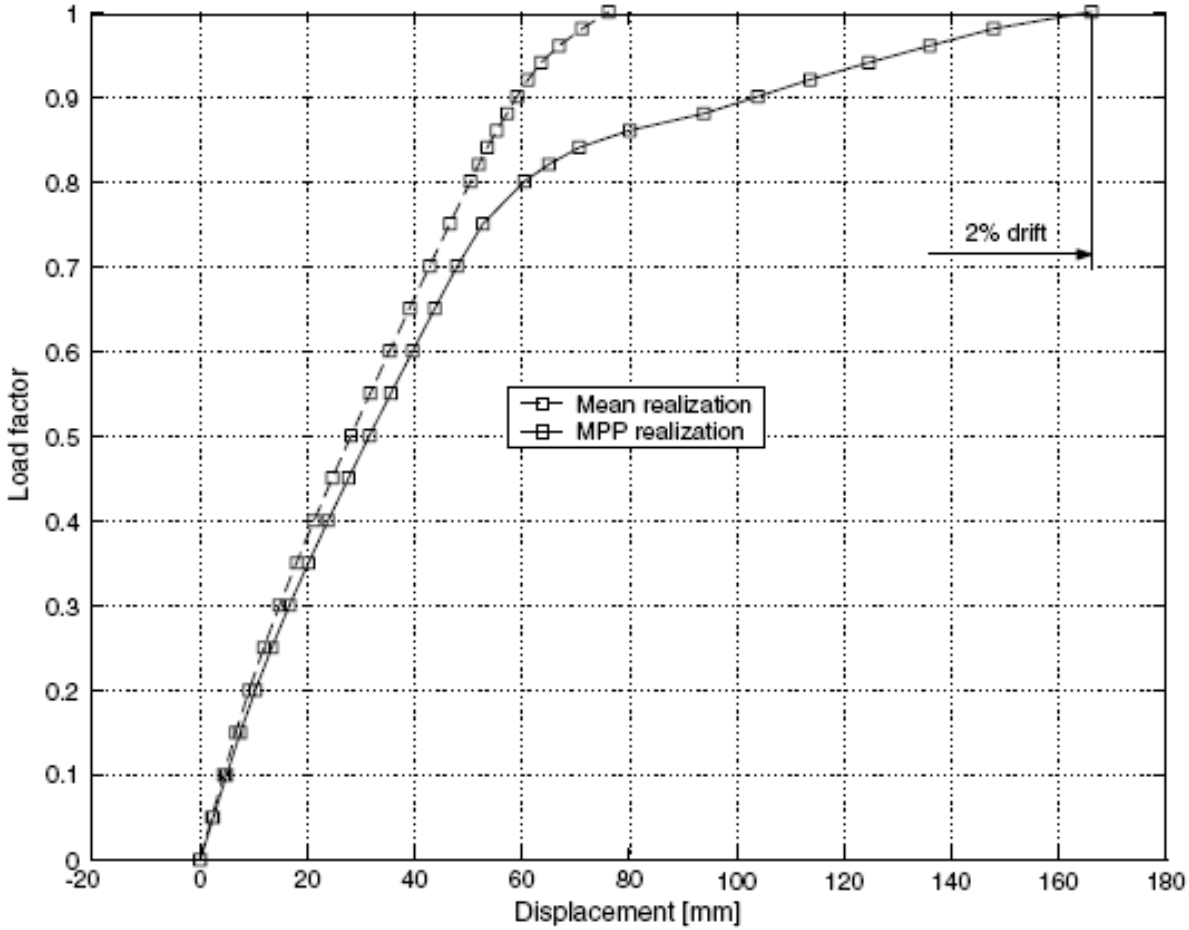
Deterministic gravity and lateral loads

Pushover Load-Displacement

At mean and design point realizations

HLRF converges in 9 iterations: $\beta = 2.76$, $p_f = 0.29\%$

- DDM – 9 limit state function evaluations
- Finite difference – 1200+ limit state function evaluations



Importance Ranking

Highest importance on depth of members framing in to interior joint

Steel yield stress and cover thickness also rank high

Object	Parameter	γ-Value
Member 3	h	−0.34697
Member 7	h	−0.34339
Member 4	h	−0.32343
Member 8	h	−0.31364
Member 4	f_y	−0.2853
Member 5	h	−0.24275
Member 8	f_y	−0.23581
Member 7	f_y	−0.23421
Member 3	f_y	−0.17896
Member 3	Cover depth	0.15903
Member 1	h	−0.15483
Member 4	Cover depth	0.14254
Member 7	Cover depth	0.1311
Member 5	Cover depth	0.12401
Member 4	ϵ_{cu}^{cover}	0.12336
Member 8	Cover depth	0.12045
Member 5	f_y	−0.12021
Member 1	f_y	−0.09227
Node 2	x-coordinate	−0.0902
Member 1	Cover depth	0.07958

Using γ importance vector instead of α as it accounts for correlation among random variables

Conclusions

- DDM is accurate and efficient alternative to finite differences in assessing structural response sensitivity
- Fewer function evaluations in reliability analysis
- Other applications in optimization and system identification

Ongoing work:

- Live load reliability analysis of bridge girders
- Particle finite element reliability analysis of fluid-structure interaction

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