

# Response Sensitivity of Geometrically Nonlinear Force-Based Frame Elements

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**Abstract:** To expand the scope of accurate and efficient gradient-based applications in structural engineering, the direct differentiation method (DDM) is applied to compute the response sensitivity of force-based frame finite elements with combined material and geometric nonlinearity where the transverse displacement field is determined by curvature-based displacement interpolation. Sensitivity is developed for element-level parameters including constitutive properties, cross-section dimensions, and integration points and weights, as well as structural-level parameters corresponding to nodal coordinates. The response sensitivity is found to be significantly more complicated than for geometrically linear force-based elements because it requires the derivative of the transverse displacement field under the condition of fixed basic forces. Finite-difference calculations verify the DDM sensitivity equations for material and geometric nonlinear force-based element response while reliability analysis of a gravity-loaded steel frame demonstrates the efficiency of the DDM sensitivity in a gradient-based application. DOI: [10.1061/\(ASCE\)ST.1943-541X.0000757](https://doi.org/10.1061/(ASCE)ST.1943-541X.0000757). © 2013 American Society of Civil Engineers.

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## Introduction

Recent literature has shown that force-based elements are superior to standard displacement-based formulations in simulating geometrically linear response of frame members with material nonlinearity (Alemdar and White 2005; Hjelmstad and Taciroglu 2005; Calabrese et al. 2010). Interpolation of the element displacement field is not required and accuracy of the computed solution depends only on the numerical integration method used in the element state determination (Spacone et al. 1996; Neuenhofer and Filippou 1997). As a result, mesh refinement is not necessary to simulate the spread of plasticity along a frame member. This is an important modeling consideration for structural engineering applications where multiple simulations are carried out to assess performance under increasing levels of demand.

The efficiency of force-based element formulations in simulating the material nonlinear response of frame structures also appeals to gradient-based applications such as reliability, optimization, and system identification. Although finite-difference approximations are readily available to find the gradient of structural response with respect to parameters, they are time consuming and prone to round-off error. The direct differentiation method (DDM) is an efficient and accurate alternative to finite differences (Zhang and Der Kiureghian 1993); however, it requires the implementation of analytic expressions for derivatives of the element response, which are often more complex to develop than the response equations themselves. In addition to a wide range of material and element formulations, the

DDM has been used in several aspects of finite-element (FE) response sensitivity analysis, including multipoint constraints (Gu et al. 2009) and follower loads (Pajot and Maute 2006) as well as the second derivative of structural response (Bebamzadeh and Haukaas 2008). Concurrent efforts by Scott et al. (2004) and Conte et al. (2004) led to DDM sensitivity implementations for geometrically linear force-based elements with material nonlinearity.

The corotational transformation (Crisfield 1991) provides an effective means of accounting for geometrically nonlinear frame response. In addition to simulating large displacement  $P$ - $\Delta$  effects of a single frame element, the corotational transformation allows a mesh of geometrically linear frame elements to be used to simulate the  $P$ - $\delta$  effects of a frame member. The corotational mesh approach makes the simulation of combined material and geometric nonlinearity straightforward; however, the necessary mesh refinement counteracts the computational advantages of force-based elements. The DDM sensitivity of the corotational transformation (Scott and Filippou 2007) enables gradient-based analysis of geometrically nonlinear frame response only if a corotational mesh is used for each member.

To maintain the coarse mesh advantages of force-based elements, Neuenhofer and Filippou (1998) developed a curvature-based displacement interpolation (CBDI) procedure that captures geometrically nonlinear response using a single element per frame member. In the CBDI procedure, the curvature field is approximated using Lagrange polynomials and then integrated to find the transverse displacement field. The CBDI procedure was extended to material nonlinearity by De Souza (2000) such that a single element can be used to simulate both material and geometric nonlinearity. Jafari et al. (2010) extended the CBDI formulation to include shear deformations in a curvature shear-based displacement interpolation (CSBDI) procedure. An alternative geometrically nonlinear force-based element state determination that uses finite-difference approximations was developed by Jeffers and Sotelino (2010) to simulate frame response to fire attack. Despite the modeling capabilities of geometrically nonlinear force-based frame elements, the associated response sensitivity has not been addressed in the literature.

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The objective of this paper is to develop DDM response sensitivity of material and geometrically nonlinear force-based elements that utilize the CDBI procedure in the determination of their transverse displacement field. After a brief overview of geometrically nonlinear force-based elements that use the CDBI procedure, analytic derivatives of the element response are presented. Material properties, cross-section dimensions, integration points and weights, and nodal coordinates are considered uncertain parameters in developing the DDM equations of the CDBI formulation, encompassing all potential element-level uncertainties in a finite response sensitivity element analysis. Numerical examples demonstrate the correctness of the analytic DDM equations as well as their efficiency over finite differences in a first-order reliability analysis.

## Geometrically Nonlinear Force-Based Element

Force-based frame elements are formulated in a basic system without rigid body displacement modes in terms of three degrees of freedom for planar elements (Filippou and Fenves 2004). As shown in Fig. 1(a), the axial force at end J is basic force  $q_1$ , and the moments at ends I and J are basic forces  $q_2$  and  $q_3$ , respectively. The work conjugate deformations are the change in element length,  $v_1$ , and rotations at each end,  $v_2$  and  $v_3$ .

The homogeneous solution for equilibrium between basic forces and internal section forces,  $\mathbf{s}$ , is expressed as a matrix-vector product

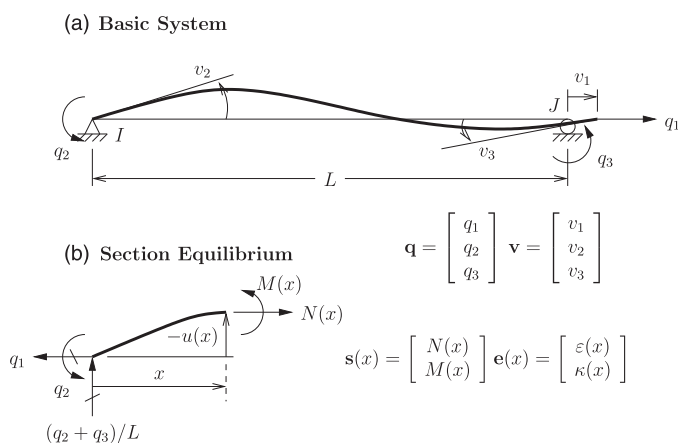
$$\mathbf{s}(x) = \mathbf{b}(x, u(x))\mathbf{q} \quad (1)$$

where the force interpolation matrix,  $\mathbf{b}$ , depends on the transverse displacement field,  $u(x)$ , as shown in Fig. 1(b)

$$\mathbf{b}(x, u(x)) = \begin{bmatrix} 1 & 0 & 0 \\ -u(\xi) & \xi - 1 & \xi \end{bmatrix}, \quad \xi = x/L \quad (2)$$

The dependence of the force interpolation matrix on the transverse displacement field accounts for geometric nonlinearity inside the basic system.

From the Hellinger-Reissner variational principle (De Souza 2000; Hjelmstad and Taciroglu 2005), compatibility of the element deformations with the internal section deformations,  $\mathbf{e}$ , is satisfied in integral form



**Fig. 1.** Frame element response definitions: (a) basic forces and deformations within the basic system; (b) equilibrium of basic forces and internal section forces

$$\mathbf{v} = \int_0^L \tilde{\mathbf{b}}^T[x, u(x)]\mathbf{e}(x)dx \quad (3)$$

with

$$\tilde{\mathbf{b}}(x, u(x)) = \begin{bmatrix} 1 & 0 & 0 \\ -u(\xi)/2 & \xi - 1 & \xi \end{bmatrix}, \quad \xi = x/L \quad (4)$$

where a factor of 1/2 arises from the Euler-Bernoulli beam theory for geometric nonlinearity.

Implementation of force-based elements requires numerical evaluation of Eq. (3), where the integrand is sampled at  $N_p$  discrete locations,  $x$ , each with associated integration weight,  $w$

$$\mathbf{v} \approx \sum_{i=1}^{N_p} \tilde{\mathbf{b}}^T(x_i, u(x_i))\mathbf{e}(x_i)w_i \quad (5)$$

The element flexibility matrix is obtained from the partial derivative of Eq. (5) with respect to basic forces

$$\mathbf{f} \equiv \frac{\partial \mathbf{v}}{\partial \mathbf{q}} = \sum_{i=1}^{N_p} \tilde{\mathbf{b}}_i^T \mathbf{f}_{si} \frac{\partial \mathbf{s}_i}{\partial \mathbf{q}} w_i + \sum_{i=1}^{N_p} \frac{\partial \tilde{\mathbf{b}}_i^T}{\partial \mathbf{q}} \mathbf{e}_i w_i \quad (6)$$

where evaluation of the section response is abbreviated; e.g.,  $\mathbf{e}_i \equiv \mathbf{e}(x_i)$ . The section flexibility matrix,  $\mathbf{f}_s$ , contains the partial derivative of the section deformations with respect to section forces

$$\mathbf{f}_s \equiv \frac{\partial \mathbf{e}}{\partial \mathbf{s}} = \begin{bmatrix} \frac{\partial \varepsilon}{\partial N} & \frac{\partial \varepsilon}{\partial M} \\ \frac{\partial \kappa}{\partial N} & \frac{\partial \kappa}{\partial M} \end{bmatrix} \quad (7)$$

The derivative of the section forces with respect to the basic forces [ $\partial \mathbf{s} / \partial \mathbf{q}$  in Eq. (6)] is obtained by differentiation of Eq. (1) with respect to  $\mathbf{q}$

$$\frac{\partial \mathbf{s}}{\partial \mathbf{q}} = \mathbf{b} + \frac{\partial \mathbf{b}}{\partial \mathbf{q}} \mathbf{q} \quad (8)$$

The section interpolation matrices depend on basic forces,  $\mathbf{q}$ , via the transverse displacement field

$$\frac{\partial \mathbf{b}}{\partial \mathbf{q}} = \frac{\partial \mathbf{b}}{\partial u} \frac{\partial u}{\partial \mathbf{q}}, \quad \frac{\partial \tilde{\mathbf{b}}}{\partial \mathbf{q}} = \frac{\partial \tilde{\mathbf{b}}}{\partial u} \frac{\partial u}{\partial \mathbf{q}} \quad (9)$$

Then, substitution of Eqs. (8) and (9) into Eq. (6) gives the element flexibility matrix

$$\mathbf{f} = \sum_{i=1}^{N_p} \tilde{\mathbf{b}}_i^T \mathbf{f}_{si} \mathbf{b}_i w_i + \sum_{i=1}^{N_p} \tilde{\mathbf{b}}_i^T \mathbf{f}_{si} \mathbf{q} \frac{\partial \mathbf{b}_i}{\partial u} \frac{\partial u_i}{\partial \mathbf{q}} w_i + \sum_{i=1}^{N_p} \frac{\partial \tilde{\mathbf{b}}_i^T}{\partial u} \mathbf{e}_i \frac{\partial u_i}{\partial \mathbf{q}} w_i \quad (10)$$

The derivatives of the section interpolation matrices with respect to transverse displacement are

$$\frac{\partial \mathbf{b}}{\partial u} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \frac{\partial \tilde{\mathbf{b}}}{\partial u} = \begin{bmatrix} 0 & 0 & 0 \\ -1/2 & 0 & 0 \end{bmatrix} \quad (11)$$

The element flexibility matrix is evaluated numerically by the same integration method used for the element compatibility relationship of Eq. (5). This matrix is inverted to basic stiffness,  $\mathbf{k} = \mathbf{f}^{-1}$ , and then transformed and assembled into the structural stiffness matrix

by standard FE procedures. Any integration method, including plastic hinge methods (Scott and Fenves 2006; Addessi and Ciampi 2007), can be used in the geometrically nonlinear force-based element state determination with the CBDI procedure described in the subsequent section; however, the Gauss-Legendre quadrature is the only approach that guarantees symmetry of the element stiffness matrix (De Souza 2000).

### Curvature-Based Displacement Interpolation

As observed in Eq. (2), the interpolation of section forces depends on the transverse displacement field, which is not tracked explicitly in the state determination of geometrically linear force-based elements. To account for geometric nonlinearity, Neuenhofer and Filippou (1998) proposed an approximation of the curvature field along an element using Lagrange polynomials

$$\kappa(\xi) = \sum_{j=1}^{N_p} l_j(\xi) \kappa_j, \quad \xi = x/L \quad (12)$$

where  $\kappa_j$  = curvature sampled at the  $j$ th integration point. The  $j$ th Lagrange basis polynomial,  $l_j(\xi)$ , evaluates to 1 at the  $j$ th integration point and zero at all other points

$$l_j(\xi) = \frac{\prod_{i=1, i \neq j}^{N_p} (\xi - \xi_i)}{\prod_{i=1, i \neq j}^{N_p} (\xi_j - \xi_i)} \quad (13)$$

Double integration of Eq. (12) and application of the boundary conditions for the basic system allow the transverse displacement at the  $N_p$  integration points,  $\mathbf{u} \equiv [u_j]$ , to be expressed as the matrix-vector product

$$\mathbf{u} = -L^2 \mathbf{I}^* \boldsymbol{\kappa} \quad (14)$$

where  $\boldsymbol{\kappa} \equiv [\kappa_j]$  = vector that collects curvatures from all  $N_p$  integration points and  $\mathbf{I}^*$  = CBDI influence matrix. The  $j$ th column of  $\mathbf{I}^*$  contains the transverse displacement field arising from a unit curvature imposed at the  $j$ th integration point. The relationship between Lagrange basis polynomials and the Vandermonde matrix (Golub and Van Loan 1996) allows the CBDI influence matrix to be computed from the product of two matrices

$$\mathbf{I}^* = \mathbf{h} \mathbf{g}^{-1} \quad (15)$$

Matrix  $\mathbf{h}$  contains polynomials that satisfy the boundary conditions of the basic system

$$\mathbf{h} \equiv [h_{ij}] = \frac{\xi_i^{j+1} - \xi_i}{j(j+1)} \quad (16)$$

Matrix  $\mathbf{g}$  is the Vandermonde matrix of monomials

$$\mathbf{g} \equiv [g_{ij}] = \xi_i^{j-1} \quad (17)$$

for which a closed-form inverse exists.

### Derivative of Transverse Displacement

The derivative of transverse displacement with respect to basic forces,  $\partial \mathbf{u} / \partial \mathbf{q}$ , is required to evaluate the element flexibility matrix

according to Eq. (10). Differentiation of Eq. (14) with respect to basic forces gives an  $N_p \times 3$  matrix in which curvature is the only term that depends on  $\mathbf{q}$  by way of the section constitutive relationship and element equilibrium

$$\frac{\partial \mathbf{u}}{\partial \mathbf{q}} = -L^2 \mathbf{I}^* \frac{\partial \boldsymbol{\kappa}}{\partial \mathbf{q}} = -L^2 \mathbf{I}^* \sum_{j=1}^{N_p} \frac{\partial \boldsymbol{\kappa}}{\partial s_j} \frac{\partial s_j}{\partial \mathbf{q}} \quad (18)$$

The  $N_p \times 2$  matrix  $\partial \boldsymbol{\kappa} / \partial s_j$  is all zeros except for the  $j$ th row, which contains the second row of the section flexibility matrix [Eq. (7)] at the  $j$ th integration point. Substituting  $\partial s_j / \partial \mathbf{q}$  from Eq. (8) into Eq. (18) gives

$$\frac{\partial \mathbf{u}}{\partial \mathbf{q}} = -L^2 \mathbf{I}^* \mathbf{F}_{\kappa s} \mathbf{B} + q_1 L^2 \mathbf{I}^* \mathbf{F}_{\kappa M} \frac{\partial \mathbf{u}}{\partial \mathbf{q}} \quad (19)$$

where  $\mathbf{B} = 2N_p \times 3$  matrix that aggregates the element's section force interpolation matrices, and  $\mathbf{F}_{\kappa s}$  and  $\mathbf{F}_{\kappa M} = N_p \times 2N_p$  and  $N_p \times N_p$  blocked diagonal matrices, respectively, of the section flexibility coefficients corresponding to curvature

$$\mathbf{B} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_{N_p} \end{bmatrix}, \quad \mathbf{F}_{\kappa s} = \begin{bmatrix} \frac{\partial \kappa_1}{\partial N_1} & \frac{\partial \kappa_1}{\partial M_1} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \frac{\partial \kappa_2}{\partial N_2} & \frac{\partial \kappa_2}{\partial M_2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \frac{\partial \kappa_{N_p}}{\partial N_{N_p}} & \frac{\partial \kappa_{N_p}}{\partial M_{N_p}} \end{bmatrix}, \quad \mathbf{F}_{\kappa M} = \begin{bmatrix} \frac{\partial \kappa_1}{\partial M_1} & 0 & \dots & 0 \\ 0 & \frac{\partial \kappa_2}{\partial M_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{\partial \kappa_{N_p}}{\partial M_{N_p}} \end{bmatrix} \quad (20)$$

Then, solving for  $\partial \mathbf{u} / \partial \mathbf{q}$  in Eq. (19) gives the following linear system of equations for the derivative of the transverse displacement field with respect to basic forces

$$(\mathbf{I} - q_1 L^2 \mathbf{I}^* \mathbf{F}_{\kappa M}) \frac{\partial \mathbf{u}}{\partial \mathbf{q}} = -L^2 \mathbf{I}^* \mathbf{F}_{\kappa s} \mathbf{B} \quad (21)$$

Eq. (21) reduces to the linear-elastic, prismatic case considered by Neuenhofer and Filippou (1998) when  $\mathbf{F}_{\kappa s}$  and  $\mathbf{F}_{\kappa M}$  of Eq. (20) are defined by the section flexibility coefficients  $\partial \kappa / \partial M = 1/EI$  and  $\partial \kappa / \partial N = 0$ . In this case, the solution for  $\partial \mathbf{u} / \partial \mathbf{q}$  depends only on the axial force,  $q_1$ . The solution to Eq. (21) depends additionally on changes to  $\mathbf{F}_{\kappa s}$  and  $\mathbf{F}_{\kappa M}$  when material nonlinearity is simulated at the section level. Upon convergence of the element state determination and solution of the structural-level equilibrium equations, derivatives of the element response can be taken with respect to uncertain model parameters, as described in the subsequent section.

## Response Sensitivity

The response sensitivity formulation for geometrically nonlinear force-based elements follows the same general approach as for geometrically linear elements (Scott et al. 2004). In this approach, the governing equations of element equilibrium and compatibility are differentiated with respect to  $\theta$ , an uncertain parameter of the structural model. Additional derivatives resulting from the dependence of the transverse displacement field on basic forces arise in the geometrically nonlinear case.

### Derivatives of Element Response Quantities

As in previous derivations of frame element response sensitivity, derivatives of key element response quantities must be defined to develop computable quantities for DDM response sensitivity analysis. First, the derivative of section forces with respect to an uncertain parameter of the structural model is

$$\frac{\partial \mathbf{s}}{\partial \theta} \equiv \mathbf{k}_s \frac{\partial \mathbf{e}}{\partial \theta} + \left. \frac{\partial \mathbf{s}}{\partial \theta} \right|_{\mathbf{e}} \quad (22)$$

where  $\mathbf{k}_s = \partial \mathbf{s} / \partial \mathbf{e}$  = section stiffness matrix, which is the inverse of section flexibility,  $\mathbf{k}_s = \mathbf{f}_s^{-1}$ . The term  $\mathbf{k}_s (\partial \mathbf{e} / \partial \theta)$  accounts for the implicit dependence of section forces on the parameter, by way of the section deformation sensitivity. Explicit dependence of section forces on the parameter is from  $\left. \partial \mathbf{s} / \partial \theta \right|_{\mathbf{e}}$ , the derivative of section forces with respect to  $\theta$  under the condition that  $\partial \mathbf{e} / \partial \theta$  is zero (Zhang and Der Kiureghian 1993). For path-dependent structural response, this conditional derivative depends on all model parameters, not just the parameters associated with the section force-deformation response (Haukaas 2006).

Second, noting that the geometrically nonlinear response depends on the transverse displacement, the derivative of  $u$  is introduced in terms of its implicit (via basic forces) and explicit dependence on  $\theta$

$$\frac{\partial u}{\partial \theta} \equiv \frac{\partial u}{\partial \mathbf{q}} \frac{\partial \mathbf{q}}{\partial \theta} + \left. \frac{\partial u}{\partial \theta} \right|_{\mathbf{q}} \quad (23)$$

where  $\partial u / \partial \mathbf{q}$  is computed according to Eq. (21) and  $\left. \partial u / \partial \theta \right|_{\mathbf{q}}$  = conditional derivative of the transverse displacement field. Finally, the derivative of basic forces,  $\mathbf{q}$ , with respect to  $\theta$  is defined in an analogous manner to the derivative of section forces in Eq. (22)

$$\frac{\partial \mathbf{q}}{\partial \theta} \equiv \mathbf{k} \frac{\partial \mathbf{v}}{\partial \theta} + \left. \frac{\partial \mathbf{q}}{\partial \theta} \right|_{\mathbf{v}} \quad (24)$$

The conditional derivative of basic forces,  $\left. \partial \mathbf{q} / \partial \theta \right|_{\mathbf{v}}$ , is the quantity that must be transformed and assembled in to the global system of response sensitivity equations. The solution for this conditional derivative is presented in the subsequent sections.

### Derivative of Governing Equations

The derivative of the equilibrium relationship [Eq. (1)] with respect to  $\theta$  is

$$\frac{\partial \mathbf{s}}{\partial \theta} = \mathbf{b} \frac{\partial \mathbf{q}}{\partial \theta} + \frac{\partial \mathbf{b}}{\partial \theta} \mathbf{q} \quad (25)$$

The derivatives of section and basic forces defined in Eqs. (22) and (24), respectively, are inserted into Eq. (25), giving

$$\mathbf{k}_s \frac{\partial \mathbf{e}}{\partial \theta} + \left. \frac{\partial \mathbf{s}}{\partial \theta} \right|_{\mathbf{e}} = \mathbf{b} \left( \mathbf{k} \frac{\partial \mathbf{v}}{\partial \theta} + \left. \frac{\partial \mathbf{q}}{\partial \theta} \right|_{\mathbf{v}} \right) + \frac{\partial \mathbf{b}}{\partial \theta} \mathbf{q} \quad (26)$$

From Eq. (26), the derivative of section deformations is

$$\frac{\partial \mathbf{e}}{\partial \theta} = \mathbf{f}_s \mathbf{b} \left( \mathbf{k} \frac{\partial \mathbf{v}}{\partial \theta} + \left. \frac{\partial \mathbf{q}}{\partial \theta} \right|_{\mathbf{v}} \right) + \mathbf{f}_s \left( \frac{\partial \mathbf{b}}{\partial \theta} \mathbf{q} - \left. \frac{\partial \mathbf{s}}{\partial \theta} \right|_{\mathbf{e}} \right) \quad (27)$$

where  $\partial \mathbf{v} / \partial \theta$  is obtained from the nodal displacement sensitivity according to the kinematic transformation between the structural system and the basic system of the element.

The derivative of the element compatibility relationship defined in Eq. (5) is

$$\frac{\partial \mathbf{v}}{\partial \theta} = \sum_{i=1}^{N_p} \tilde{\mathbf{b}}_i^T \frac{\partial \mathbf{e}_i}{\partial \theta} w_i + \sum_{i=1}^{N_p} \frac{\partial \tilde{\mathbf{b}}_i^T}{\partial \theta} \mathbf{e}_i w_i + \sum_{i=1}^{N_p} \tilde{\mathbf{b}}_i^T \mathbf{e}_i \frac{\partial w_i}{\partial \theta} \quad (28)$$

into which  $\partial \mathbf{e} / \partial \theta$  from Eq. (27) is inserted

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial \theta} = & \left( \sum_{i=1}^{N_p} \tilde{\mathbf{b}}_i^T \mathbf{f}_{si} \mathbf{b}_i w_i \right) \left( \mathbf{k} \frac{\partial \mathbf{v}}{\partial \theta} + \left. \frac{\partial \mathbf{q}}{\partial \theta} \right|_{\mathbf{v}} \right) + \sum_{i=1}^{N_p} \tilde{\mathbf{b}}_i^T \mathbf{f}_{si} \left( \frac{\partial \mathbf{b}_i}{\partial \theta} \mathbf{q} - \left. \frac{\partial \mathbf{s}_i}{\partial \theta} \right|_{\mathbf{e}} \right) w_i \\ & + \sum_{i=1}^{N_p} \frac{\partial \tilde{\mathbf{b}}_i^T}{\partial \theta} \mathbf{e}_i w_i + \sum_{i=1}^{N_p} \tilde{\mathbf{b}}_i^T \mathbf{e}_i \frac{\partial w_i}{\partial \theta} \end{aligned} \quad (29)$$

To isolate the conditional derivative of basic forces,  $\left. \partial \mathbf{q} / \partial \theta \right|_{\mathbf{v}}$ , the terms involving  $\partial \mathbf{v} / \partial \theta$  must be removed from Eq. (29), which is possible after differentiation of the section interpolation matrices,  $\mathbf{b}$  and  $\tilde{\mathbf{b}}$ , with respect to  $\theta$ .

### Derivative of Section Interpolation Matrices

As indicated in Eq. (2), the section force interpolation matrix,  $\mathbf{b}$ , depends on the section locations,  $\xi$ , and the transverse displacement field. The derivative of  $\mathbf{b}$  with respect to  $\theta$  then consists of two terms

$$\frac{\partial \mathbf{b}}{\partial \theta} = \frac{\partial \mathbf{b}}{\partial u} \frac{\partial u}{\partial \theta} + \frac{\partial \mathbf{b}}{\partial \xi} \frac{\partial \xi}{\partial \theta} = \frac{\partial \mathbf{b}}{\partial u} \left( \frac{\partial u}{\partial \mathbf{q}} \frac{\partial \mathbf{q}}{\partial \theta} + \left. \frac{\partial u}{\partial \theta} \right|_{\mathbf{q}} \right) + \frac{\partial \mathbf{b}}{\partial \xi} \frac{\partial \xi}{\partial \theta} \quad (30)$$

where  $\partial u / \partial \theta$  has been expanded using Eq. (23). An expression identical to Eq. (30) exists for the derivative of  $\tilde{\mathbf{b}}$  defined in Eq. (4). Inserting these derivatives into Eq. (29) yields

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial \theta} = & \left( \sum_{i=1}^{N_p} \tilde{\mathbf{b}}_i^T \mathbf{f}_{si} \mathbf{b}_i w_i + \sum_{i=1}^{N_p} \tilde{\mathbf{b}}_i^T \mathbf{f}_{si} \frac{\partial \mathbf{b}_i}{\partial u_i} \mathbf{q} \frac{\partial u_i}{\partial \mathbf{q}} w_i + \sum_{i=1}^{N_p} \frac{\partial \tilde{\mathbf{b}}_i^T}{\partial u_i} \mathbf{e}_i \frac{\partial u_i}{\partial \mathbf{q}} w_i \right) \\ & \times \left( \mathbf{k} \frac{\partial \mathbf{v}}{\partial \theta} + \left. \frac{\partial \mathbf{q}}{\partial \theta} \right|_{\mathbf{v}} \right) - \sum_{i=1}^{N_p} \tilde{\mathbf{b}}_i^T \mathbf{f}_{si} \left. \frac{\partial \mathbf{s}_i}{\partial \theta} \right|_{\mathbf{e}} w_i + \sum_{i=1}^{N_p} \tilde{\mathbf{b}}_i^T \mathbf{f}_{si} \frac{\partial \mathbf{b}_i}{\partial u_i} \mathbf{q} \left. \frac{\partial u_i}{\partial \theta} \right|_{\mathbf{q}} w_i \\ & + \sum_{i=1}^{N_p} \frac{\partial \tilde{\mathbf{b}}_i^T}{\partial u_i} \mathbf{e}_i \left. \frac{\partial u_i}{\partial \theta} \right|_{\mathbf{q}} w_i + \sum_{i=1}^{N_p} \tilde{\mathbf{b}}_i^T \mathbf{e}_i \frac{\partial w_i}{\partial \theta} + \sum_{i=1}^{N_p} \tilde{\mathbf{b}}_i^T \mathbf{f}_{si} \frac{\partial \mathbf{b}_i}{\partial \xi_i} \mathbf{q} \frac{\partial \xi_i}{\partial \theta} w_i \\ & + \sum_{i=1}^{N_p} \frac{\partial \tilde{\mathbf{b}}_i^T}{\partial \xi_i} \mathbf{e}_i \frac{\partial \xi_i}{\partial \theta} w_i \end{aligned} \quad (31)$$

With the definition of the element flexibility matrix in Eq. (10) and the identity  $\mathbf{f} \mathbf{k} = \mathbf{I}$ , the terms involving  $\partial \mathbf{v} / \partial \theta$  in Eq. (31) cancel. This leaves the following expression for the conditional derivative of basic forces:

$$\begin{aligned} \frac{\partial \mathbf{q}}{\partial \theta} \Big|_{\mathbf{v}} = & \mathbf{k} \left( \sum_{i=1}^{N_p} \tilde{\mathbf{b}}_i^T \mathbf{f}_{si} \frac{\partial \mathbf{s}_i}{\partial \theta} \Big|_{\mathbf{e}} w_i - \sum_{i=1}^{N_p} \tilde{\mathbf{b}}_i^T \mathbf{f}_{si} \frac{\partial \mathbf{b}_i}{\partial u_i} \mathbf{q} \frac{\partial u_i}{\partial \theta} \Big|_{\mathbf{q}} w_i \right. \\ & - \sum_{i=1}^{N_p} \frac{\partial \tilde{\mathbf{b}}_i^T}{\partial u_i} \mathbf{e}_i \frac{\partial u_i}{\partial \theta} \Big|_{\mathbf{q}} w_i - \sum_{i=1}^{N_p} \tilde{\mathbf{b}}_i^T \mathbf{e}_i \frac{\partial w_i}{\partial \theta} \\ & \left. - \sum_{i=1}^{N_p} \tilde{\mathbf{b}}_i^T \mathbf{f}_{si} \frac{\partial \mathbf{b}_i}{\partial \xi_i} \mathbf{q} \frac{\partial \xi_i}{\partial \theta} w_i - \sum_{i=1}^{N_p} \frac{\partial \tilde{\mathbf{b}}_i^T}{\partial \xi_i} \mathbf{e}_i \frac{\partial \xi_i}{\partial \theta} w_i \right) \quad (32) \end{aligned}$$

The derivatives  $\partial w/\partial \theta$  and  $\partial \xi/\partial \theta$  will be nonzero if  $\theta$  corresponds to a differentiable parameter of the element integration rule; e.g., a plastic hinge length. This case requires derivatives of the section interpolation matrices with respect to  $\xi$

$$\frac{\partial \mathbf{b}}{\partial \xi} = \frac{\partial \tilde{\mathbf{b}}}{\partial \xi} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad (33)$$

If Gaussian-based quadrature is used along the entire element domain, the derivatives  $\partial w/\partial \theta$  and  $\partial \xi/\partial \theta$  will be zero because the locations and weights of the integration points are determined solely from the analyst-specified  $N_p$ , which is not differentiable. Regardless of the type of parameter, the conditional derivative  $\partial u/\partial \theta \Big|_{\mathbf{q}}$  must be computed to solve for the conditional derivative of basic forces according to Eq. (32).

### Derivative of Transverse Displacement Field

The derivative of the transverse displacement field requires differentiation of the CBDI approximation defined in Eq. (14)

$$\frac{\partial \mathbf{u}}{\partial \theta} \Big|_{\mathbf{q}} = -L^2 \mathbf{1}^* \frac{\partial \boldsymbol{\kappa}}{\partial \theta} \Big|_{\mathbf{q}} - 2L \frac{\partial L}{\partial \theta} \mathbf{1}^* \boldsymbol{\kappa} - L^2 \frac{\partial \mathbf{1}^*}{\partial \theta} \boldsymbol{\kappa} \quad (34)$$

where the derivative of  $\boldsymbol{\kappa}$  carries the conditional derivative notation because it is the only term in Eq. (14) that depends on basic forces,  $\mathbf{q}$ . To find this conditional derivative of curvature,  $\partial \mathbf{q}/\partial \theta = 0$  is imposed in Eq. (27)

$$\frac{\partial \mathbf{e}}{\partial \theta} \Big|_{\mathbf{q}} = \mathbf{f}_s \left( \frac{\partial \mathbf{b}}{\partial \theta} \Big|_{\mathbf{q}} \mathbf{q} - \frac{\partial \mathbf{s}}{\partial \theta} \Big|_{\mathbf{e}} \right) \quad (35)$$

where  $\partial \mathbf{b}/\partial \theta \Big|_{\mathbf{q}}$ , the conditional derivative of the section force interpolation matrix, appears in turn. Then, applying the condition  $\partial \mathbf{q}/\partial \theta = \mathbf{0}$  to Eq. (30) and inserting this result into the previous equation gives

$$\frac{\partial \mathbf{e}}{\partial \theta} \Big|_{\mathbf{q}} = \mathbf{f}_s \left( \frac{\partial \mathbf{b}}{\partial u} \frac{\partial u}{\partial \theta} \Big|_{\mathbf{q}} + \frac{\partial \mathbf{b}}{\partial \xi} \mathbf{q} \frac{\partial \xi}{\partial \theta} - \frac{\partial \mathbf{s}}{\partial \theta} \Big|_{\mathbf{e}} \right) \quad (36)$$

Exploiting the sparsity of  $\partial \mathbf{b}/\partial u$  and  $\partial \mathbf{b}/\partial \xi$  defined in Eqs. (11) and (33), respectively, this equation simplifies to

$$\frac{\partial \mathbf{e}}{\partial \theta} \Big|_{\mathbf{q}} = \mathbf{f}_s \left( \begin{bmatrix} 0 \\ -q_1 \end{bmatrix} \frac{\partial u}{\partial \theta} \Big|_{\mathbf{q}} + \begin{bmatrix} 0 \\ q_2 + q_3 \end{bmatrix} \frac{\partial \xi}{\partial \theta} - \frac{\partial \mathbf{s}}{\partial \theta} \Big|_{\mathbf{e}} \right) \quad (37)$$

Then, selecting the derivative of curvature from this equation and aggregating this result over all, the  $N_p$  integration points gives

$$\frac{\partial \boldsymbol{\kappa}}{\partial \theta} \Big|_{\mathbf{q}} = -q_1 \mathbf{F}_{\kappa M} \frac{\partial u}{\partial \theta} \Big|_{\mathbf{q}} + (q_2 + q_3) \mathbf{F}_{\kappa M} \frac{\partial \xi}{\partial \theta} - \mathbf{F}_{\kappa s} \frac{\partial \mathbf{s}}{\partial \theta} \Big|_{\mathbf{e}} \quad (38)$$

Eq. (38) contains the block-diagonal section flexibility matrices,  $\mathbf{F}_{\kappa M}$  and  $\mathbf{F}_{\kappa s}$ , defined in Eq. (20), along with block vectors,  $\partial \xi/\partial \theta$  and  $\partial \mathbf{s}/\partial \theta \Big|_{\mathbf{e}}$ , of the derivatives of the integration point locations and section forces, respectively.

The conditional derivative of curvature defined in Eq. (38) is then substituted in to Eq. (34), and the solution for  $\partial \mathbf{u}/\partial \theta \Big|_{\mathbf{q}}$  ensues from the following linear system of equations:

$$\begin{aligned} (\mathbf{I} - q_1 L^2 \mathbf{1}^* \mathbf{F}_{\kappa M}) \frac{\partial \mathbf{u}}{\partial \theta} \Big|_{\mathbf{q}} = & L^2 \mathbf{1}^* \mathbf{F}_{\kappa s} \frac{\partial \mathbf{s}}{\partial \theta} \Big|_{\mathbf{e}} - 2L \frac{\partial L}{\partial \theta} \mathbf{1}^* \boldsymbol{\kappa} \\ & - (q_2 + q_3) L^2 \mathbf{1}^* \mathbf{F}_{\kappa M} \frac{\partial \xi}{\partial \theta} - L^2 \frac{\partial \mathbf{1}^*}{\partial \theta} \boldsymbol{\kappa} \quad (39) \end{aligned}$$

where the left-hand side matrix is the same as that required for computing  $\partial \mathbf{u}/\partial \mathbf{q}$  in Eq. (21). All terms on the right-hand side of this equation are computable at the converged element state, where  $\partial \mathbf{s}/\partial \theta \Big|_{\mathbf{e}}$  is computed from the section force-deformation response and the derivative of the element length  $\partial L/\partial \theta$  is computed from uncertain coordinates of the element nodes. The vector  $\partial \xi/\partial \theta$  is nonzero only if  $\theta$  corresponds to the location of an element integration point, in which case  $\partial \mathbf{1}^*/\partial \theta$ , the derivative of the CBDI influence matrix, will also be nonzero. Differentiation of Eq. (15) with respect to  $\theta$  gives

$$\frac{\partial \mathbf{1}^*}{\partial \theta} = \frac{\partial \mathbf{h}}{\partial \theta} \mathbf{g}^{-1} + \mathbf{h} \frac{\partial \mathbf{g}^{-1}}{\partial \theta} \quad (40)$$

Using the derivative of a matrix inverse, Eq. (40) can be expressed in terms of  $\partial \mathbf{g}/\partial \theta$

$$\frac{\partial \mathbf{1}^*}{\partial \theta} = \left( \frac{\partial \mathbf{h}}{\partial \theta} - \mathbf{h} \mathbf{g}^{-1} \frac{\partial \mathbf{g}}{\partial \theta} \right) \mathbf{g}^{-1} \quad (41)$$

The derivatives  $\partial \mathbf{h}/\partial \theta$  and  $\partial \mathbf{g}/\partial \theta$  are obtained by differentiation of Eqs. (16) and (17)

$$\frac{\partial h_{ij}}{\partial \theta} = \left( \frac{(j+1)\xi_i^j - 1}{j(j+1)} \right) \frac{\partial \xi_i}{\partial \theta}, \quad \frac{\partial g_{ij}}{\partial \theta} = [(j-1)\xi_i^{j-2}] \frac{\partial \xi_i}{\partial \theta} \quad (42)$$

With the solution for  $\partial \mathbf{u}/\partial \theta \Big|_{\mathbf{q}}$  established, all element-level components to the two-phase DDM analysis are computable (Zhang and Der Kiureghian 1993). In Phase I, the conditional derivative  $\partial \mathbf{q}/\partial \theta \Big|_{\mathbf{v}}$  from Eq. (32) is assembled into the global system of sensitivity equations. Then, after solution for the nodal response sensitivity,  $\partial \mathbf{e}/\partial \theta$  from Eq. (27) is computed to update the section response sensitivity for path dependency in Phase II.

### Examples

The DDM response sensitivity equations for geometrically nonlinear force-based elements have been implemented in the *OpenSees* FE software framework (McKenna et al. 2010). The first example verifies that the DDM implementation is correct for a simply supported beam with geometric nonlinearity and path-dependent material response. The second example involves a first-order reliability analysis of a steel frame with geometric and material nonlinearity.

#### Geometrically Nonlinear Elastoplastic Beam

The prismatic simply supported beam shown in Fig. 2 with an eccentric axial load is a commonly used illustrative example of CBDI procedures (Neuenhofer and Filippou 1998; Jafari et al. 2010) where a single geometrically nonlinear force-based element

simulates linear-elastic buckling. Additional modeling features were added to this example to verify the DDM response sensitivity for the material nonlinear response assumed in the aforementioned derivations.

The member is  $W14 \times 90$  with length  $L = 5,080$  mm. To account for material properties and cross-section dimensions as uncertain parameters, a fiber-discretization of the cross section is used with 20 fibers along the web depth and two fibers in each flange. The stress-strain response is elastoplastic with yield stress  $\sigma_y = 240$  MPa and elastic and hardening moduli of  $E = 200,000$  MPa and  $H = 11,000$  MPa, respectively. Gauss-Radau plastic hinge integration is used, giving the six integration points shown in Fig. 2, each of which is assigned the aforementioned fiber-discretized cross section. The plastic hinge lengths are assumed equal to 1.5 times the member depth; i.e.,  $l_{pI} = l_{pJ} = 1.5d$ . Eccentricity of the axial load is assumed to be equal to half the member depth to accentuate the axial-moment interaction in the CBDI state determination.

The eccentric axial load,  $P(t) = P_{\max} \sin(t)$ , was applied through pseudotime steps in the range  $t = [0, 2]$  such that a state of unloading was reached and path dependency was activated. The peak load value,  $P_{\max} = 0.8 EI/L^2$ , was well below the critical Euler buckling load  $\pi^2 EI/L^2$ . The time history of  $U$ , the rotation at end J of the beam, is shown in Fig. 3(a). Material yielding occurred at  $t = 0.78$ , which corresponded to a load of  $P = 0.56 EI/L^2$ . After yielding, the rotation increased rapidly until unloading began at  $t = \pi/2$ . Significant amplification of the internal bending moment over the geometrically linear solution is observed in Fig. 3(b) at the peak load,  $P_{\max}$ .

Given the computed response shown in Fig. 3, the standard approach to verifying its sensitivity with respect to an uncertain parameter is to compare the DDM sensitivity to that obtained by finite-difference calculations. In the limit, as a parameter perturbation,

$\Delta\theta$ , decreases to zero, the finite-difference approximation of the nodal displacement sensitivity should approach the DDM result

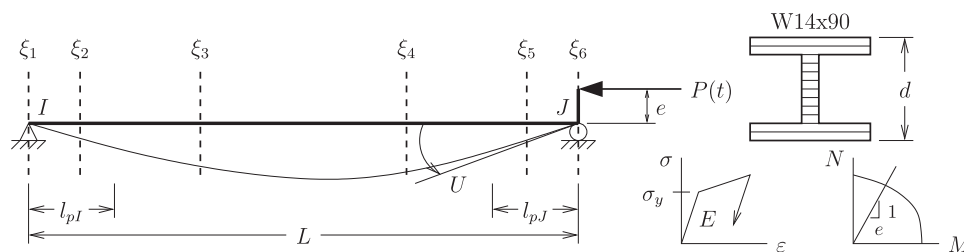
$$\lim_{\Delta\theta \rightarrow 0} \frac{U(\theta + \Delta\theta) - U(\theta)}{\Delta\theta} = \frac{\partial U}{\partial \theta} \quad (43)$$

where a forward finite-difference calculation is shown.

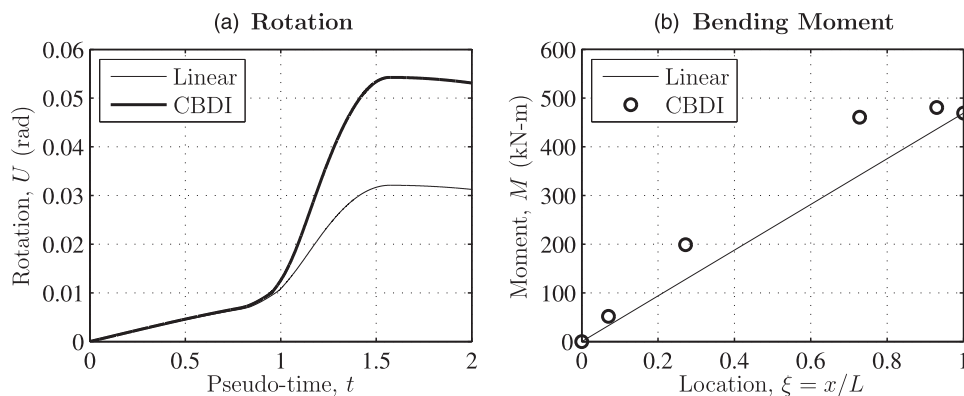
The sensitivity of the beam end rotation with respect to the section-level parameters of the elastic modulus and the member depth is shown in Fig. 4, where the vertical axis is scaled by the assumed parameter values. The finite-difference perturbation is 0.0001 times the assumed parameter value, which is sufficiently small to satisfy Eq. (43). The agreement of the finite-difference calculations with the DDM sensitivity verifies the geometrically nonlinear DDM implementation. Both of these section-level parameters act as resistance variables, which is evidenced by the negative sensitivity values. For reference, the geometrically linear DDM response sensitivity is shown. Discrete jumps in the sensitivity are observed as the individual fibers switch from elastic to plastic states (Conte et al. 2003).

The sensitivity of the beam rotation to elastoplastic material parameters yield stress,  $\sigma_y$ , and hardening modulus,  $H$ , is shown in Fig. 5. The DDM sensitivity for the CBDI formulation is compared with the finite-difference computations, as well as to the geometrically linear response sensitivity. For both parameters, the sensitivity is zero prior to material yielding. Again, there is agreement of the DDM and finite-difference sensitivity. Both parameters act as resistance variables, with the response being more sensitive to changes in yield stress than to the hardening modulus.

Sensitivity with respect to nodal coordinates at end J of the beam is shown in Fig. 6, where it is observed that both coordinates act as load variables. An increase in either the X- or Y-coordinate at end J makes the beam more unstable because of the loss of material and



**Fig. 2.** Simply supported elastoplastic beam with eccentric axial load, Gauss-Radau plastic hinge integration, fiber-discretized cross section, and stress-strain behavior



**Fig. 3.** Beam response with linear and nonlinear geometry: (a) rotation time history; (b) distribution of bending moment along the beam at peak load

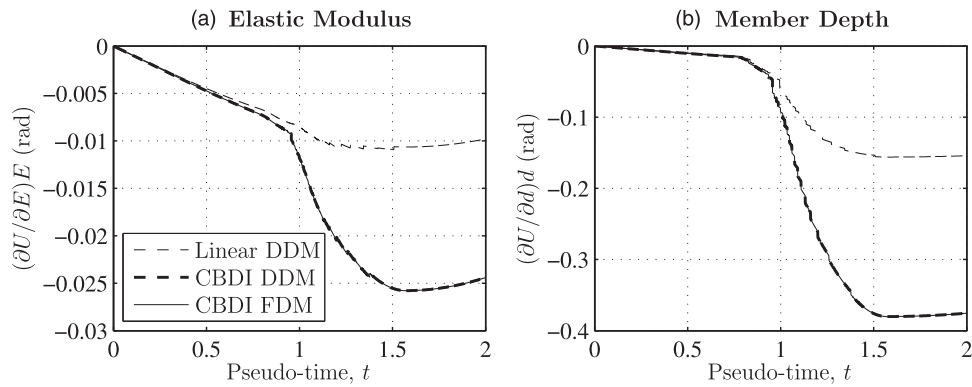


Fig. 4. Sensitivity of beam response to elastic modulus and member depth

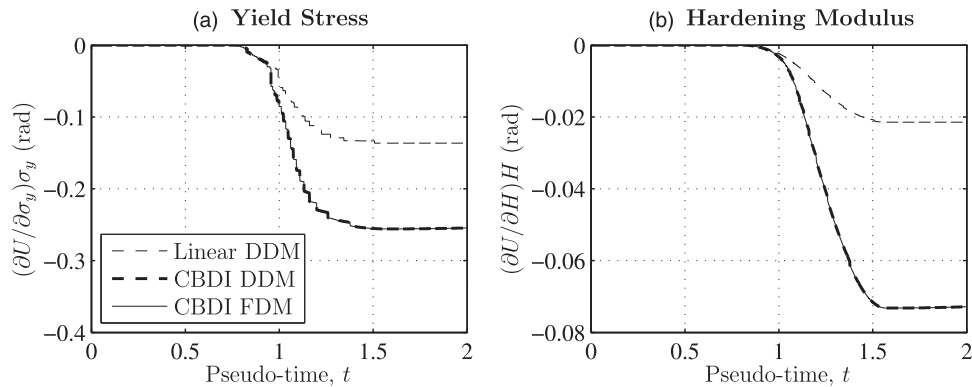


Fig. 5. Sensitivity of beam response to material yield stress and hardening modulus

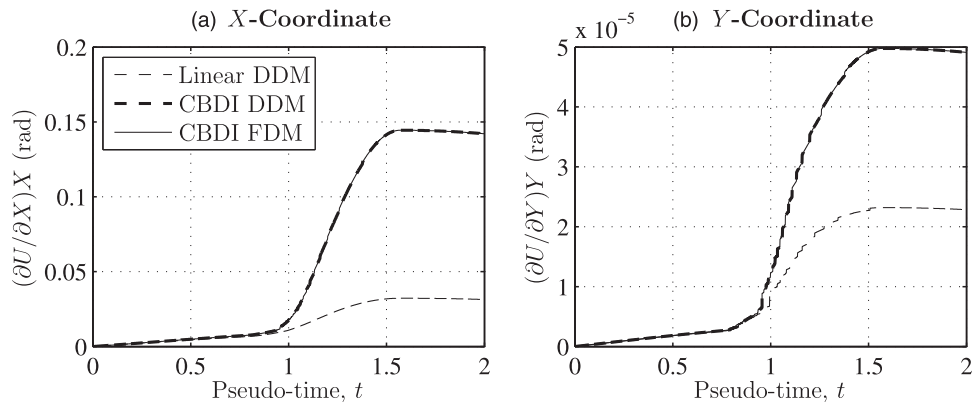


Fig. 6. Sensitivity of beam response to nodal coordinates at end J

geometric stiffness that results from an increase in beam length. It is observed in Fig. 6, as well as for the section-level parameters shown in Figs. 4 and 5, that the amplification of geometrically nonlinear response sensitivity over the geometrically linear case is generally greater than the amplification of response shown in Fig. 3(a).

The final set of sensitivity results is for the assumed plastic hinge lengths,  $l_{pl}$  and  $l_{pJ}$ , at each end of the beam. As shown in Fig. 7, there is agreement between the finite differences and the DDM for the CBDI response sensitivity. It is observed that, after material yielding, the hinge lengths act as either load or resistance variables depending on the load level. This is the case for both linear and nonlinear geometry; however, there is no correlation between the formulations and

the load levels at which the plastic hinge length parameters switch between load and resistance behavior.

### Steel Frame Reliability

With the DDM response sensitivity verified using finite-difference calculations, a FE reliability analysis is shown in this example. The structural model, originally developed by Malek (2001), is of an industrial building where two columns provide lateral support for 11 bays and system strength is controlled by the loss of lateral stability under gravity loading. In quantifying the effect of uncertain geometric and material properties on the frame's lateral strength, Buonopane

(2008) simplified the model to three bays with each outer column representing a group of five leaning columns, as shown in Fig. 8. An initial sway equal to 1/500 of the frame height was imposed to trigger lateral instability under the factored load combination  $1.2D + 1.6L$ .

A single force-based CDBI element is used for each column member. The response of the interior columns is obtained using Gauss-Radau plastic hinge integration, and the hinge lengths are equal to 1.5 times the member depth. A fiber discretized cross section is used for all column members with an elastic-perfectly plastic stress-strain relationship (20 web fibers and two fibers per flange).

The vertical load-lateral displacement response of the frame with the mean parameter values listed in Table 1 is shown in Fig. 9. The mean response reveals that the frame is stable under the factored design load (analysis load factor  $\lambda = 1.0$ ) and loses stability when the load factor increases to  $\lambda = 1.3$  at a lateral displacement of 42 mm. At this peak response, the axial force in Column C2 is 1,232 kN and the maximum end moment is 213 kN·m at the top. Although Column C3 sees a higher axial force of 1,270 kN, the maximum end moment is lower, at 139 kN·m. These results agree with the analysis by Buonopane (2008), who used a corotational mesh of four geometrically linear displacement-based elements to simulate the response of each column member.

With the mean response established for the given parameter values, first-order reliability method (FORM) analysis (Ditlevsen and Madsen 1996) is carried out for a performance function,  $g$ ,

which indicates failure ( $g \leq 0$ ) when the lateral roof displacement,  $U$ , exceeds 50 mm

$$g = 50 - U \quad (44)$$

and where the factored load combination is treated as deterministic. This performance function addresses the probability that the frame will become unstable when the analysis load factor is  $\lambda = 1.0$ . The yield strength, elastic modulus, section depth, and plastic hinge lengths of Columns C2 and C3, as well as the  $X$ - and  $Y$ -coordinates at the top of all four columns are considered to be uncertain model parameters with the statistical distributions listed in Table 1. All random variables are uncorrelated.

Using the DDM to evaluate  $\partial U / \partial \theta$  in the gradient of the performance function of Eq. (44), the FORM analysis converges in four iterations to the most probable point of failure, or design point, with a reliability index of  $\beta = 3.435$ . The FORM analysis requires  $2.7 \times 10^5$  clicks per iteration, where the number of clicks comes from a high-resolution, system-dependent counter in  $Tcl/Tk$  (Welch 2000). When using finite differences to evaluate the gradient of the performance function, the FORM analysis converges in the same number of iterations; however, it requires  $4.7 \times 10^6$  clicks per iteration, which is about 17.4 times higher than the analysis where gradients were evaluated by the DDM. The response of the frame at the design point (i.e., with values of the random variables that correspond to the most probable point of failure) is shown in Fig. 9

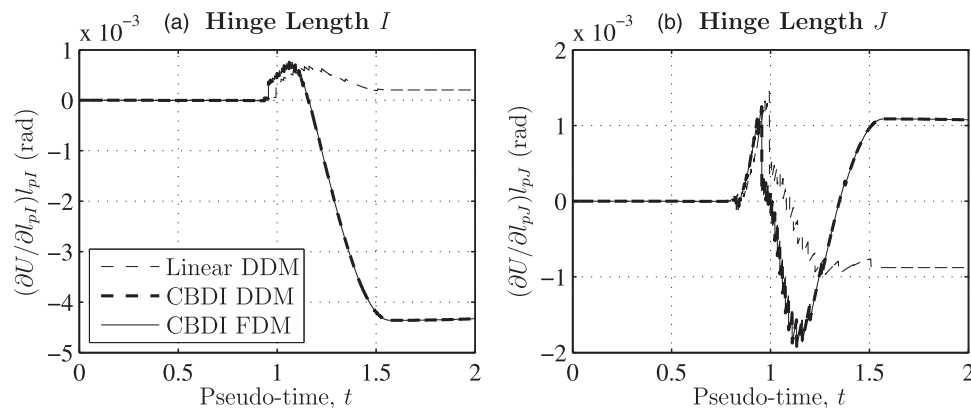


Fig. 7. Sensitivity of beam response to plastic hinge lengths

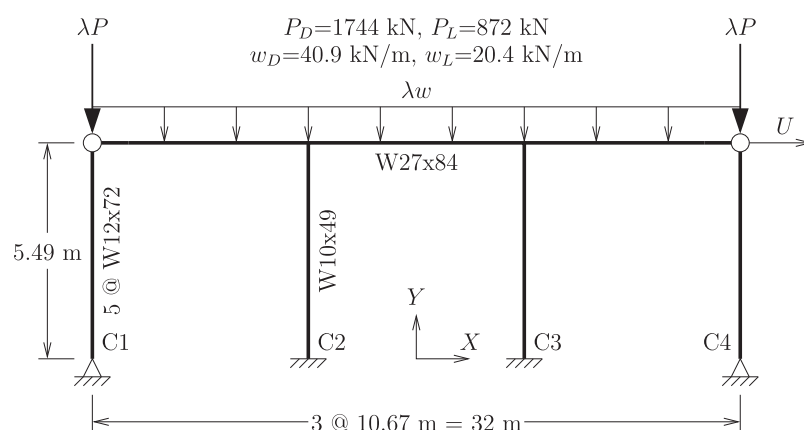


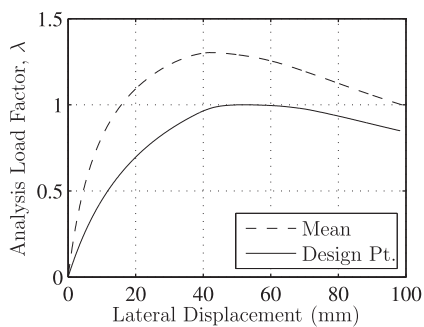
Fig. 8. Model of industrial building from Buonopane (2008, © ASCE)



**Table 1.** Assumed Statistical Distribution and Importance Ranking of Random Variables at the Design Point

Parameter	Unit	Probability density function	Mean	Standard deviation	Design point	Importance
$X_{C4}$	m	N	$1.600 \times 10^1$	$1.098 \times 10^{-2}$	$1.603 \times 10^1$	$6.014 \times 10^{-1}$
$X_{C1}$	m	N	$-1.600 \times 10^1$	$1.098 \times 10^{-2}$	$-1.597 \times 10^1$	$5.996 \times 10^{-1}$
$f_{y,C2}$	kPa	LN	$3.450 \times 10^5$	$2.070 \times 10^4$	$3.110 \times 10^5$	$-3.593 \times 10^{-1}$
$d_{C2}$	m	N	$2.535 \times 10^{-1}$	$5.070 \times 10^{-3}$	$2.487 \times 10^{-1}$	$-2.013 \times 10^{-1}$
$d_{C3}$	m	N	$2.535 \times 10^{-1}$	$5.070 \times 10^{-3}$	$2.488 \times 10^{-1}$	$-1.957 \times 10^{-1}$
$E_{C3}$	kPa	LN	$2.000 \times 10^8$	$6.800 \times 10^6$	$1.952 \times 10^8$	$-1.489 \times 10^{-1}$
$X_{C3}$	m	N	$5.333 \times 10$	$1.098 \times 10^{-2}$	$5.341 \times 10$	$1.479 \times 10^{-1}$
$X_{C2}$	m	N	$-5.333 \times 10$	$1.098 \times 10^{-2}$	$-5.326 \times 10$	$1.455 \times 10^{-1}$
$E_{C2}$	kPa	LN	$2.000 \times 10^8$	$6.800 \times 10^6$	$1.978 \times 10^8$	$-6.678 \times 10^{-2}$
$Y_{C3}$	m	N	$5.490 \times 10$	$1.098 \times 10^{-2}$	$5.491 \times 10$	$2.716 \times 10^{-2}$
$Y_{C2}$	m	N	$5.490 \times 10$	$1.098 \times 10^{-2}$	$5.491 \times 10$	$1.793 \times 10^{-2}$
$Y_{C4}$	m	N	$5.490 \times 10$	$1.098 \times 10^{-2}$	$5.490 \times 10$	$-8.877 \times 10^{-3}$
$Y_{C1}$	m	N	$5.490 \times 10$	$1.098 \times 10^{-2}$	$5.490 \times 10$	$-8.872 \times 10^{-3}$
$l_{p,bottom,C2}$	m	N	$3.802 \times 10^{-1}$	$3.802 \times 10^{-2}$	$3.807 \times 10^{-1}$	$2.775 \times 10^{-3}$
$l_{p,top,C2}$	m	N	$3.802 \times 10^{-1}$	$3.802 \times 10^{-2}$	$3.806 \times 10^{-1}$	$2.701 \times 10^{-3}$
$l_{p,top,C3}$	m	N	$3.802 \times 10^{-1}$	$3.802 \times 10^{-2}$	$3.800 \times 10^{-1}$	$-1.275 \times 10^{-3}$
$l_{p,bottom,C3}$	m	N	$3.802 \times 10^{-1}$	$3.802 \times 10^{-2}$	$3.800 \times 10^{-1}$	$-1.255 \times 10^{-3}$
$f_{y,C3}$	kPa	LN	$3.450 \times 10^5$	$2.070 \times 10^4$	$3.444 \times 10^5$	0.000

Note: LN = lognormal; N = normal.



**Fig. 9.** Vertical load-lateral displacement response of industrial building at parameter values corresponding to the mean and design point

along with the mean response. At the design point, the axial load in Column C2 reduced to 944 kN with a maximum end moment of 229 kN · m. For Column C3 the axial load was 982 kN and the maximum end moment was 196 kN · m.

The importance measures listed in Table 1 reveal that the  $X$ -coordinates of the outer column groups (C1 and C4) have the largest influence on the frame response at the design point. This indicates the frame is sensitive to initial sway; however, the importance values are large because of the model simplification where five columns and tributary gravity load are concentrated on each of the outer column lines. In the next group of importance variables are the depth and yield stress of Column C2, which have a significant influence on this member's plastic flexural response. The yield stress of Column C3 has zero importance at the design point, indicating this member remains elastic; however, the depth and elastic modulus of this column rank high in importance and act as resistance variables. In the next group of importance variables are the  $X$ -coordinates of the interior Columns C2 and C3, followed by the elastic modulus of C2. The  $Y$ -coordinates of all four columns comprise the next group of importance variables with the  $Y$ -coordinates of the inner columns acting as load variables and those for the outer columns acting as resistance variables. The assumed plastic hinge lengths of the interior column members rank low in importance, with those for Column C2 acting as load variables because of the plastic response of this member.

## Conclusions

Response sensitivity for geometrically nonlinear force-based frame FEs has been developed by direct differentiation of the governing equations of equilibrium, compatibility, and transverse displacement with the CBDI procedure. The dependence of the element response on the transverse displacement field made the response sensitivity derivation significantly more complex than in the geometrically linear case. The analytic sensitivity equations were verified against finite-difference approximations for material, cross-section, plastic hinge, and nodal coordinate parameters. In addition, the response sensitivity was used in the reliability analysis of a steel frame with material and geometric nonlinearity, where random variables corresponding to initial sway and column strength ranked highest in importance. Use of the DDM sensitivity in the reliability analysis resulted in an order of magnitude reduction in computational expense compared with the same analysis using finite-difference sensitivity.

The response sensitivity equations lay the foundation for the use of geometrically nonlinear force-based elements in other gradient-based structural engineering applications such as optimization and system identification. Extensions of the response sensitivity equations to a three-dimensional CBDI implementation and CSBDI are straightforward. Both of these extensions require an increase in the matrix dimensions of the response sensitivity equations but do not affect their underlying functional form.

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