Inverse Kinematics

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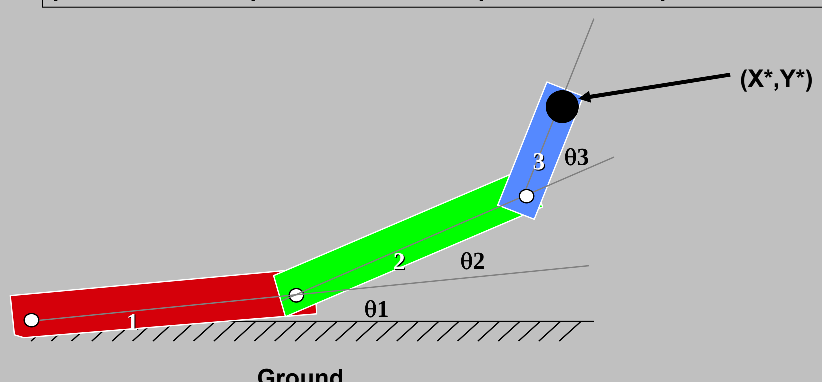
Inverse Kinematics

Forward Kinematics solves the problem "if I know the link transformation parameters, where are the links?".

Inverse Kinematics (IK) solves the problem "If I know where I want the parts to be, what part transformation parameters will put them there?"



Inverse Kinematics

Note that you can differentiate the equations

X* = f(Θ1, Θ2, Θ3) and Y* = g(Θ1, Θ2, Θ3)

If you have a good guess for (Θ1, Θ2, Θ3) which currently produce an X* and Y* that are not quite what you want, then you can refine your values for (Θ1, Θ2, Θ3) and try again. The fundamental equations for this are:

$$X^* - X^*_{approx} = \Delta X = \frac{\partial X^*}{\partial \theta_1}\Delta\theta_1 + \frac{\partial X^*}{\partial \theta_2}\Delta\theta_2 + \frac{\partial X^*}{\partial \theta_3}\Delta\theta_3$$

$$Y^* - Y^*_{approx} = \Delta Y = \frac{\partial Y^*}{\partial \theta_1}\Delta\theta_1 + \frac{\partial Y^*}{\partial \theta_2}\Delta\theta_2 + \frac{\partial Y^*}{\partial \theta_3}\Delta\theta_3$$

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Or, in matrix form:

$$\begin{bmatrix} \frac{\partial X^*}{\partial \theta_1} & \frac{\partial X^*}{\partial \theta_2} & \frac{\partial X^*}{\partial \theta_3} \\ \frac{\partial Y^*}{\partial \theta_1} & \frac{\partial Y^*}{\partial \theta_2} & \frac{\partial Y^*}{\partial \theta_3} \end{bmatrix} \begin{Bmatrix} \Delta\theta_1 \\ \Delta\theta_2 \\ \Delta\theta_3 \end{Bmatrix} = \begin{Bmatrix} \Delta X \\ \Delta Y \end{Bmatrix}$$

2 x 3 3 x 1 2 x 1

The matrix:

$$\begin{bmatrix} \frac{\partial X^*}{\partial \theta_1} & \frac{\partial X^*}{\partial \theta_2} & \frac{\partial X^*}{\partial \theta_3} \\ \frac{\partial Y^*}{\partial \theta_1} & \frac{\partial Y^*}{\partial \theta_2} & \frac{\partial Y^*}{\partial \theta_3} \end{bmatrix}$$

is called the Jacobian, and is abbreviated as [J]:

$$[J]\begin{Bmatrix} \Delta\theta_1 \\ \Delta\theta_2 \\ \Delta\theta_3 \end{Bmatrix} = \begin{Bmatrix} \Delta X \\ \Delta Y \end{Bmatrix}$$

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Note that [J] is not a square matrix, so this system of equations cannot be solved for directly. But, if we pre-multiply by the transpose of [J], we get:

$$\left([J]^T [J]\right)\begin{Bmatrix}\Delta\theta_1\\ \Delta\theta_2\\ \Delta\theta_3\end{Bmatrix} = [J]^T \begin{Bmatrix}\Delta X\\ \Delta Y\end{Bmatrix}$$

(3 x 2   2 x 3)   3 x 1   (3 x 2   2 x 1)

3 x 3   3 x 1   3 x 1

which is solvable.

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$$\begin{Bmatrix}\Delta\theta_1\\ \Delta\theta_2\\ \Delta\theta_3\end{Bmatrix} = \left([J]^T [J]\right)^{-1} [J]^T \begin{Bmatrix}\Delta X\\ \Delta Y\end{Bmatrix}$$

3 x 1   3 x 3   3 x 2   2 x 1

It is not obvious, but this is the *Least Squares* formulation. It will give an optimum ($\Delta\Theta_1$, $\Delta\Theta_2$, $\Delta\Theta_3$) to make (X*,Y*) move closer to the desired values.

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Differentiate the equation X* = f(Θ1, Θ2, Θ3) and Y* = g(Θ1, Θ2, Θ3)

Pick a starting Θ1, Θ2, and Θ3

1. Compute X^*_{approx} and Y^*_{approx} from Θ1, Θ2, Θ3
2. Compute $\Delta X = X^* - X^*_{approx}$ and $\Delta Y = Y^* - Y^*_{approx}$
3. If ΔX and ΔY are "small enough", we're done
4. Compute: $\frac{\partial X^*}{\partial \theta_1}, \frac{\partial X^*}{\partial \theta_2}, \frac{\partial X^*}{\partial \theta_3}, \frac{\partial Y^*}{\partial \theta_1}, \frac{\partial Y^*}{\partial \theta_2}, \frac{\partial Y^*}{\partial \theta_3}$
5. Form the Jacobian [J]
6. Solve the system of equations: $\left([J]^T [J]\right) \begin{Bmatrix} \Delta\theta_1 \\ \Delta\theta_2 \\ \Delta\theta_3 \end{Bmatrix} = [J]^T \begin{Bmatrix} \Delta X \\ \Delta Y \end{Bmatrix}$
7. Refine: Θ1= Θ1+ ΔΘ1 ; Θ2= Θ2+ ΔΘ2 ; Θ3= Θ3+ ΔΘ3

Numerical Differentiation

Rather then explicitly differentiating an equation, it I soften easier to calculate the derivative numerically. This is known as the ***Central Difference*** method, where you look both forward and backwards to see how the dependent variable is changing with respect to the independent variable:

$$\frac{\partial X}{\partial \theta} \approx \frac{X(\theta + \Delta\theta) - X(\theta - \Delta\theta)}{2\Delta\theta}$$

Pick a delta that is small, but not so small that floating point accuracy becomes an issue

Numerical Differentiation

If there are other independent variables, hold them constant:

$$\frac{\partial X}{\partial \theta_1} \approx \frac{X(\theta_1 + \Delta\theta_1, \theta_2, \theta_3) - X(\theta_1 - \Delta\theta_1, \theta_2, \theta_3)}{2\Delta\theta_1}$$

Numerical Differentiation

In case you ever need it, here is how to compute the second derivative

$$\frac{\partial^2 X}{\partial \theta^2} \approx \frac{X(\theta + \Delta\theta) - 2X(\theta) + X(\theta - \Delta\theta)}{\Delta\theta^2}$$

As before, if there are other independent variables, hold them constant: