Solving a Nonlinear Equation: Newton's Method

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Newton's Method for Solving a Nonlinear Equation

Scenario: You have an equation y(x) = 0, that is, there is a certain x value that makes the function equal to zero. But, the equation is too messy to solve directly. You do have an initial guess of the correct value of x. It is close, but it is wrong.

For example, solve this equation for x:

$$y(x) = \cos^3 x + \log_{10} x = 0$$

Starting with an initial guess of x = 6

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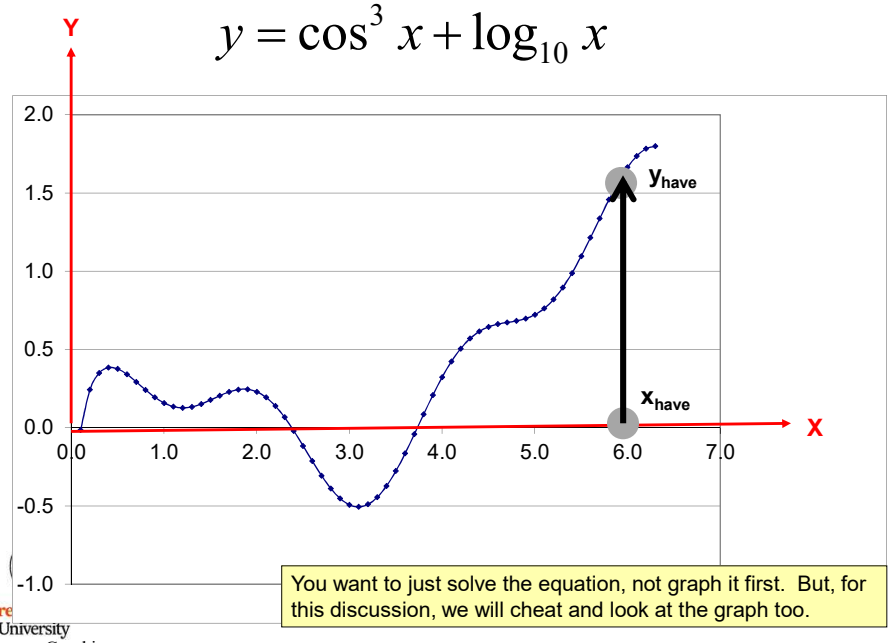
$$y = \cos^3 x + \log_{10} x$$



You want to just solve the equation, not graph it first. But, for this discussion, we will cheat and look at the graph too.

Newton's Method for Solving a Nonlinear Equation

You can take the x you have, x_{have} , and plug it into the equation to produce y_{have} and thus see how close you are to y = 0. But now what?

From calculus, we know that:

$$\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} \quad \text{or} \quad \frac{dy}{dx}\Delta x = \Delta y$$

The required Δy to get from $y = y_{have}$ to y = 0. is:

$$\Delta y = y_{want} - y_{have} = 0. - y_{have}$$

So that: $$\frac{dy}{dx}\Delta x = \Delta y = y_{want} - y_{have} = 0. - y_{have}$$

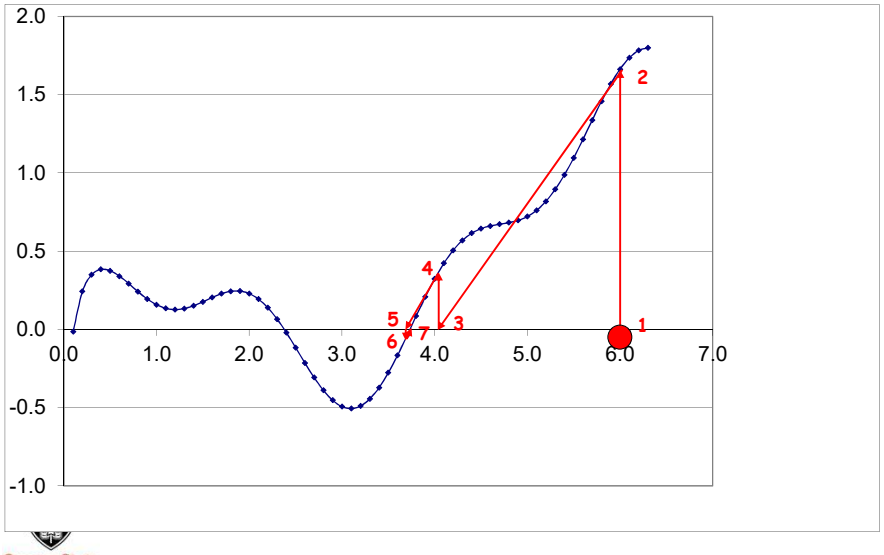
which gives us: $$\Delta x = \frac{-y_{have}}{\frac{dy}{dx}}$$

We will use that to find the next value of x to try, and then repeat the process:

$$x_{have} = x_{have} + \Delta x = x_{have} + \frac{-y_{have}}{\frac{dy}{dx}}$$

$$y_{have} = y(x_{have})$$

Here's what is really going on



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Watching Newton's Method Work in Excel

$$y = \cos^3 x + \log_{10} x = 0$$

$$\frac{dy}{dx} = -3 * \sin x * \cos^2 x + \frac{1}{x \ln(10)}$$

x_{have}	y_{have}	dydx	x_{next}
6.00000	1.66336	0.84518	4.03196
4.03196	0.35651	1.03069	3.68607
3.68607	-0.05934	1.25483	3.73336
3.73336	0.00040	1.26907	3.73304
3.73304	0.00000	1.26903	3.73304
3.73304	0.00000	1.26903	3.73304
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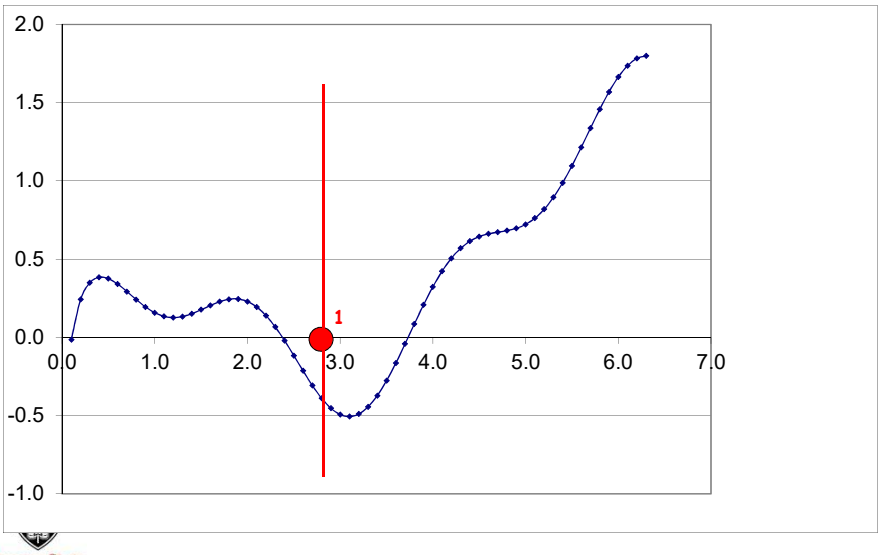
What would have happened if we had started with x=2.75?



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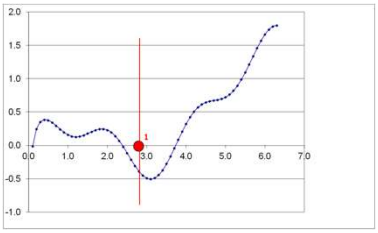

x_{have}	y_{have}	dydx	x_{next}
2.75000	-0.35033	-0.82027	2.32291
2.32291	0.04716	-0.83554	2.37936
2.37936	-0.00193	-0.90125	2.37721
2.37721	0.00000	-0.89907	2.37721
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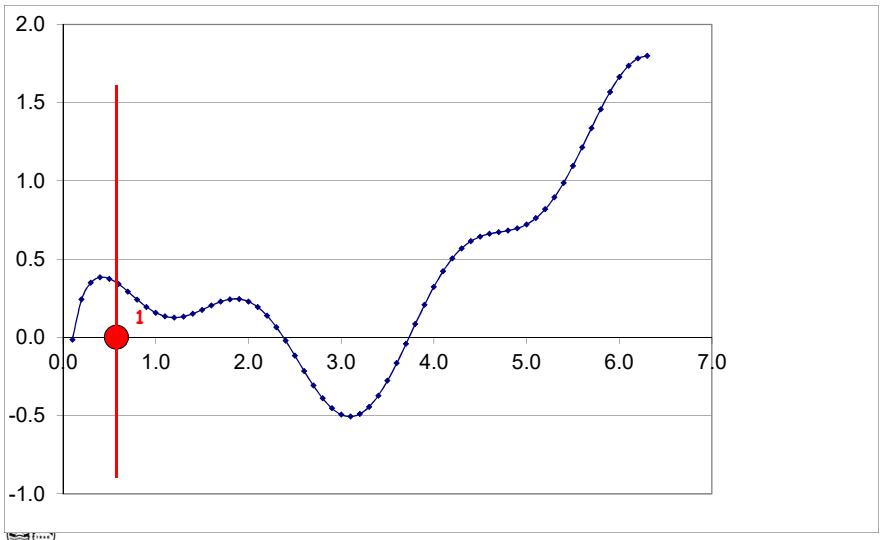
What would have happened if we had started with x=0.55?



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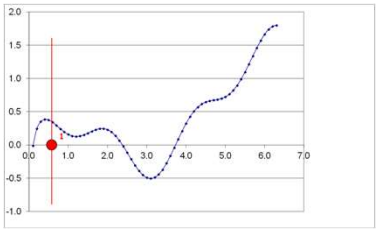

x_{have}	y_{have}	dydx	x_{next}
0.55000	0.35998	-0.35004	1.57839
1.57839	0.19821	0.27498	0.85755
0.85755	0.21336	-0.46480	1.31659
1.31659	0.13535	0.14624	0.39100
0.39100	0.38242	0.13344	-2.47476
-2.47476	#NUM!	0.97020	#NUM!
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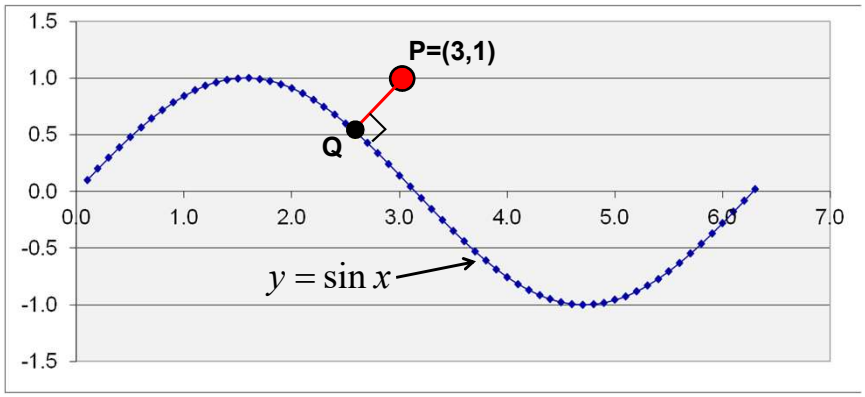
A Collision Detection Example

Let's say we have a nonlinear surface. How close is the point (3,1) to that surface?



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A Collision Detection Problem Example

Using our friend, the dot product:

$$(P_x - Q_x, P_y - Q_y) \bullet slope = 0$$

where the vector slope is:

$$slope = (dx, dy) = (1, \frac{dy}{dx}) = (1, \frac{d \sin x}{dx}) = (1, \cos x)$$

substituting for Q_x, Q_y, and the slope:

$$f(x) = (P_x - x, P_y - \sin x) \bullet (1, \cos x) = 0$$

Note that in this case, we are solving f(x)=0, not y(x)=0 !

and expanding:

$$f(x) = \left(P_x - x\right) + \cos x * \left(P_y - \sin x\right) = 0$$

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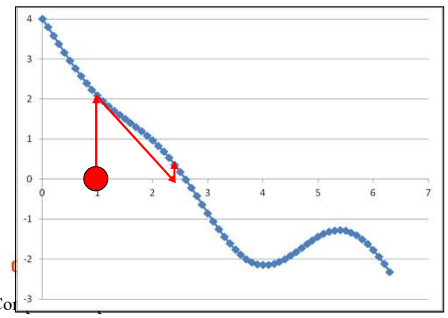
A Collision Detection Problem Example

$$f(x) = \left(P_x - x\right) + \cos x * \left(P_y - \sin x\right) = 0$$

xhave	yhave	fhave	dfdx	xnext
1.00000	0.84147	2.08565	-1.42532	2.46328
2.46328	0.62748	0.24666	-1.84002	2.59733
2.59733	0.51778	-0.00988	-1.98158	2.59235
2.59235	0.52204	-0.00001	-1.97699	2.59234
2.59234	0.52205	0.00000	-1.97698	2.59234
2.59234	0.52205	0.00000	-1.97698	2.59234
2.59234	0.52205	0.00000	-1.97698	2.59234
2.59234	0.52205	0.00000	-1.97698	2.59234
2.59234	0.52205	0.00000	-1.97698	2.59234
2.59234	0.52205	0.00000	-1.97698	2.59234
2.59234	0.52205	0.00000	-1.97698	2.59234



$$dist = \sqrt{(3 - 2.59234)^2 + (1 - .52205)^2}$$

$$dist = .62819$$