Solving a Nonlinear Equation: Newton’s Method

Mike Bailey
mjb@cs.oregonstate.edu
Oregon State University

Newton’s Method for Solving a Nonlinear Equation

Scenario: You have an equation $y(x) = 0$, but it is too messy to solve directly. You do have an initial guess of the correct value of $x$. It is close, but it is wrong.

For example, solve this equation for $x$:

$$y(x) = \cos^3 x + \log_{10} x = 0$$

Starting with an initial guess of $x = 6$
You want to just solve the equation, not graph it first. But, for this discussion, we will cheat and look at the graph too.

### Newton's Method for Solving a Nonlinear Equation

You can take the x you have, $x_{\text{have}}$, and plug it into the equation to produce $y_{\text{have}}$ and thus see how close you are to $y = 0$. But now what?

From calculus, we know that:

$$\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} \quad \text{or} \quad \frac{dy}{dx} \Delta x = \Delta y$$

So that:

$$\Delta x = \Delta y = y_{\text{want}} - y_{\text{have}} = 0 - y_{\text{have}}$$

which gives us:

$$\Delta x = -\frac{y_{\text{have}}}{\frac{dy}{dx}}$$

We will use that to find the next value of x to try, and then repeat the process:

$$x'_{\text{have}} = x_{\text{have}} + \Delta x = x_{\text{have}} + -\frac{y_{\text{have}}}{\frac{dy}{dx}}$$

$$y'_{\text{have}} = y(x'_{\text{have}})$$
Watching Newton’s Method Work

\[ y = \cos^3 x + \log_{10} x = 0 \]

\[ \frac{dy}{dx} = -3 \sin x \cos^2 x + \frac{1}{x \ln(10)} \]

<table>
<thead>
<tr>
<th>(x_{\text{have}})</th>
<th>(y_{\text{have}})</th>
<th>(dy/dx)</th>
<th>(x_{\text{next}})</th>
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Here’s what is really going on
What would have happened if we had started with $x=2.75$?

![Graph showing the path with $x=2.75$.]
What would have happened if we had started with $x=0.55$?

\[ \begin{array}{cccc}
    x_{\text{have}} & y_{\text{have}} & dydx & x_{\text{next}} \\
    0.55000 & 0.35998 & -0.35004 & 1.57839 \\
    1.57839 & 0.19821 & 0.27498 & 0.85755 \\
    0.85755 & 0.21336 & -0.46480 & 1.31659 \\
    1.31659 & 0.13535 & 0.14624 & 0.39100 \\
    0.39100 & 0.38242 & 0.13344 & -2.47476 \\
    -2.47476 & \#NUM! & 0.97020 & \#NUM! \\
    \#NUM! & \#NUM! & \#NUM! & \#NUM! \\
    \#NUM! & \#NUM! & \#NUM! & \#NUM! \\
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\end{array} \]
A Collision Detection Problem Example

Let's say we have a nonlinear surface. How close is the point (3,1) to that surface?

\[ P=(3,1) \]

Using our friend, the dot product:

\[(P_x - Q_x, P_y - Q_y) \cdot \text{slope} = 0\]

where the vector slope is:

\[
\text{slope} = (dx,dy) = (1, \frac{dy}{dx}) = (1, \frac{d\sin x}{dx}) = (1, \cos x)
\]

substituting for \(Q_x, Q_y\) and the slope:

\[ f(x) = (P_x - x, P_y - \sin x) \cdot (1, \cos x) = 0 \]

and expanding:

\[ f(x) = (P_x - x) + \cos x \cdot (P_y - \sin x) = 0 \]

Note that in this case, we are solving \(f(x)=0\), not \(y(x)=0\)!
A Collision Detection Problem Example

\[ f(x) = (P_x - x) + \cos x \ast (P_y - \sin x) = 0 \]

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<thead>
<tr>
<th>xhave</th>
<th>yhave</th>
<th>fhave</th>
<th>dfdx</th>
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\[ \text{dist} = \sqrt{(3 - 2.59234)^2 + (1 - .52205)^2} \]

\[ \text{dist} = .62819 \]