Transformations

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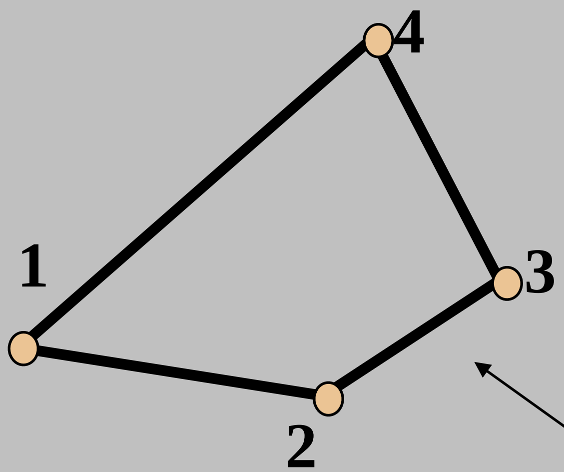
Geometry vs. Topology

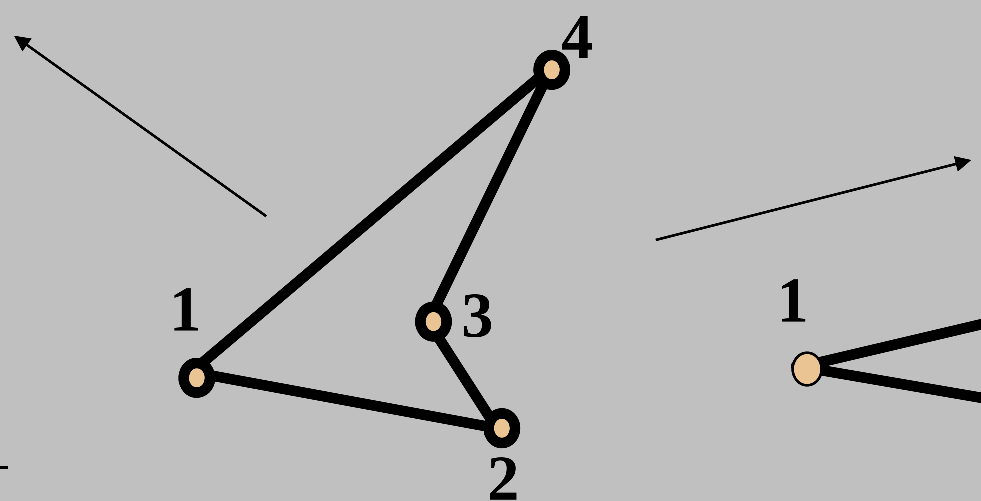
Geometry:

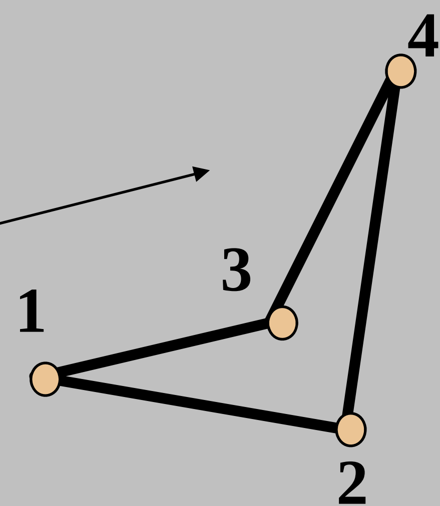
Where things are (e.g., coordinates)



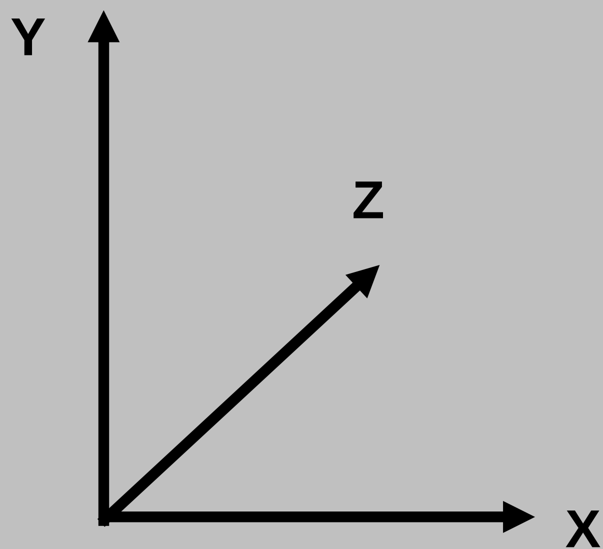
Topology:

How things are connected

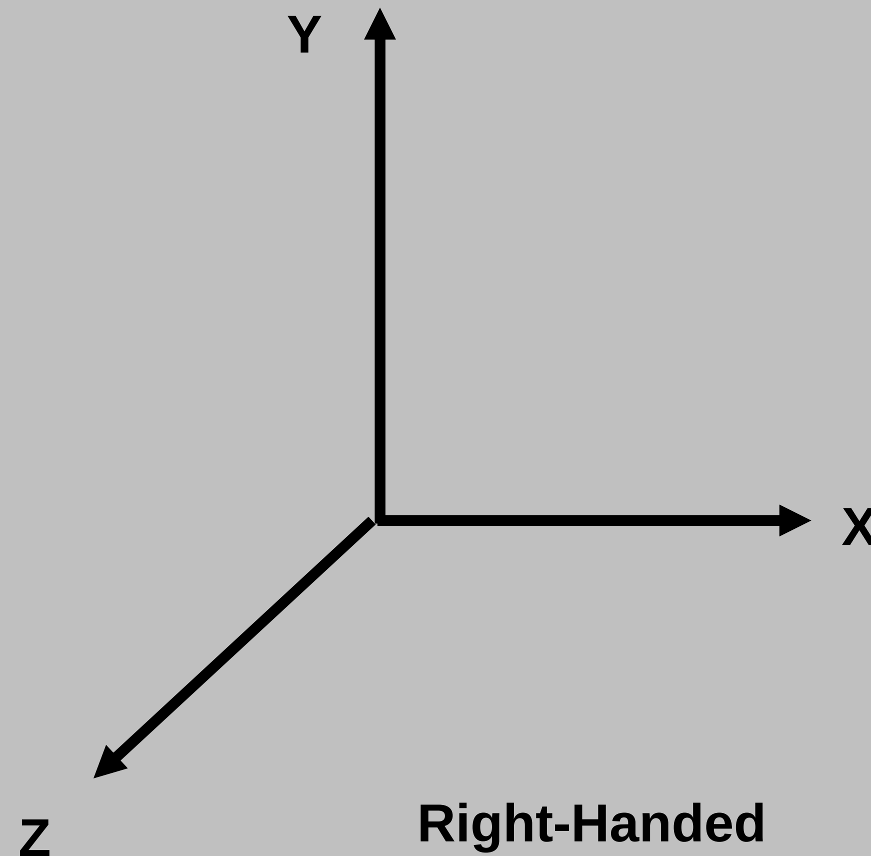




3D Coordinate Systems

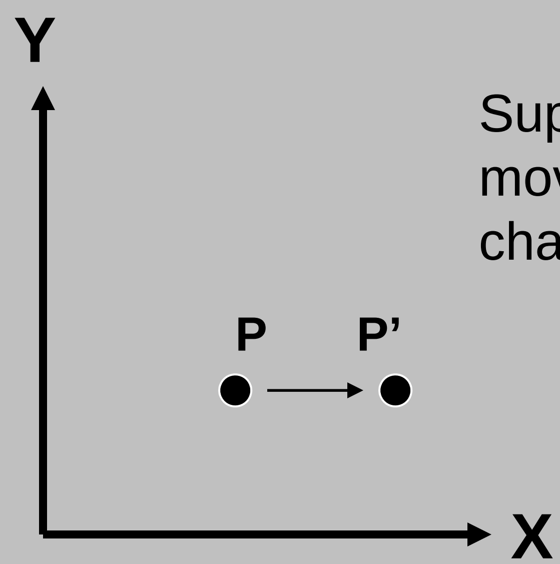


Left-Handed



Right-Handed

Transformations



Suppose you have a point P and you want to move it over by 2 units in X – how would you change P's coordinates?

$$P' = (P'_x, P'_y) = (P_x + 2., P_y)$$

This is known as a coordinate *transformation*

General Form of 3D Linear Transformations

$$x' = Ax + By + Cz + D$$

$$y' = Ex + Fy + Gz + H$$

$$z' = Ix + Jy + Kz + L$$

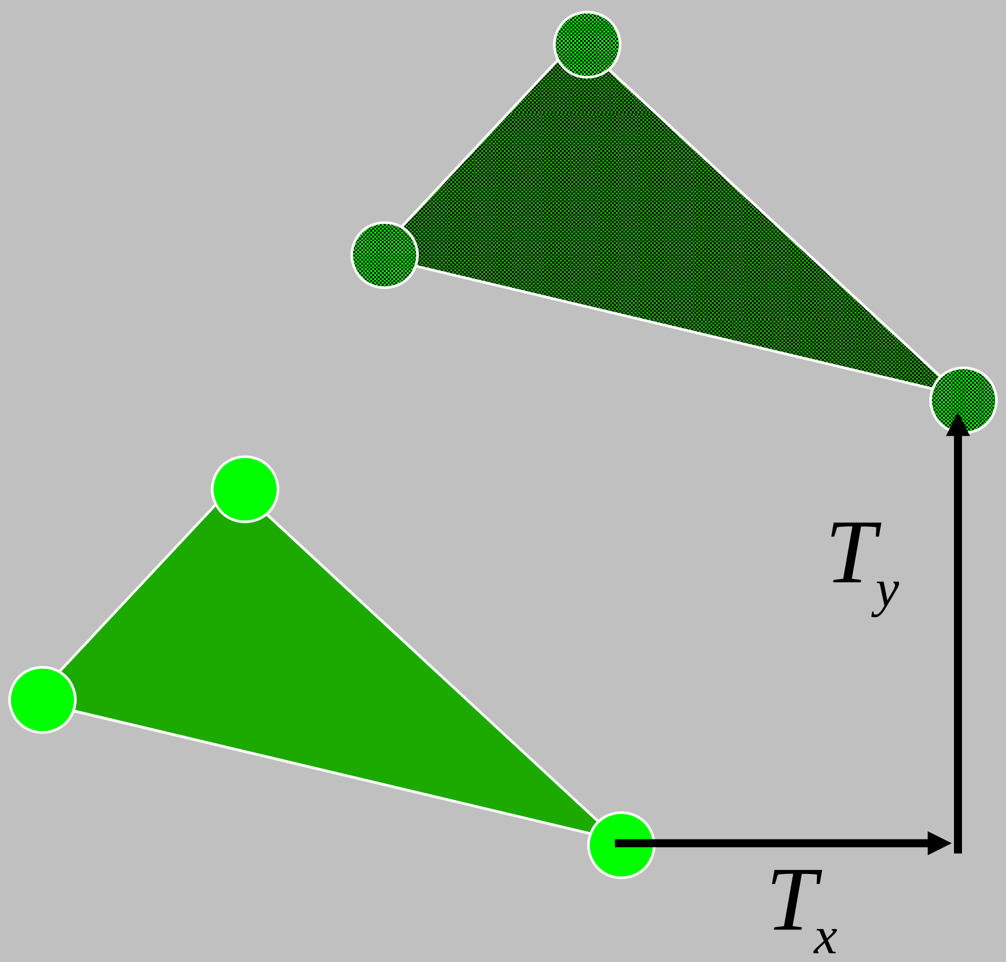
Transform the geometry – leave the topology as is

Translation

$$x' = x + T_x$$

$$y' = y + T_y$$

$$z' = z + T_z$$

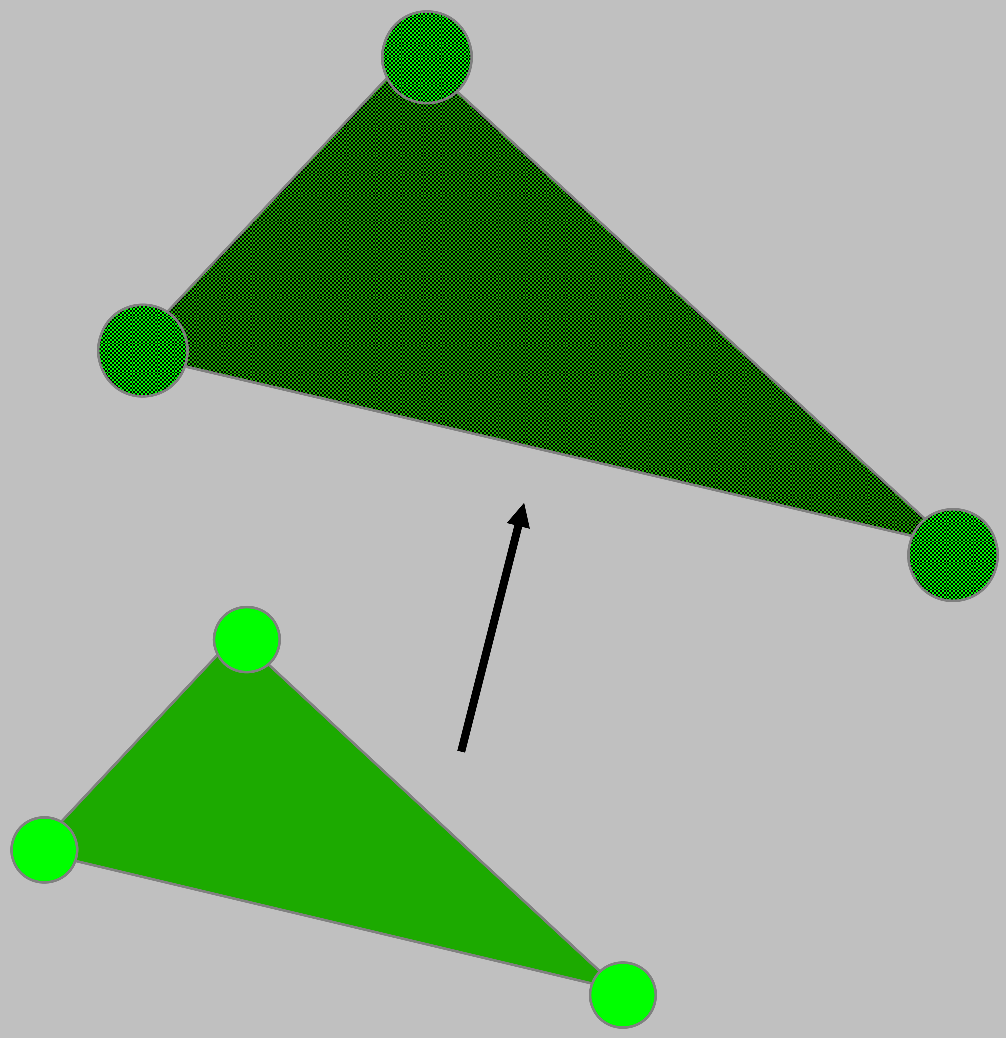


Scaling

$$x' = x \bullet S_x$$

$$y' = y \bullet S_y$$

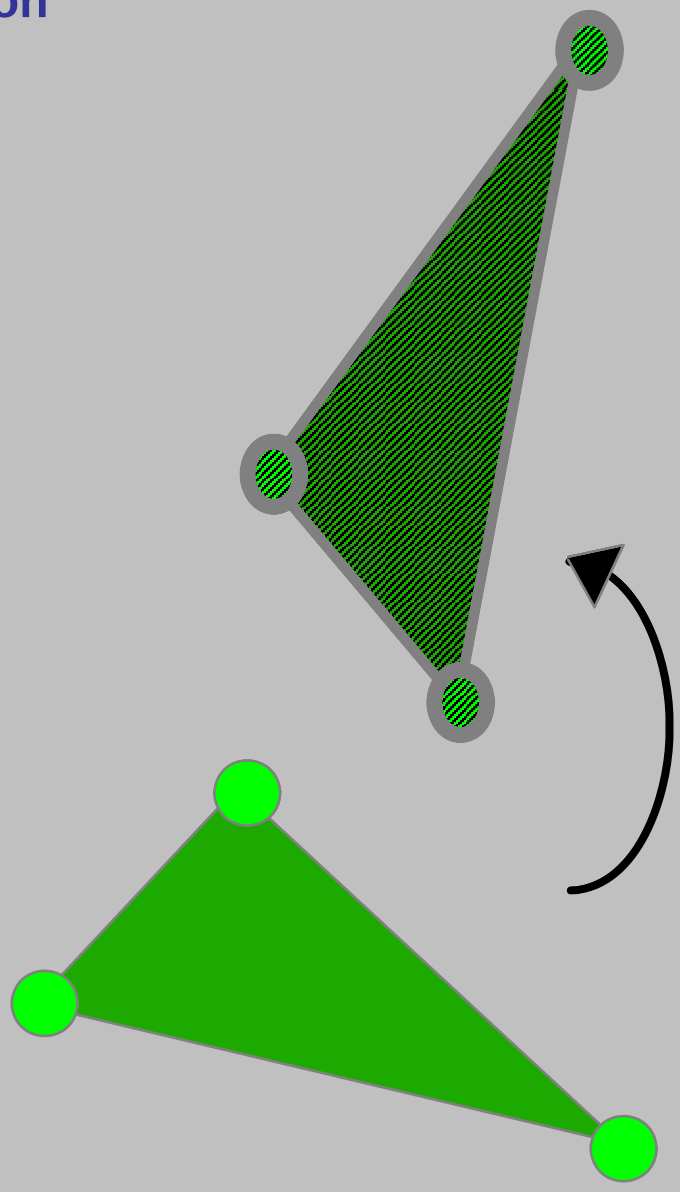
$$z' = z \bullet S_z$$



2D Rotation

$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta$$



Linear Equations in Matrix Form

$$x' = Ax + By + Cz + D$$
$$y' = Ex + Fy + Gz + H$$
$$z' = Ix + Jy + Kz + L$$

x consuming column

x' producing row →

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

Identity Matrix

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

[I] signifies that "Nothing has changed"

Matrix Inverse

$$[M] \bullet [M]^{-1} = [I]$$

$$[M] \bullet [M]^{-1} = \text{“Nothing has changed”}$$

“Whatever [M] does, $[M]^{-1}$ undoes”

Translation

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

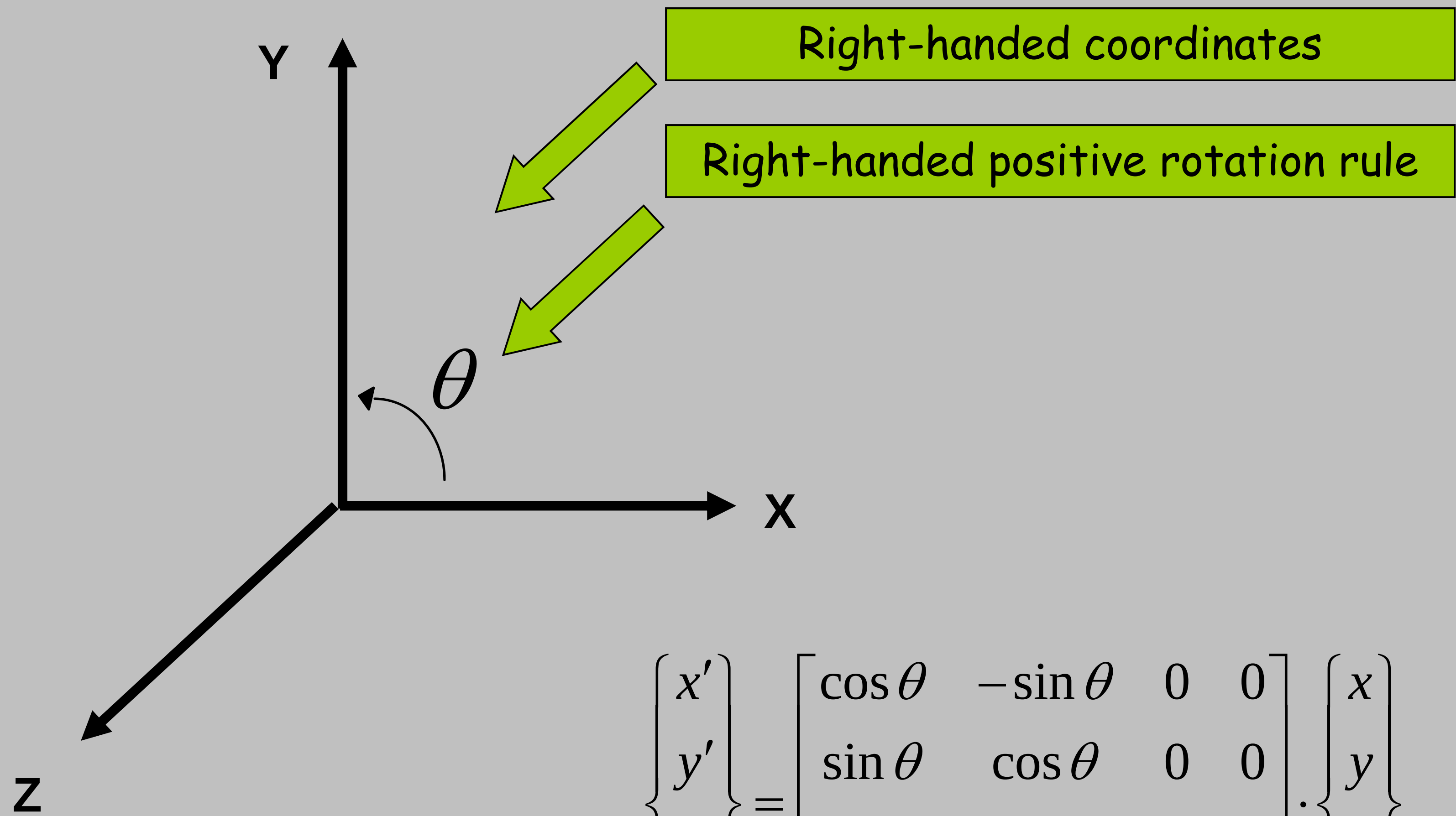
Quick! What is the inverse of this matrix?

Scaling

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

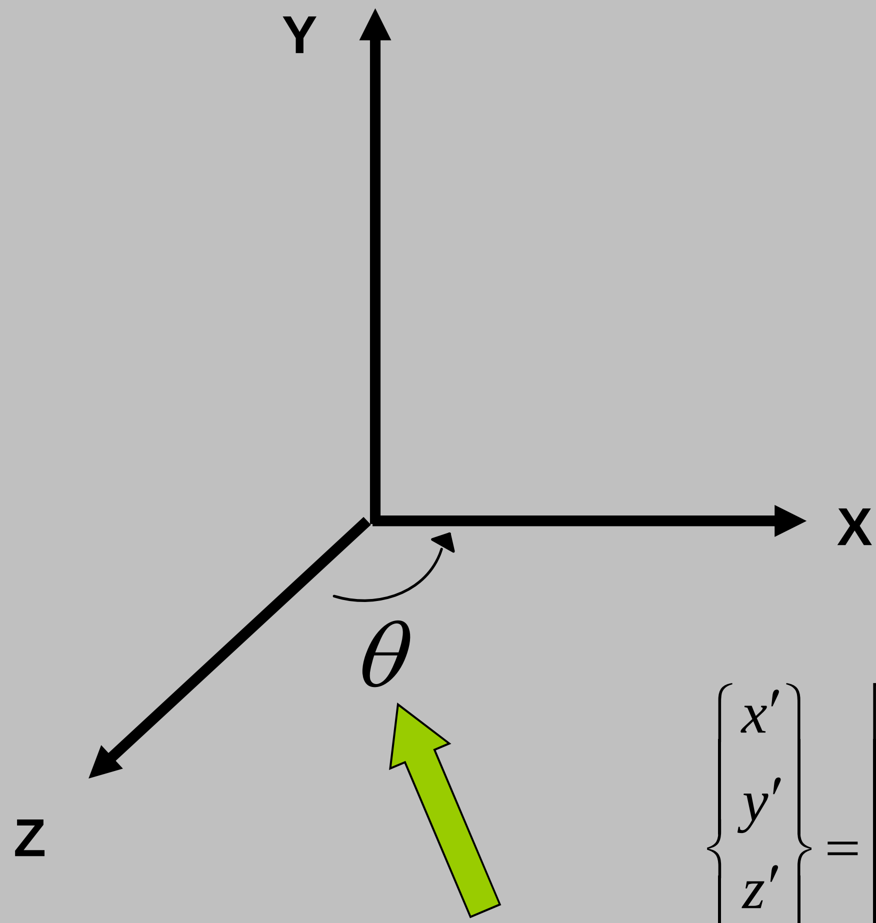
Quick! What is the inverse of this matrix?

3D Rotation About Z



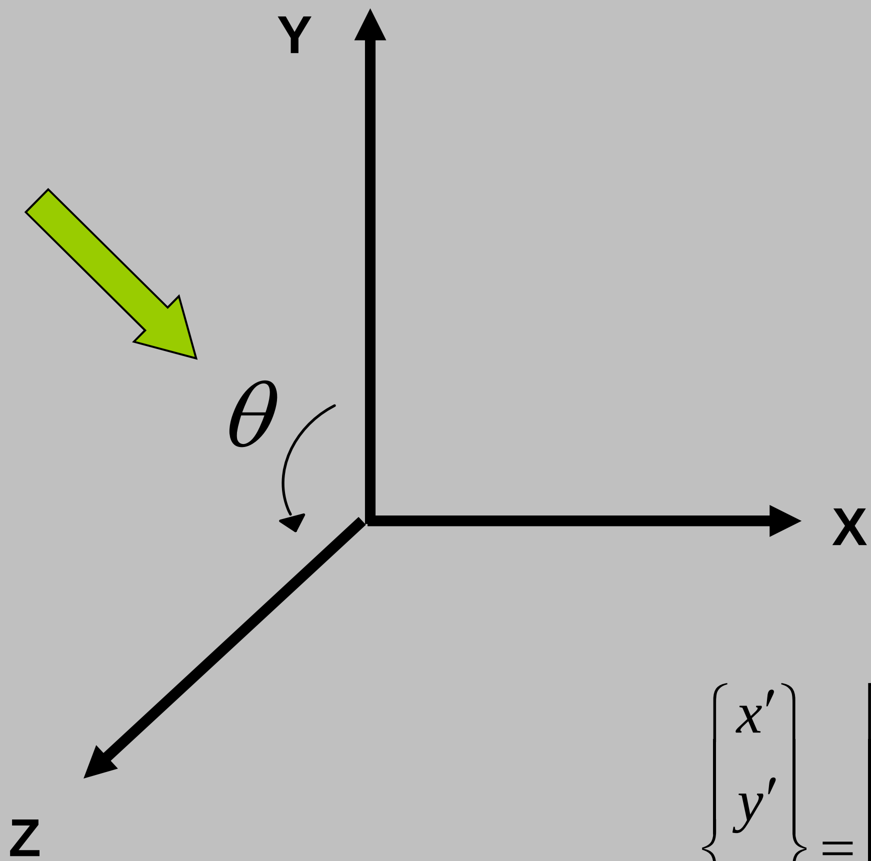
$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

3D Rotation About Y



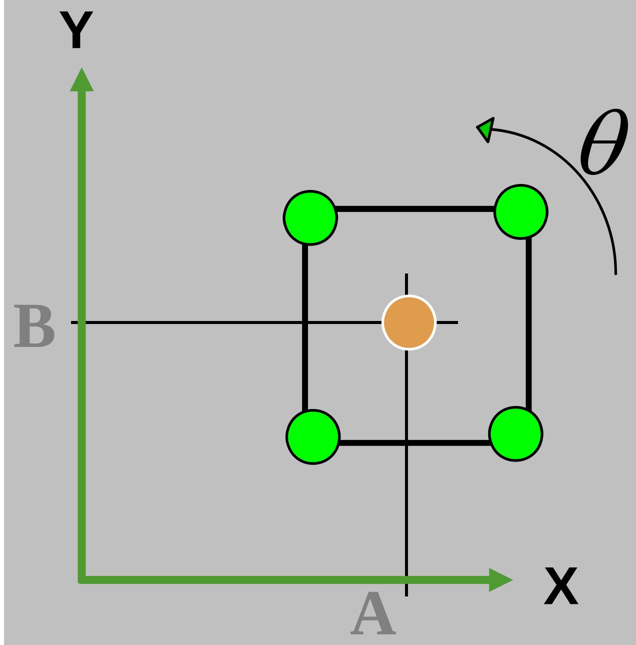
$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

3D Rotation About X



$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

Compound Transformations

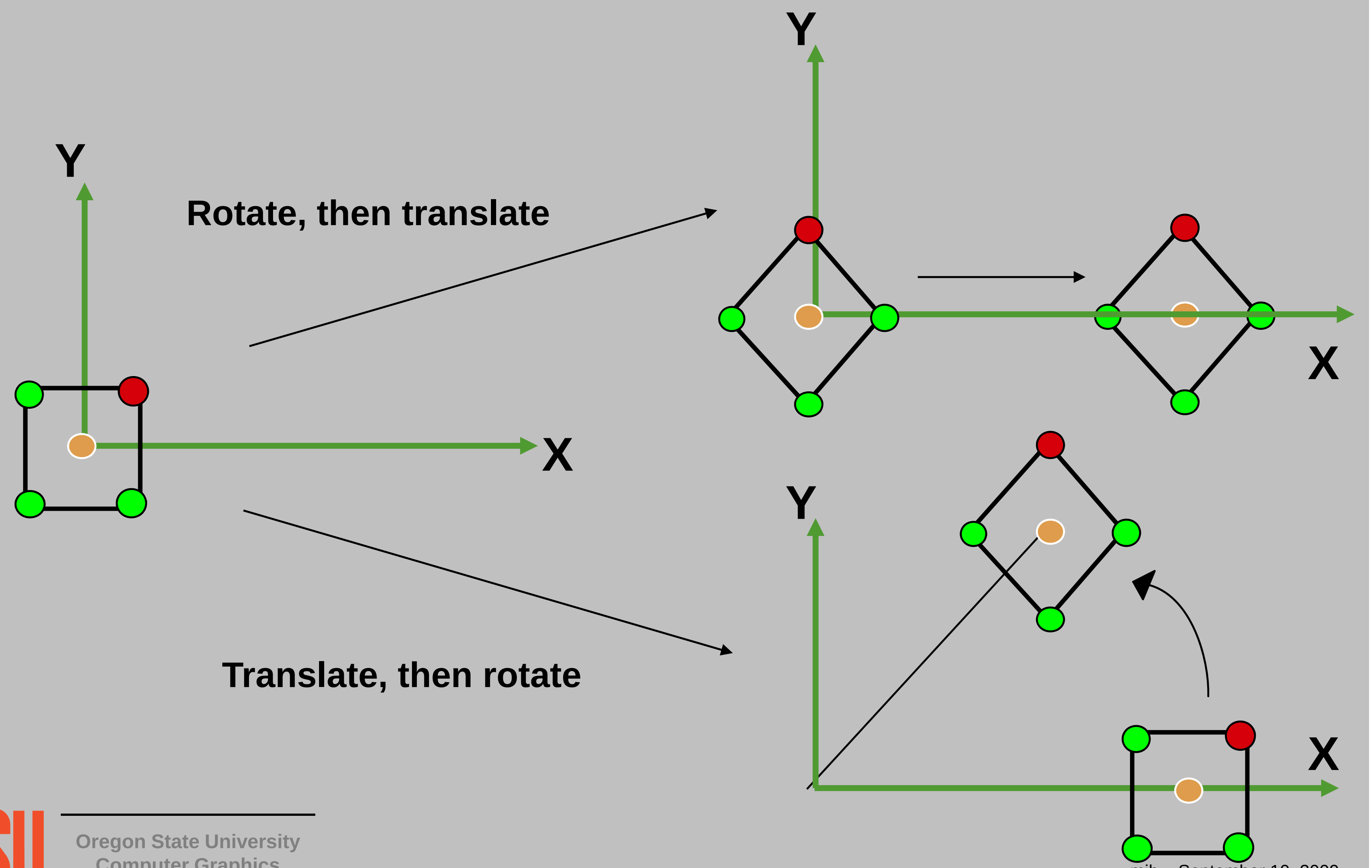


Write it →

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \left(\left[T_{+A,+B} \right] \cdot \left(\left[R_{\theta} \right] \cdot \left(\left[T_{-A,-B} \right] \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \right) \right) \right)$$

← Say it

Matrix Multiplication is Not Commutative

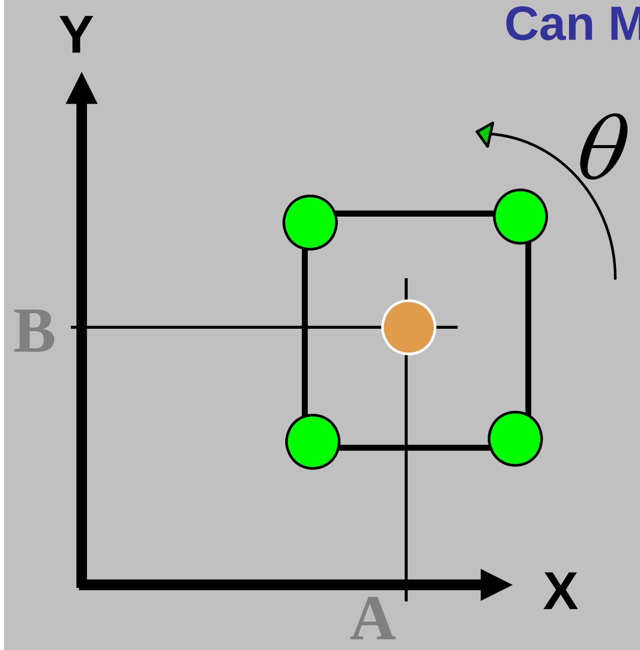


Matrix Multiplication is Associative

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \left(\left[T_{+A,+B} \right] \cdot \left(\left[R_\theta \right] \cdot \left(\left[T_{-A,-B} \right] \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \right) \right) \right)$$

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \left(\left[T_{+A,+B} \right] \cdot \left[R_\theta \right] \cdot \left[T_{-A,-B} \right] \right) \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

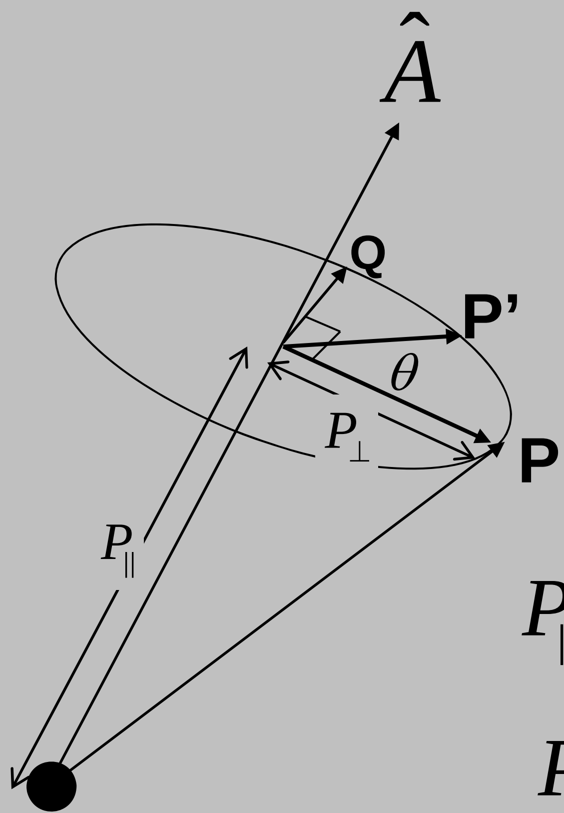
Can Multiply All Geometry by One Matrix !



$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = [M] \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Graphics hardware can do this very quickly!

The Rotation Matrix for an Arbitrary Axis and Angle



$$P_\parallel = \hat{A}(\hat{A} \bullet P)$$

$$P_\perp = P - P_\parallel = P - \hat{A}(\hat{A} \bullet P)$$

$$Q = \hat{A} \times P_\perp = \hat{A} \times P_\perp + 0 = \hat{A} \times P_\perp + \hat{A} \times P_\parallel = \hat{A} \times \left(P_\perp + P_\parallel\right) = \hat{A} \times P$$

The Rotation Matrix for an Arbitrary Axis and Angle

$$P_{\parallel}' = P_{\parallel} \qquad P_{\perp}' = P_{\perp}\cos\theta + Q\sin\theta$$

$$P' = P'_{\parallel} + P'_{\perp}$$

$$P' = \left[\hat{A}(\hat{A} \bullet P)\right] + \cos\theta\left[P - \hat{A}(\hat{A} \bullet P)\right] + \sin\theta\left[\hat{A} \times P\right]$$

The Rotation Matrix for an Arbitrary Axis and Angle

$$\hat{A}(\hat{A} \bullet P) = \begin{bmatrix} A_x A_x & A_x A_y & A_x A_z \\ A_y A_x & A_y A_y & A_y A_z \\ A_z A_x & A_z A_y & A_z A_z \end{bmatrix} \{P\}$$

$$\hat{A} \times P = \begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix} \{P\}$$

The Rotation Matrix for an Arbitrary Axis and Angle

$$P' = \left[\hat{A}(\hat{A} \bullet P) \right] + \cos\theta \left[P - \hat{A}(\hat{A} \bullet P) \right] + \sin\theta \left[\hat{A} \times P \right]$$

$$P' = \left(\begin{bmatrix} A_x A_x & A_x A_y & A_x A_z \\ A_y A_x & A_y A_y & A_y A_z \\ A_z A_x & A_z A_y & A_z A_z \end{bmatrix} + \cos\theta \left[[I] - \begin{bmatrix} A_x A_x & A_x A_y & A_x A_z \\ A_y A_x & A_y A_y & A_y A_z \\ A_z A_x & A_z A_y & A_z A_z \end{bmatrix} \right] + \sin\theta \begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix} \right) \{P\}$$

$$[M] = \left(\begin{bmatrix} A_x A_x & A_x A_y & A_x A_z \\ A_y A_x & A_y A_y & A_y A_z \\ A_z A_x & A_z A_y & A_z A_z \end{bmatrix} + \cos\theta \begin{bmatrix} 1 - A_x A_x & -A_x A_y & -A_x A_z \\ -A_y A_x & 1 - A_y A_y & -A_y A_z \\ -A_z A_x & -A_z A_y & 1 - A_z A_z \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix} \right)$$

The Rotation Matrix for an Arbitrary Axis and Angle

$$[M] = \left(\begin{bmatrix} A_xA_x & A_xA_y & A_xA_z \\ A_yA_x & A_yA_y & A_yA_z \\ A_zA_x & A_zA_y & A_zA_z \end{bmatrix} + \cos\theta \begin{bmatrix} 1-A_xA_x & -A_xA_y & -A_xA_z \\ -A_yA_x & 1-A_yA_y & -A_yA_z \\ -A_zA_x & -A_zA_y & 1-A_zA_z \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix} \right)$$

$$[M] = \begin{bmatrix} A_xA_x + \cos\theta(1-A_xA_x) & A_xA_y - \cos\theta(A_xA_y) - \sin\theta A_z & A_xA_z - \cos\theta(A_xA_z) + \sin\theta A_y \\ A_yA_x - \cos\theta(A_yA_x) + \sin\theta A_z & A_yA_y + \cos\theta(1-A_yA_y) & A_yA_z - \cos\theta(A_yA_z) - \sin\theta A_x \\ A_zA_x - \cos\theta(A_zA_x) - \sin\theta A_y & A_zA_y - \cos\theta(A_zA_y) + \sin\theta A_x & A_zA_z + \cos\theta(1-A_zA_z) \end{bmatrix}$$

For this to be correct, A must be a unit vector