

Transformations

Mike Bailey

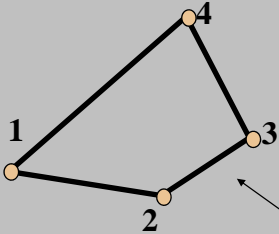
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Geometry vs. Topology

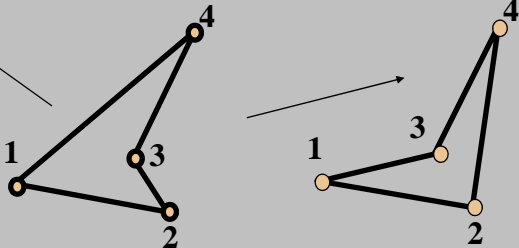
Geometry:

Where things are (e.g., coordinates)

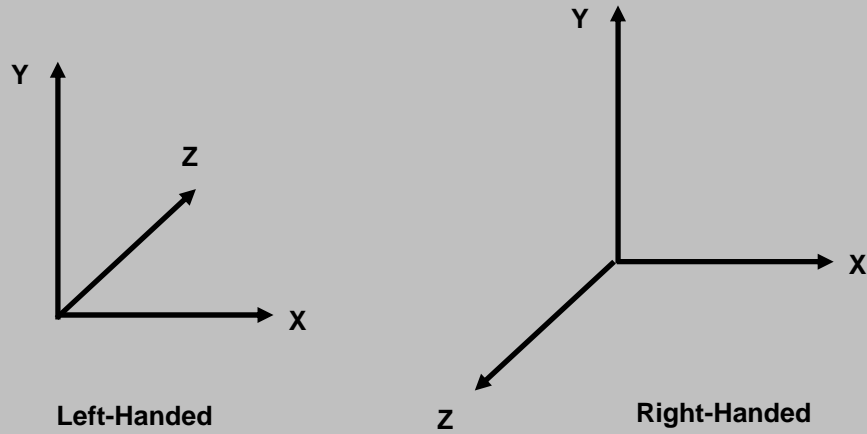


Topology:

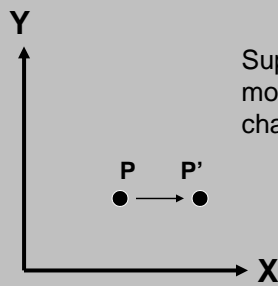
How things are connected



3D Coordinate Systems



Transformations



Suppose you have a point P and you want to move it over by 2 units in X – how would you change P's coordinates?

$$P' = (P'_x, P'_y) = (P_x + 2., P_y)$$

This is known as a coordinate *transformation*

General Form of 3D Linear Transformations

$$x' = Ax + By + Cz + D$$

$$y' = Ex + Fy + Gz + H$$

$$z' = Ix + Jy + Kz + L$$

Transform the geometry – leave the topology as is



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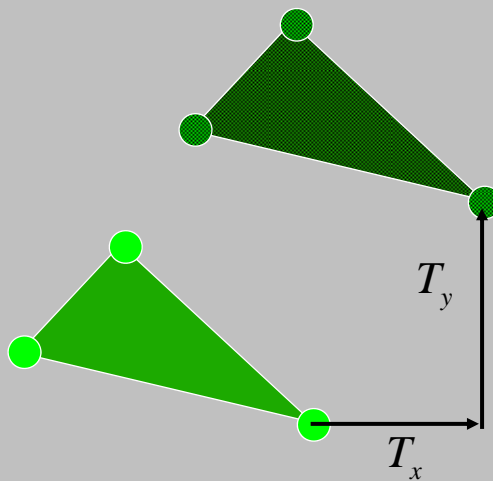
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Translation

$$x' = x + T_x$$

$$y' = y + T_y$$

$$z' = z + T_z$$



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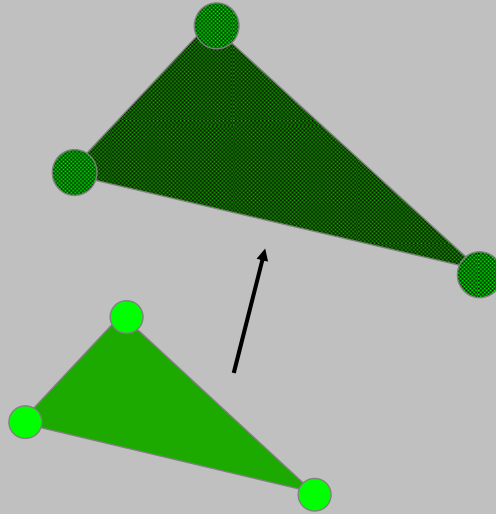
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Scaling

$$x' = x \cdot S_x$$

$$y' = y \cdot S_y$$

$$z' = z \cdot S_z$$



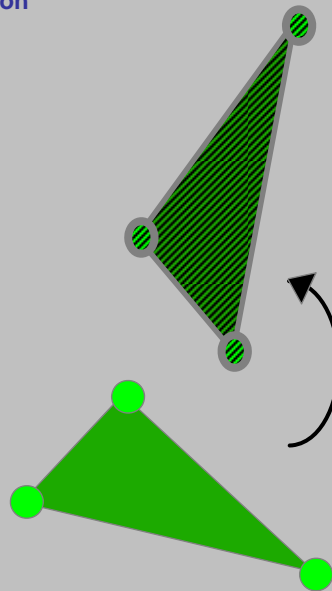
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2D Rotation

$$x' = x \cos \theta - y \sin \theta$$

$$y' = x \sin \theta + y \cos \theta$$



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Linear Equations in Matrix Form

$$x' = Ax + By + Cz + D$$

$$y' = Ex + Fy + Gz + H$$

$$z' = Ix + Jy + Kz + L$$

x consuming column

x' producing row

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$



Identity Matrix

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

[I] signifies that "Nothing has changed"



Matrix Inverse

$$[M] \cdot [M]^{-1} = [I]$$

$$[M] \cdot [M]^{-1} = \text{“Nothing has changed”}$$

“Whatever $[M]$ does, $[M]^{-1}$ undoes”



Translation

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

Quick! What is the inverse of this matrix?



Scaling

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

Quick! What is the inverse of this matrix?



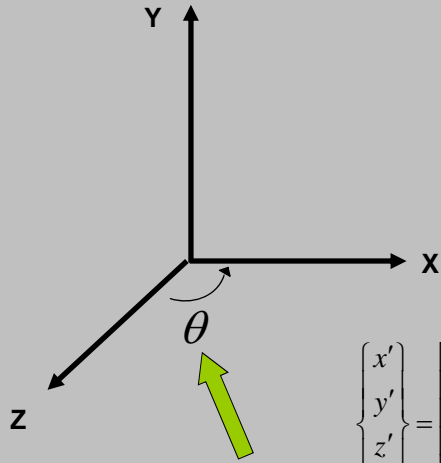
3D Rotation About Z

Right-handed coordinates

Right-handed positive rotation rule

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$


3D Rotation About Y



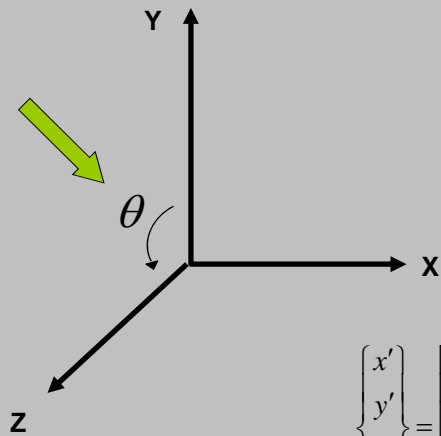
$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} \cos \theta & 0 & \sin \theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \theta & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$



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3D Rotation About X



$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$



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Compound Transformations

Write it

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} [T_{+A,+B}] \cdot [R_\theta] \cdot [T_{-A,-B}] \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \end{pmatrix}$$

Say it

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Matrix Multiplication is Not Commutative

Rotate, then translate

Translate, then rotate

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Matrix Multiplication is Associative

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \left(\left[T_{+A,+B} \right] \cdot \left(\left[R_\theta \right] \cdot \left(\left[T_{-A,-B} \right] \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \right) \right) \right)$$

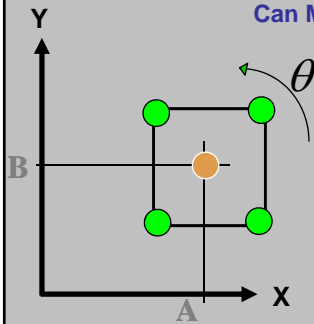
$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \left(\left[T_{+A,+B} \right] \cdot \left[R_\theta \right] \cdot \left[T_{-A,-B} \right] \right) \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$



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Can Multiply All Geometry by One Matrix !



$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = [M] \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

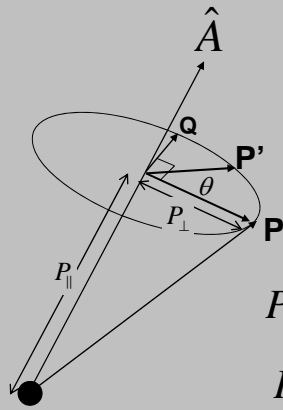
Graphics hardware can do this very quickly!



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The Rotation Matrix for an Arbitrary Axis and Angle



$$P_{\parallel} = \hat{A}(\hat{A} \bullet P)$$

$$P_{\perp} = P - P_{\parallel} = P - \hat{A}(\hat{A} \bullet P)$$

$$Q = \hat{A} \times P_{\perp} = \hat{A} \times P_{\perp} + 0 = \hat{A} \times P_{\perp} + \hat{A} \times P_{\parallel} = \hat{A} \times (P_{\perp} + P_{\parallel}) = \hat{A} \times P$$



The Rotation Matrix for an Arbitrary Axis and Angle

$$P'_{\parallel} = P_{\parallel}$$

$$P'_{\perp} = P_{\perp} \cos \theta + Q \sin \theta$$

$$P' = P'_{\parallel} + P'_{\perp}$$

$$P' = \left[\hat{A}(\hat{A} \bullet P) \right] + \cos \theta \left[P - \hat{A}(\hat{A} \bullet P) \right] + \sin \theta \left[\hat{A} \times P \right]$$



The Rotation Matrix for an Arbitrary Axis and Angle

$$\hat{A}(\hat{A} \bullet P) = \begin{bmatrix} A_x A_x & A_x A_y & A_x A_z \\ A_y A_x & A_y A_y & A_y A_z \\ A_z A_x & A_z A_y & A_z A_z \end{bmatrix} \{P\}$$

$$\hat{A} \times P = \begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix} \{P\}$$



The Rotation Matrix for an Arbitrary Axis and Angle

$$P' = \left[\hat{A}(\hat{A} \bullet P) \right] + \cos \theta \left[P - \hat{A}(\hat{A} \bullet P) \right] + \sin \theta \left[\hat{A} \times P \right]$$

$$P' = \left(\begin{bmatrix} A_x A_x & A_x A_y & A_x A_z \\ A_y A_x & A_y A_y & A_y A_z \\ A_z A_x & A_z A_y & A_z A_z \end{bmatrix} + \cos \theta \left[I - \begin{bmatrix} A_x A_x & A_x A_y & A_x A_z \\ A_y A_x & A_y A_y & A_y A_z \\ A_z A_x & A_z A_y & A_z A_z \end{bmatrix} \right] + \sin \theta \begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix} \right) \{P\}$$

$$[M] = \left(\begin{bmatrix} A_x A_x & A_x A_y & A_x A_z \\ A_y A_x & A_y A_y & A_y A_z \\ A_z A_x & A_z A_y & A_z A_z \end{bmatrix} + \cos \theta \begin{bmatrix} 1 - A_x A_x & -A_x A_y & -A_x A_z \\ -A_y A_x & 1 - A_y A_y & -A_y A_z \\ -A_z A_x & -A_z A_y & 1 - A_z A_z \end{bmatrix} + \sin \theta \begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix} \right)$$



The Rotation Matrix for an Arbitrary Axis and Angle

$$[M] = \begin{bmatrix} A_x A_x & A_x A_y & A_x A_z \\ A_y A_x & A_y A_y & A_y A_z \\ A_z A_x & A_z A_y & A_z A_z \end{bmatrix} + \cos \theta \begin{bmatrix} 1 - A_x A_x & -A_x A_y & -A_x A_z \\ -A_y A_x & 1 - A_y A_y & -A_y A_z \\ -A_z A_x & -A_z A_y & 1 - A_z A_z \end{bmatrix} + \sin \theta \begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix}$$

$$[M] = \begin{bmatrix} A_x A_x + \cos \theta (1 - A_x A_x) & A_x A_y - \cos \theta (A_x A_y) - \sin \theta A_z & A_x A_z - \cos \theta (A_x A_z) + \sin \theta A_y \\ A_y A_x - \cos \theta (A_y A_x) + \sin \theta A_z & A_y A_y + \cos \theta (1 - A_y A_y) & A_y A_z - \cos \theta (A_y A_z) - \sin \theta A_x \\ A_z A_x - \cos \theta (A_z A_x) - \sin \theta A_y & A_z A_y - \cos \theta (A_z A_y) + \sin \theta A_x & A_z A_z + \cos \theta (1 - A_z A_z) \end{bmatrix}$$

For this to be correct, A must be a unit vector

