


Transformations

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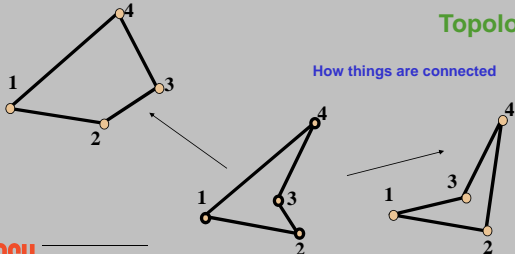


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Geometry vs. Topology

Geometry:
Where things are (e.g., coordinates)

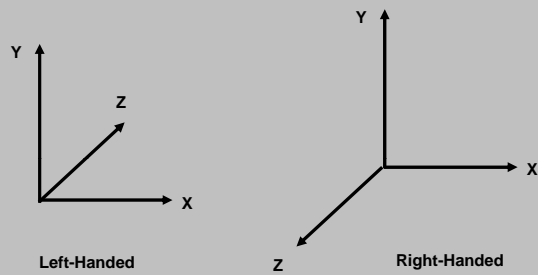
Topology:
How things are connected



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3D Coordinate Systems



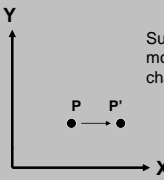
Left-Handed **Right-Handed**

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Transformations

Suppose you have a point P and you want to move it over by 2 units in X – how would you change P's coordinates?



$$P' = (P'_x, P'_y) = (P_x + 2., P_y)$$

This is known as a coordinate *transformation*

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General Form of 3D Linear Transformations

$$x' = Ax + By + Cz + D$$

$$y' = Ex + Fy + Gz + H$$

$$z' = Ix + Jy + Kz + L$$

Transform the geometry – leave the topology as is

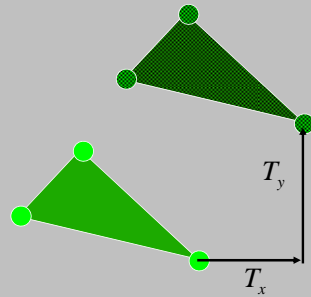
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Translation

$$x' = x + T_x$$

$$y' = y + T_y$$

$$z' = z + T_z$$


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Scaling

$$x' = x \cdot S_x$$

$$y' = y \cdot S_y$$

$$z' = z \cdot S_z$$

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2D Rotation

$$x' = x \cos \theta - y \sin \theta$$

$$y' = x \sin \theta + y \cos \theta$$

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Linear Equations in Matrix Form

$$x' = Ax + By + Cz + D$$

$$y' = Ex + Fy + Gz + H$$

$$z' = Ix + Jy + Kz + L$$

x consuming column

x' producing row →

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

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Identity Matrix

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

[I] signifies that "Nothing has changed"

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Matrix Inverse

$$[M] \cdot [M]^{-1} = [I]$$

$$[M] \cdot [M]^{-1} = \text{"Nothing has changed"}$$

"Whatever [M] does, [M]⁻¹ undoes"

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Translation

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

Quick! What is the inverse of this matrix?

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Scaling

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

Quick! What is the inverse of this matrix?

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3D Rotation About Z

Right-handed coordinates
Right-handed positive rotation rule

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

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3D Rotation About Y

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} \cos \theta & 0 & \sin \theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \theta & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

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3D Rotation About X

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

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Compound Transformations

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} T_{+A,+B} \\ R_\theta \\ T_{-A,-B} \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

Say it

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Matrix Multiplication is Not Commutative

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Matrix Multiplication is Associative

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{bmatrix} T_{+A,+B} \\ R_\theta \\ T_{-A,-B} \end{bmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \left(\begin{bmatrix} T_{+A,+B} \\ R_\theta \\ T_{-A,-B} \end{bmatrix} \right) \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

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Can Multiply All Geometry by One Matrix !

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = [M] \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Graphics hardware can do this very quickly!

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The Rotation Matrix for an Arbitrary Axis and Angle

$$P_{\parallel} = \hat{A}(\hat{A} \cdot P)$$

$$P_{\perp} = P - P_{\parallel} = P - \hat{A}(\hat{A} \cdot P)$$

$$Q = \hat{A} \times P_{\perp} = \hat{A} \times P - 0 = \hat{A} \times P_{\perp} + \hat{A} \times P_{\parallel} = \hat{A} \times (P_{\perp} + P_{\parallel}) = \hat{A} \times P$$

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The Rotation Matrix for an Arbitrary Axis and Angle

$$P'_{\parallel} = P_{\parallel} \quad P'_{\perp} = P_{\perp} \cos \theta + Q \sin \theta$$

$$P' = P'_{\parallel} + P'_{\perp}$$

$$P' = [\hat{A}(\hat{A} \cdot P)] + \cos \theta [P - \hat{A}(\hat{A} \cdot P)] + \sin \theta [\hat{A} \times P]$$

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The Rotation Matrix for an Arbitrary Axis and Angle

$$\hat{A}(\hat{A} \cdot P) = \begin{bmatrix} A_x A_x & A_x A_y & A_x A_z \\ A_y A_x & A_y A_y & A_y A_z \\ A_z A_x & A_z A_y & A_z A_z \end{bmatrix} \{P\}$$

$$\hat{A} \times P = \begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix} \{P\}$$

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The Rotation Matrix for an Arbitrary Axis and Angle

$$P' = [\hat{A}(\hat{A} \cdot P)] + \cos \theta [P - \hat{A}(\hat{A} \cdot P)] + \sin \theta [\hat{A} \times P]$$

$$P' = \left(\begin{bmatrix} A_x A_x & A_x A_y & A_x A_z \\ A_y A_x & A_y A_y & A_y A_z \\ A_z A_x & A_z A_y & A_z A_z \end{bmatrix} + \cos \theta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} A_x A_x & A_x A_y & A_x A_z \\ A_y A_x & A_y A_y & A_y A_z \\ A_z A_x & A_z A_y & A_z A_z \end{bmatrix} + \sin \theta \begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix} \right) \{P\}$$

$$[M] = \left(\begin{bmatrix} A_x A_x & A_x A_y & A_x A_z \\ A_y A_x & A_y A_y & A_y A_z \\ A_z A_x & A_z A_y & A_z A_z \end{bmatrix} + \cos \theta \begin{bmatrix} 1 - A_x A_x & -A_x A_y & -A_x A_z \\ -A_y A_x & 1 - A_y A_y & -A_y A_z \\ -A_z A_x & -A_z A_y & 1 - A_z A_z \end{bmatrix} + \sin \theta \begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix} \right)$$

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The Rotation Matrix for an Arbitrary Axis and Angle

$$[M] = \begin{bmatrix} A_x A_x & A_x A_y & A_x A_z \\ A_x A_y & A_y A_y & A_y A_z \\ A_x A_z & A_y A_z & A_z A_z \end{bmatrix} + \cos \theta \begin{bmatrix} 1 - A_x A_x & -A_x A_y & -A_x A_z \\ -A_x A_y & 1 - A_y A_y & -A_y A_z \\ -A_x A_z & -A_y A_z & 1 - A_z A_z \end{bmatrix} + \sin \theta \begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix}$$

$$[M] = \begin{bmatrix} A_x A_x + \cos \theta (1 - A_x A_x) & A_x A_y - \cos \theta (A_x A_y) - \sin \theta A_z & A_x A_z - \cos \theta (A_x A_z) + \sin \theta A_y \\ A_x A_y - \cos \theta (A_x A_y) + \sin \theta A_z & A_y A_y + \cos \theta (1 - A_y A_y) & A_y A_z - \cos \theta (A_y A_z) - \sin \theta A_x \\ A_x A_z - \cos \theta (A_x A_z) - \sin \theta A_y & A_y A_z - \cos \theta (A_y A_z) + \sin \theta A_x & A_z A_z + \cos \theta (1 - A_z A_z) \end{bmatrix}$$

For this to be correct, A must be a unit vector

