Vectors

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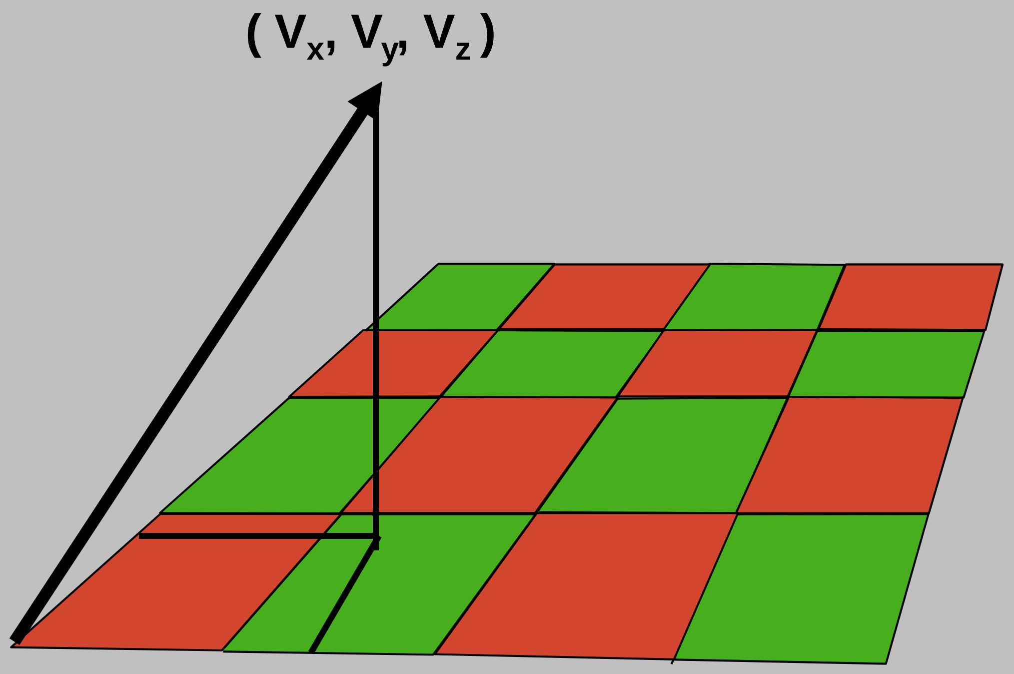
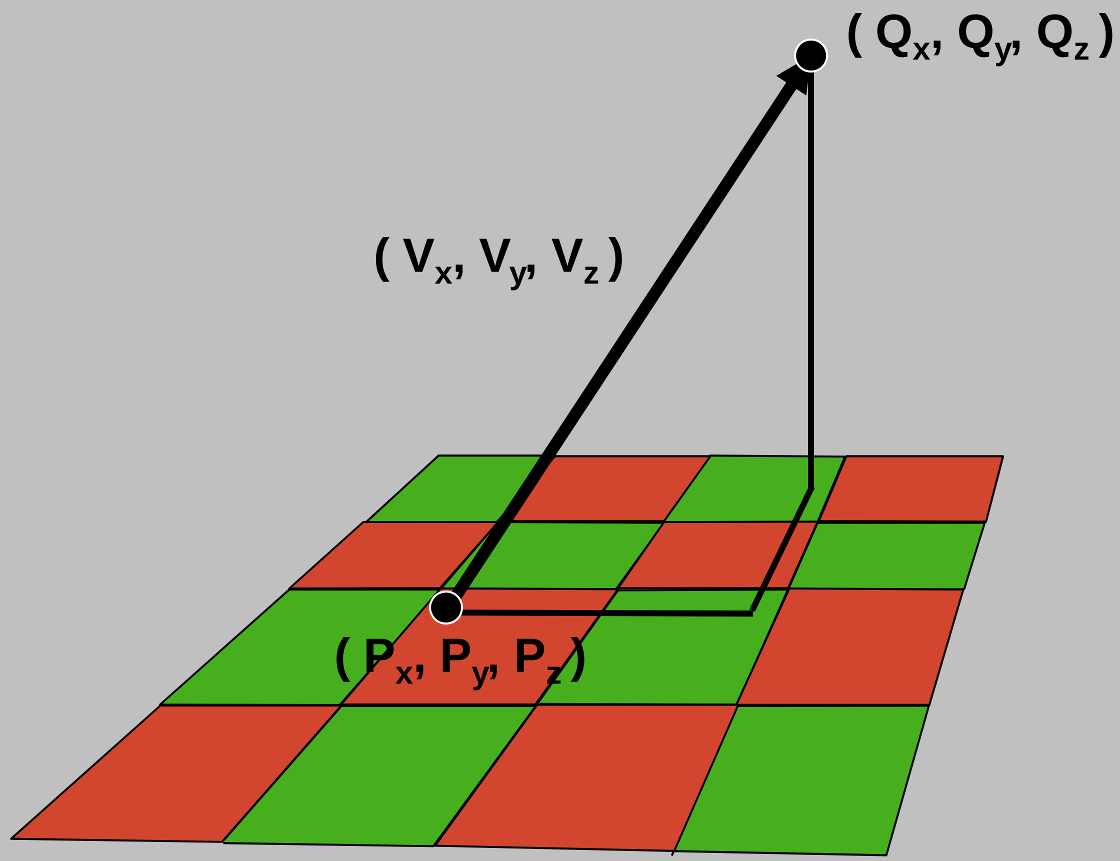
Vectors have Direction and Magnitude



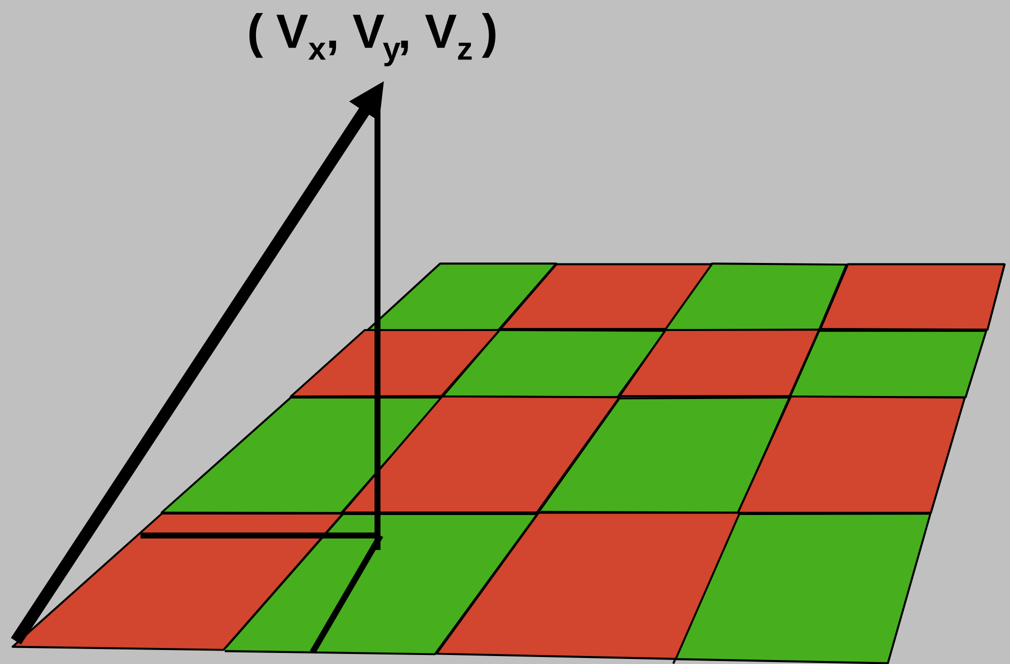
$$\|V\| = \sqrt{V_x^2 + V_y^2 + V_z^2}$$

A Vector Can Also Be Defined as the Positional Difference Between Two Points



$$(V_x, V_y, V_z) = (Q_x - P_x, Q_y - P_y, Q_z - P_z)$$

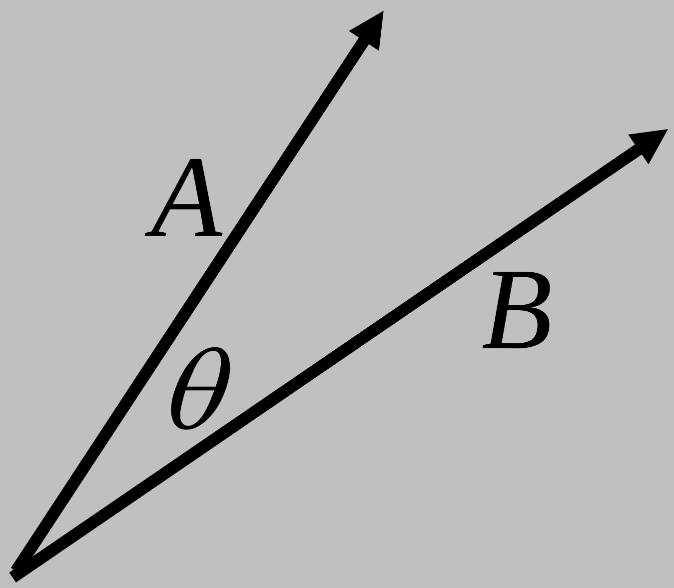
Unit Vectors have a Magnitude = 1.0



$$\|V\| = \sqrt{V_x^{\,2} + V_y^{\,2} + V_z^{\,2}}$$

$$\hat{V} = \frac{V}{\|V\|}$$

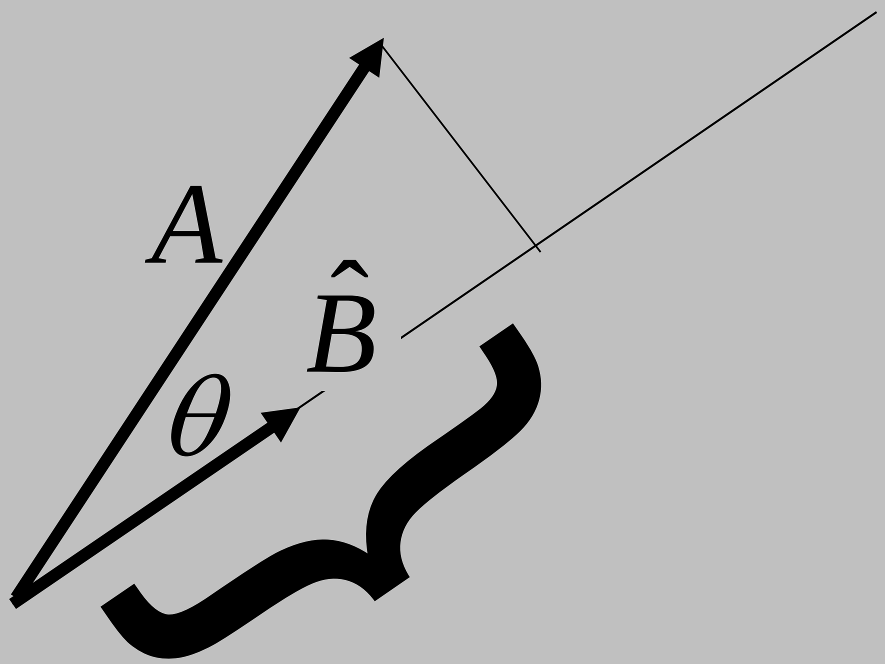
Dot Product



$$A \bullet B = (A_x B_x + A_y B_y + A_z B_z) = \|A\| \|B\| \cos\theta$$

Because it produces a scalar result (i.e., a single number), this is also called the *Scalar Product*

A Physical Interpretation of the Dot Product



$$A \bullet \hat{B} = \|A\| \cos\theta$$

= How much of A lives in the $\hat{B}$ direction

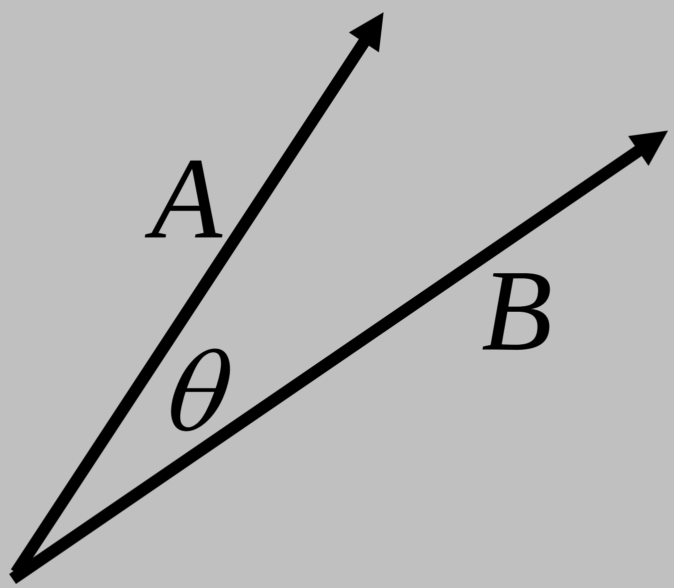
Dot Products are Commutative

$$A \bullet B = B \bullet A$$

Dot Products are Distributive

$$A \bullet (B + C) = (A \bullet B) + (A \bullet C)$$

Cross Product

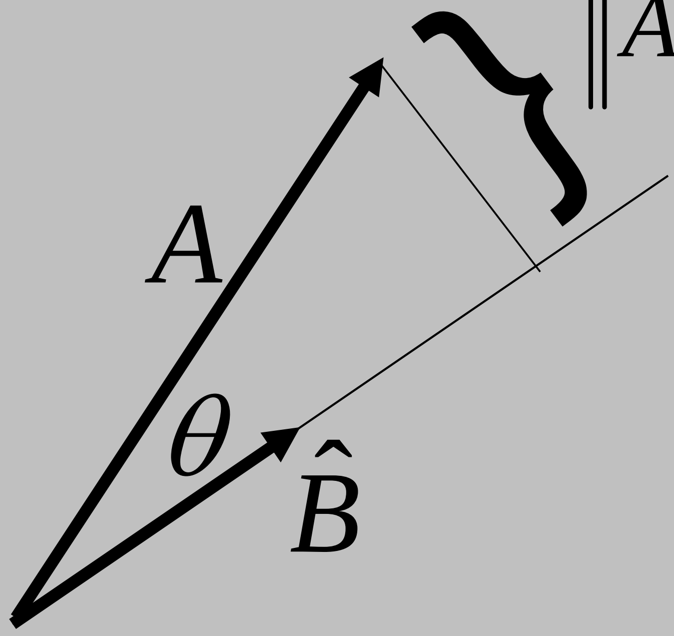


$$A \times B = (A_y B_z - A_z B_y, A_z B_x - A_x B_z, A_x B_y - A_y B_x)$$

$$\|A \times B\| = \|A\| \|B\| \sin\theta$$

Because it produces a vector result (i.e., three numbers), this is also called the *Vector Product*

A Physical Interpretation of the Cross Product

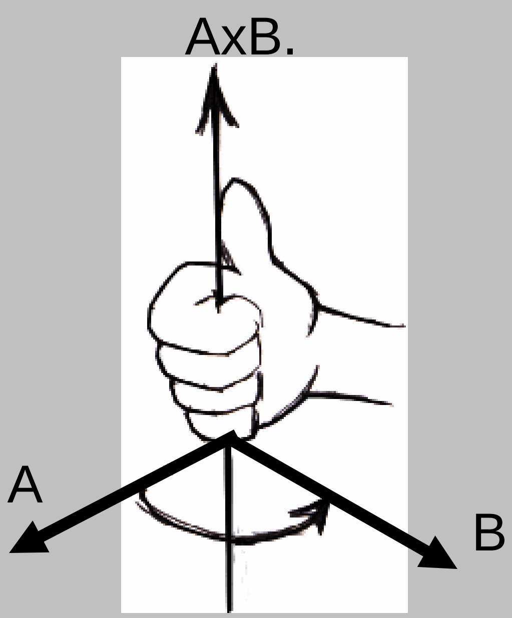


$$\left\| A \times \hat{B} \right\| = \left\| A \right\| \sin\theta$$

= *How much of A lives perpendicular to the* $\hat{B}$ *direction*

The Perpendicular Property of the Cross Product

The vector $A \times B$ is both perpendicular to A and perpendicular to B



The Right-Hand-Rule Property of the Cross Product

Curl the fingers of your right hand in the direction that starts at A and heads towards B. Your thumb points in the direction of AxB.

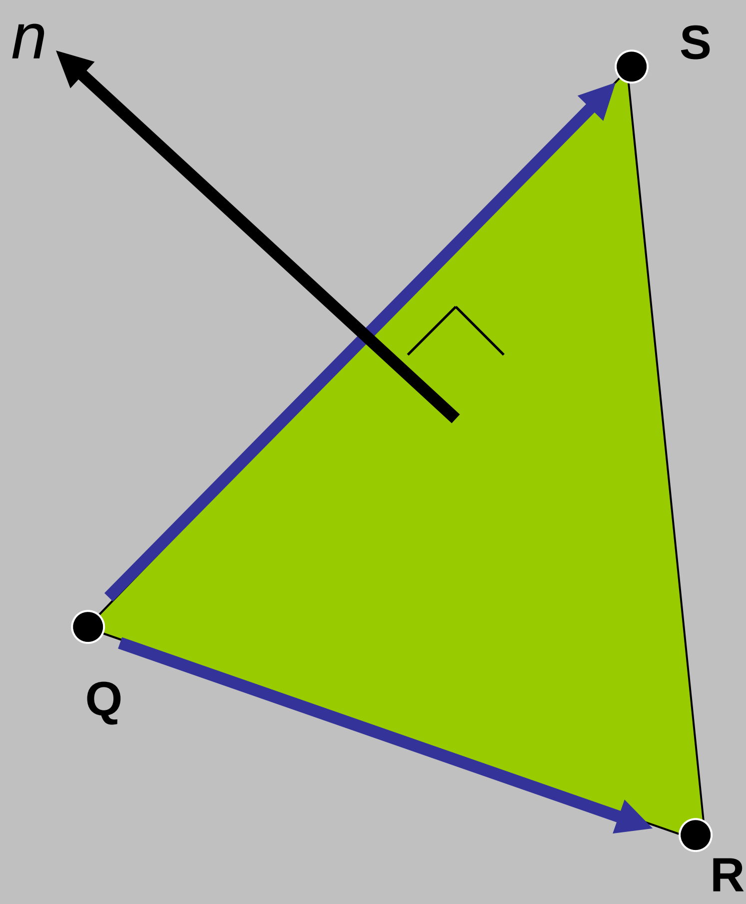
Cross Products are *Not* Commutative

$$A \times B = -B \times A$$

Cross Products are Distributive

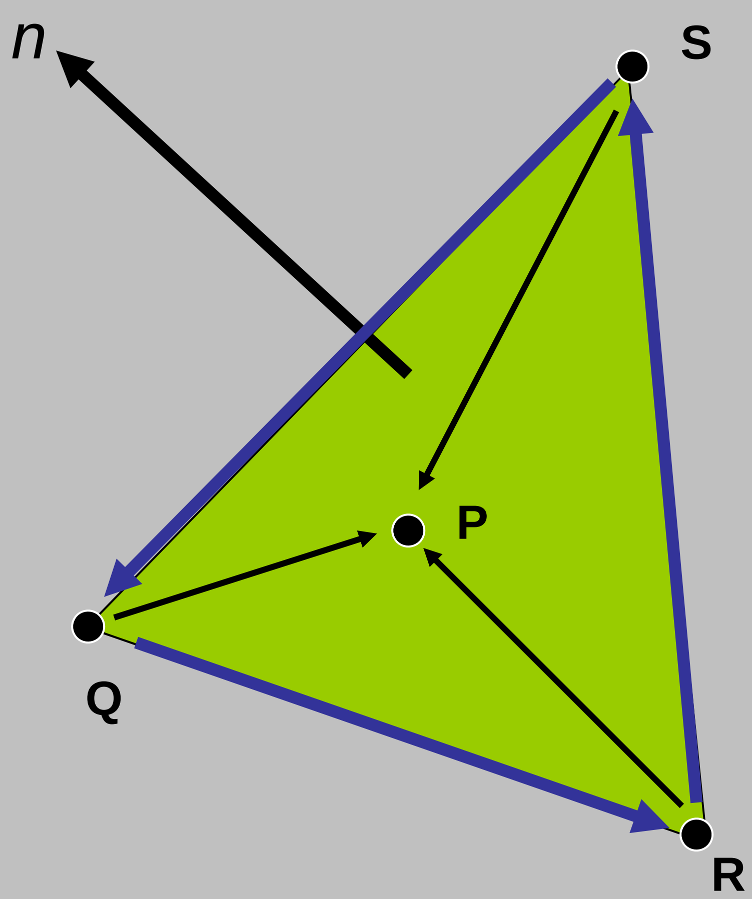
$$A \times (B + C) = (A \times B) + (A \times C)$$

A Use for the Cross Product :
Find a Vector Perpendicular to a Plane (=the Surface Normal)



$$n = (R - Q) \times (S - Q)$$

A Use for the Cross and Dot Products :
Is a Point inside a Triangle?



Let:

$$n = (R - Q) \times (S - Q)$$

$$n_Q = (R - Q) \times (P - Q)$$

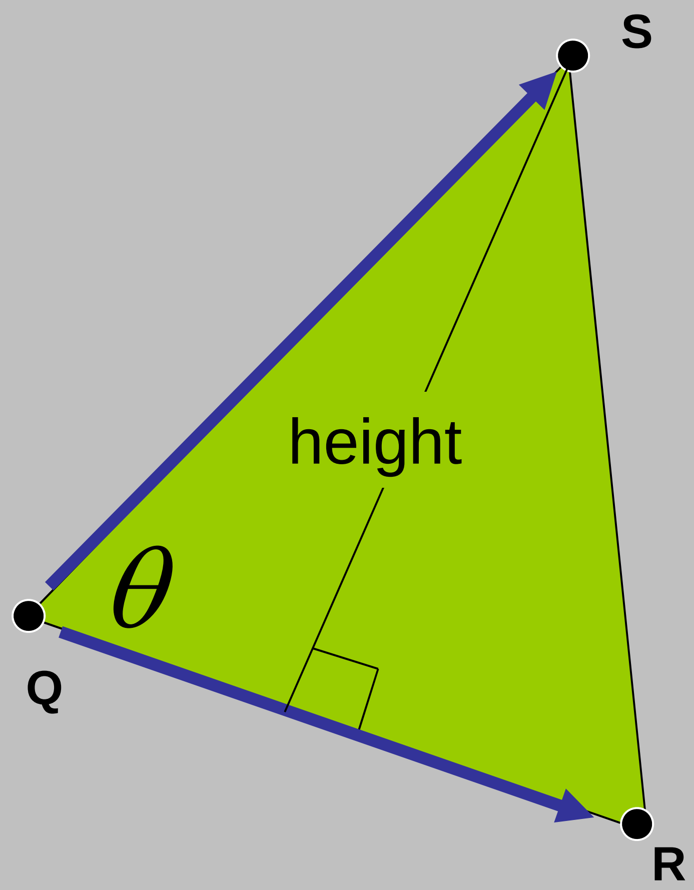
$$n_R = (S - R) \times (P - R)$$

$$n_s = (Q - S) \times (P - S)$$

If $(n \bullet n_q), (n \bullet n_r), and (n \bullet n_s)$

are all positive, then P is inside the triangle QRS

A Use for the Cross Product : Area of a Triangle



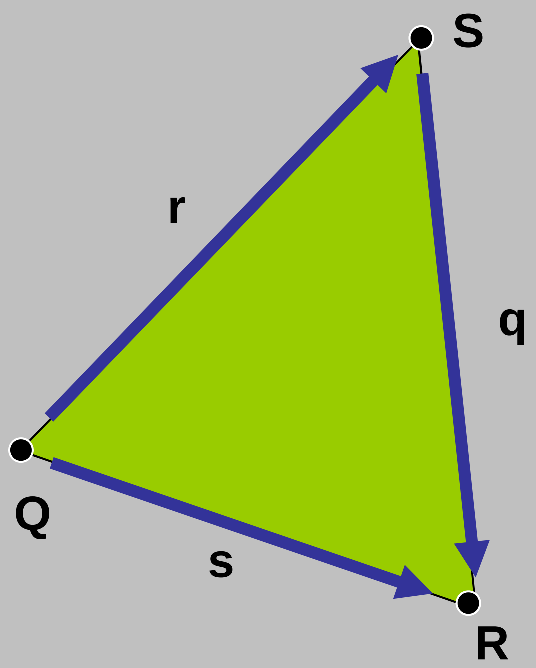
$$Area = \frac{1}{2} \cdot Base \cdot Height$$

$$Base = \left\| QR \right\|$$

$$Height = \left\| QS \right\| \sin\theta$$

$$Area = \frac{1}{2} \cdot \left\| QR \right\| \cdot \left\| QS \right\| \cdot \sin\theta = \frac{1}{2} \cdot \left\| (R-Q) \times (S-Q) \right\|$$

Derivation of the Law of Cosines



$$s = R - Q$$

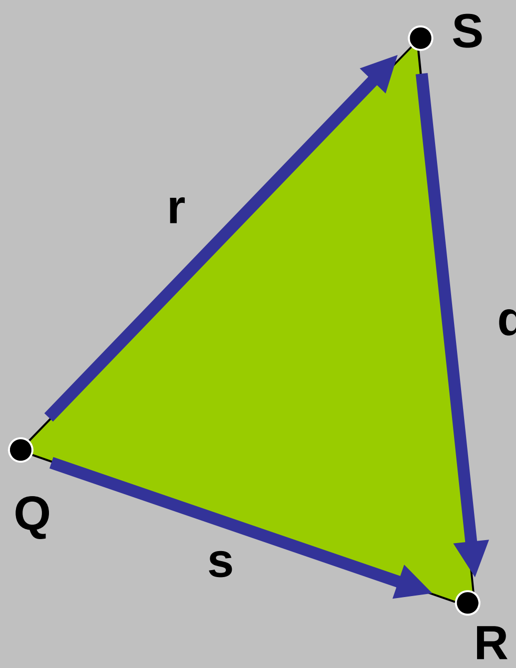
$$s^2 = \|R - Q\|^2$$

$$s^2 = (R - Q) \bullet (R - Q)$$

$$s^2 = [(R - S) + (S - Q)] \bullet [(R - S) + (S - Q)]$$

$$s^2 = [(R - S)(R - S)] + [(S - Q)(S - Q)] - 2(R - S) \bullet (S - Q)$$

$$s^2 = q^2 + r^2 - 2qr \cos S$$



Derivation of the Law of Sines

$$2 * Area(\Delta QRS) = \left\| (S - Q) \times (R - Q) \right\|$$
$$= rs \sin Q$$

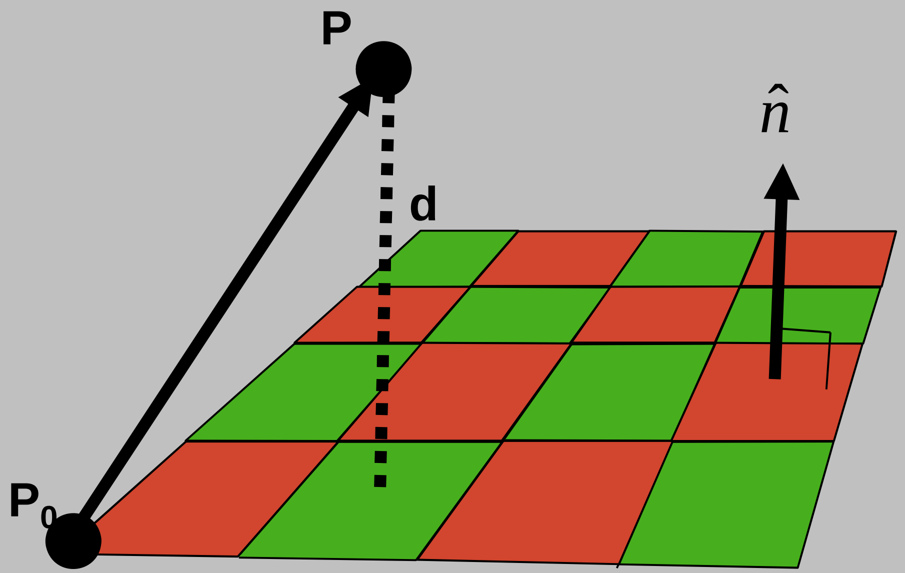
But, the area is the same regardless of which two sides we use to compute it, so:

$$rs \sin Q = qs \sin R = qr \sin S$$

Dividing by *(qrs)* gives:

$$\frac{\sin Q}{q} = \frac{\sin R}{r} = \frac{\sin S}{s}$$

Distance from a Point to a Plane



The equation of the plane is:

$$\left(\left(x,y,z\right)-\left(P_{0x},P_{0y},P_{0z}\right)\right)\bullet(n_x,n_y,n_z)=0$$

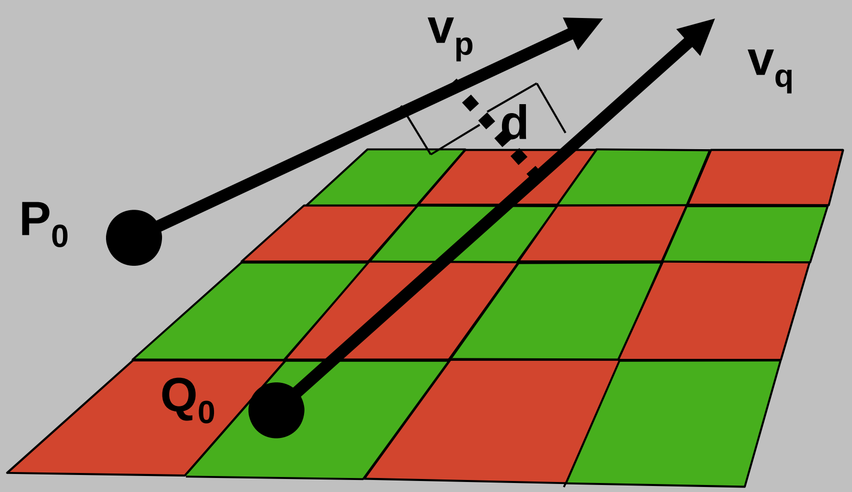
which expands out to become the more familiar Ax + By + Cz + D = 0

The distance from the point P to the plane is based on this:

$$d=\left(P-P_0\right)\bullet\hat{n}$$

The dot product is answering the question “How much of (P-P_0) is in the normal direction?”. Note that this gives a *signed distance*. If d > 0, then P is on the same side of the plane as the normal.:

Distance from a 3D Line to a 3D Line



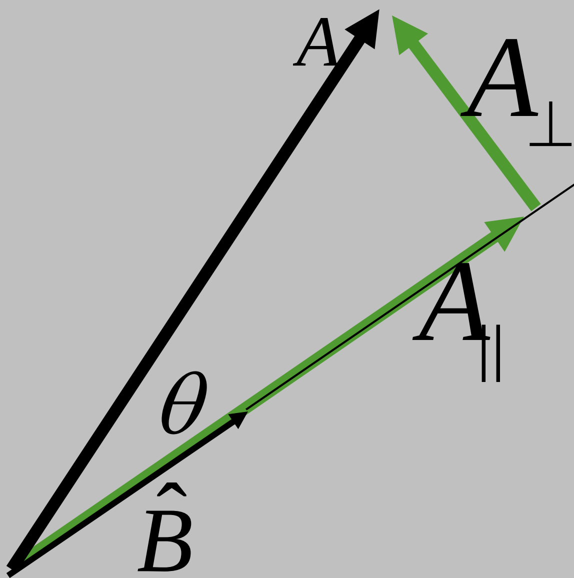
The equation of the lines are : $P = P_0 + t \cdot v_p$ $\quad Q = Q_0 + t \cdot v_q$

A vector between them that is perpendicular to both is: $v_\perp = v_p \times v_q$

We need to answer the question "How much of (Q_0-P_0) is in the v direction?". To do this, we once again use the dot product:

$$d = \left(P_0 - Q_0\right) \bullet \hat{v}_\perp$$

Another use for Dot Products :
Force One Vector to be Perpendicular to Another Vector



The strategy is to get rid of the parallel component, leaving just the perpendicular

$$A = A_\parallel + A_\perp$$

$$A_\perp = A - A_\parallel$$

But, $$A_\parallel = (A \bullet \hat{B})\hat{B}$$

So that $$A' = A_\perp = A - (A \bullet \hat{B})\hat{B}$$

This is known as *Gram-Schmidt orthogonalization*