Vectors have Direction and Magnitude

Vectors have Direction and Magnitude

Magnitude: $\|V\| = \sqrt{V_x^2 + V_y^2 + V_z^2}$
A Vector Can Also Be Defined as the Positional Difference Between Two Points

\[(V_x, V_y, V_z) = (Q_x - P_x, Q_y - P_y, Q_z - P_z)\]

Unit Vectors have a Magnitude = 1.0

\[\|V\| = \sqrt{V_x^2 + V_y^2 + V_z^2}\]

\[\hat{V} = \frac{V}{\|V\|}\]

The circumflex (^) tells us this is a unit vector.
Dot Product

\[ A = (A_x, A_y, A_z) \]
\[ B = (B_x, B_y, B_z) \]

\[ A \cdot B = (A_x B_x + A_y B_y + A_z B_z) = \|A\|\|B\| \cos \theta \]

Because it produces a scalar result (i.e., a single number), this is also called the Scalar Product.

A Physical Interpretation of the Dot Product

\[ A \cdot \hat{B} = \|A\| \cos \theta \]

This is important – memorize this phrase!

= How much of A lives in the B direction
The amount of the force accelerating the car along the road is “how much of $F$ is in the horizontal direction?”

\[ F_{\text{road}} = F \cos \theta \]

This is easy to see in 2D, but a 3D version of the same problem is trickier.

The amount of the force accelerating the car along the road is “how much of $F$ is in the $R$ direction?”

\[ F_{\text{road}} = F \cos \theta = F \cdot \hat{R} \]
A Physical Interpretation of the Dot Product

\[ F_{road} = F \cos \theta = F \cdot \hat{R} \]

Dot Products are Commutative

\[ A \cdot B = B \cdot A \]

Dot Products are Distributive

\[ A \cdot (B + C) = (A \cdot B) + (A \cdot C) \]
The Perpendicular to a 2D Vector

If \( V = (x, y) \)

then \( V_{\perp} = (-y, x) \)

You can tell that this is true because

\[
V \cdot V_{\perp} = (x, y) \cdot (-y, x) = -xy + xy = 0 = \cos 90^\circ
\]

Cross Product

\( A = (A_x, A_y, A_z) \)

\( B = (B_x, B_y, B_z) \)

\[
A \times B = (A_yB_z - A_zB_y, A_zB_x - A_xB_z, A_xB_y - A_yB_x)
\]

\[
\|A \times B\| = \|A\|\|B\|\sin \theta
\]

Because it produces a vector result (i.e., three numbers), this is also called the Vector Product.
A Physical Interpretation of the Cross Product

\[ \|A \times \hat{B}\| = \|A\| \sin \theta \]

= How much of \( A \) lives perpendicular to the \( \hat{B} \) direction

This is important – memorize this phrase!

The Perpendicular Property of the Cross Product

The vector \( A \times B \) is both perpendicular to \( A \) and perpendicular to \( B \)

The Right-Hand-Rule Property of the Cross Product

Curl the fingers of your right hand in the direction that starts at \( A \) and heads towards \( B \). Your thumb points in the direction of \( A \times B \).
Cross Products are Not Commutative

\[ \mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \]

Cross Products are Distributive

\[ \mathbf{A} \times (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C}) \]

A Use for the Cross Product:
Finding a Vector Perpendicular to a Plane (= the Surface Normal)

\[ \mathbf{n} = (\mathbf{R} - \mathbf{Q}) \times (\mathbf{S} - \mathbf{Q}) \]
A Use for the Cross Product:
Finding a Vector Perpendicular to a Plane (= the Surface Normal) –
This is used in CG Lighting

A Use for the Cross and Dot Products:
Is a Point Inside a Triangle? – 3D (X-Y-Z) Version

Let:
\[ n = (R - Q) \times (S - Q) \]
\[ n_q = (R - Q) \times (P - Q) \]
\[ n_r = (S - R) \times (P - R) \]
\[ n_s = (Q - S) \times (P - S) \]

If \( n \cdot n_q \), \( n \cdot n_r \), and \( n \cdot n_s \) are all positive, then P is inside the triangle QRS
Is a Point Inside a Triangle?  
This can be simplified if you are in 2D (X-Y)

\[ E_{RS} = (P - R) \cdot (RS) \perp \]

where:  
\[ RS = (S_x - R_x, S_y - R_y) \]

and:  
\[ (RS) \perp = (R_y - S_y, S_x - R_x) \]

Similarly,

\[ E_{SQ} = (P - S) \cdot (SQ) \perp \]

\[ E_{QR} = (P - Q) \cdot (QR) \perp \]

If \( E_{RS}, E_{SQ}, E_{QR} \) are all positive, then P is inside the triangle QRS

A Use for the Cross Product :  
Finding the Area of a 3D Triangle

\[ Area = \frac{1}{2} \cdot \text{Base} \cdot \text{Height} \]

\[ Base = \|QR\| \]

\[ Height = \|QS\| \sin \theta \]

\[ Area = \frac{1}{2} \cdot \|QR\| \cdot \|QS\| \cdot \sin \theta = \frac{1}{2} \cdot \|(R - Q) \times (S - Q)\| \]
Derivation of the Law of Cosines

\[ s = R - Q \]
\[ s^2 = \| R - Q \|^2 \]
\[ s^2 = (R - Q) \cdot (R - Q) \]

\[ s^2 = [(R - S) + (S - Q)] \cdot [(R - S) + (S - Q)] \]
\[ s^2 = [(R - S)(R - S)] + [(S - Q)(S - Q)] - 2(R - S) \cdot (S - Q) \]
\[ s^2 = q^2 + r^2 - 2qr \cos S \]

Derivation of the Law of Sines

\[ 2 \times \text{Area}(\triangle QRS) = \| (S - Q) \times (R - Q) \| \]
\[ = rs \sin Q \]

But, the area is the same regardless of which two sides we use to compute it, so:

\[ rs \sin Q = qs \sin R = qr \sin S \]

Dividing by \((qrs)\) gives:

\[ \frac{\sin Q}{q} = \frac{\sin R}{r} = \frac{\sin S}{s} \]
Distance from a Point to a Plane

In high school, you defined a plane by:

\[ Ax + By + Cz + D = 0 \]

It is more useful to define it by a point on the plane combined with the plane’s normal vector.

If you want the familiar equation of the plane, it is:

\[ \left( (x, y, z) - \left( Q_x, Q_y, Q_z \right) \right) \cdot (n_x, n_y, n_z) = 0 \]

which expands out to become the more familiar \(Ax + By + Cz + D = 0\).

The perpendicular distance from the point \(P\) to the plane is based on the plane equation:

\[ d = \left( \mathbf{P} - \mathbf{Q} \right) \cdot \mathbf{n} \]

The dot product is answering the question “How much of \((\mathbf{P} - \mathbf{Q})\) is in the \(\mathbf{n}\) direction?”. Note that this gives a signed distance. If \(d > 0\), then \(P\) is on the same side of the plane as the normal points. This is very useful.

Where does a line segment intersect an infinite plane?

The equation of the line segment is:

\[ P = (1-t)P_0 + tP_1 \]

If point \(P\) is in the plane, then:

\[ \left( \left( P_x, P_y, P_z \right) - \left( Q_x, Q_y, Q_z \right) \right) \cdot (n_x, n_y, n_z) = 0 \]

If we substitute the parametric expression for \(P\) into the plane equation, then the only thing we don’t know in that equation is \(t\). Solve it for \(t^*\). Knowing \(t^*\) will let us compute the \((x, y, z)\) of the actual intersection using the line equation. If \(t^*\) has a zero in the denominator, then that tells us that \(t^*=-\), and the line must be parallel to the plane.

This gives us the point of intersection with the infinite plane. We could now use the method covered a few slides ago to see if \(P\) lies inside a particular triangle.
Minimal Distance Between Two 3D Lines

The equation of the lines are: 

\[ P = P_0 + t \cdot v_p \quad Q = Q_0 + t \cdot v_q \]

The minimal distance vector between the two lines must be perpendicular to both

A vector between them that is perpendicular to both is: \[ v_{\perp} = v_p \times v_q \]

We need to answer the question “How much of (Q_0-P_0) is in the \( v_{\perp} \) direction?”. To do this, we once again use the dot product:

\[ d = (P_0 - Q_0) \cdot \hat{v}_{\perp} \]

Another use for Dot Products:
Force One Vector to be Perpendicular to Another Vector

Here, we want to force \( A \) to become perpendicular to \( B \)

The strategy is to get rid of the parallel component, leaving just the perpendicular

\[ A = A_{\parallel} + A_{\perp} \]
\[ A_{\perp} = A - A_{\parallel} \]

But, \[ A_{\perp} = (A \cdot \hat{B}) \hat{B} \]

So that \[ A' = A_{\perp} = A - (A \cdot \hat{B}) \hat{B} \]

This is known as **Gram-Schmidt orthogonalization**