Vectors

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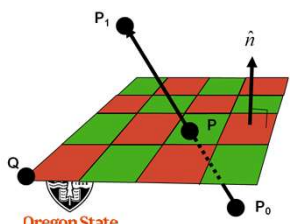
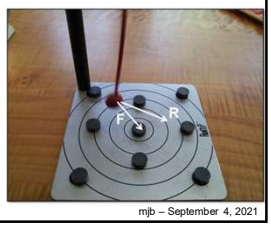
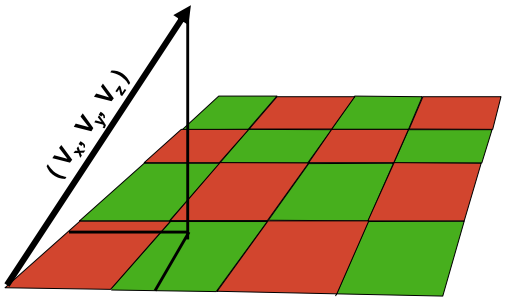


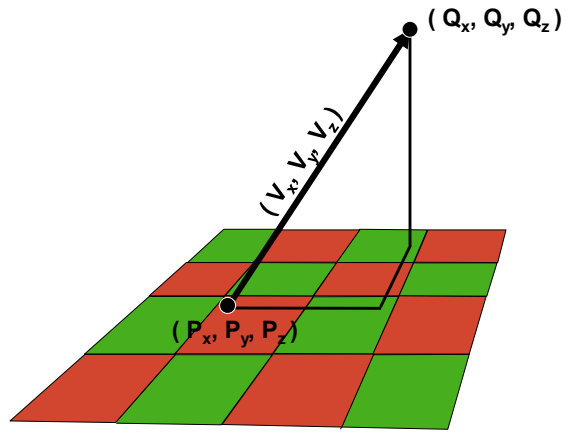


Vectors have Direction and Magnitude



Magnitude: $\|V\| = \sqrt{V_x^2 + V_y^2 + V_z^2}$

A Vector Can Also Be Defined as the Positional Difference Between Two Points



$$(V_x, V_y, V_z) = (Q_x - P_x, Q_y - P_y, Q_z - P_z)$$

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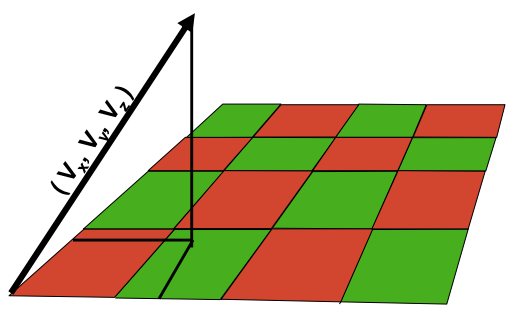
Unit Vectors have a Magnitude = 1.0



$$\|V\| = \sqrt{V_x^2 + V_y^2 + V_z^2}$$

$$\hat{V} = \frac{V}{\|V\|}$$

The circumflex (^) tells us this is a unit vector

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Dot Product

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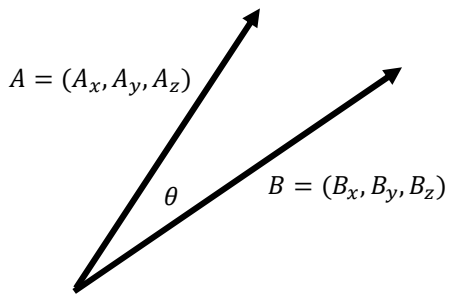

$$A \bullet B = (A_x B_x + A_y B_y + A_z B_z) = \|A\|\|B\| \cos\theta$$

Because it produces a scalar result (i.e., a single number), this is also called the *Scalar Product*

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A Physical Interpretation of the Dot Product

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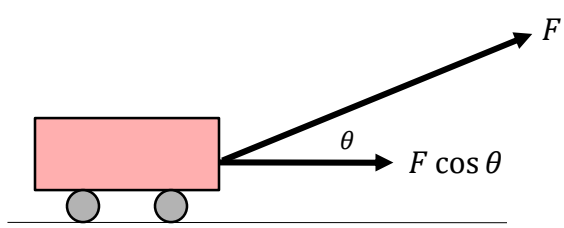

The amount of the force accelerating the car along the road is "how much of *F* is in the horizontal direction?"

$$F_{road} = F \cos\theta$$

This is easy to see in 2D, but a 3D version of the same problem is trickier.

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A Physical Interpretation of the Dot Product

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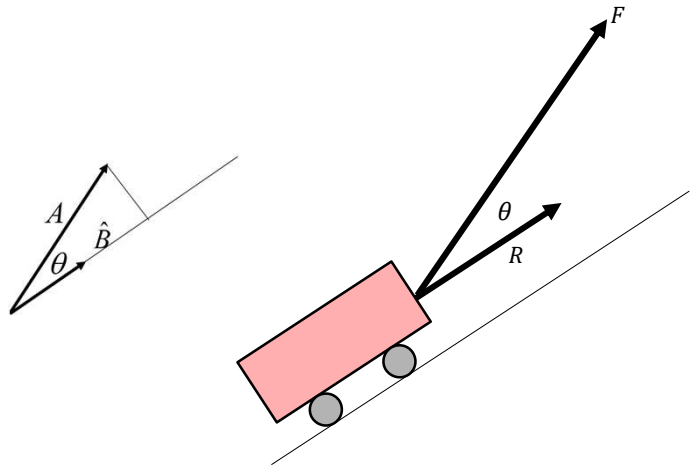

The amount of the force accelerating the car along the road is "how much of *F* is in the R direction?"

$$F_{road} = F\cos\theta = F \bullet \hat{R}$$

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Even Trickier…

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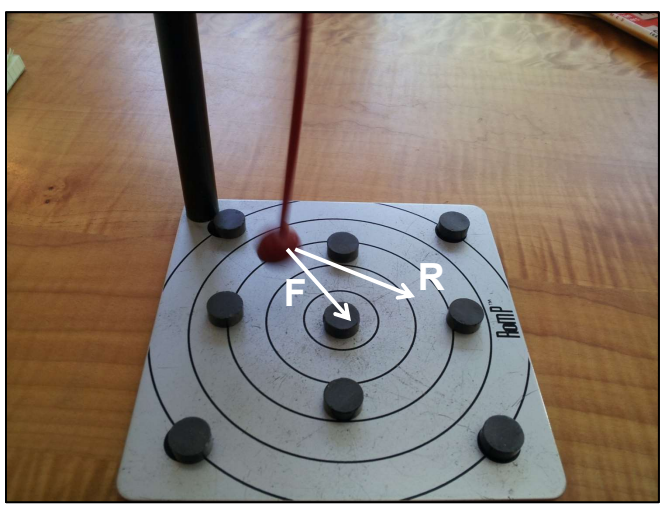

$$F_{road} = F\cos\theta = F \bullet \hat{R}$$

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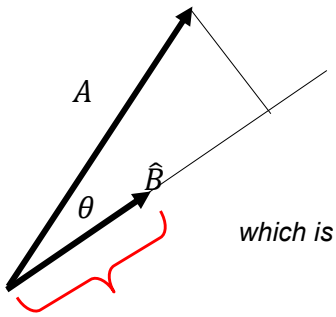
Generalizing How Much of A Lives in the B Direction

9



$$A \cdot B = \|A\|\|B\|cos\theta$$

$$A \cdot \hat{B} = \|A\|cos\theta$$

which is the length of the projection of A onto the B line

So, how much of A lives in the $\hat{B}$ direction is that magnitude times the B unit vector:

$$(A \cdot \hat{B})\ \hat{B}$$

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Generalizing How Much of A Lives Perpendicular to the B Direction

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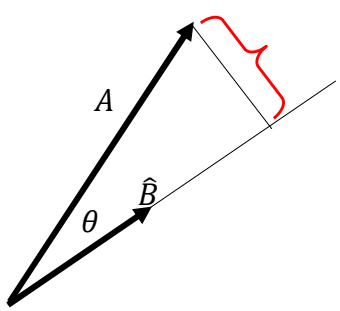

From the previous slide, how much of A lives in the $\hat{B}$ direction is :

$$(A \cdot \hat{B})\ \hat{B}$$

That, plus the perpendicular vector equals A, so that how much of A is perpendicular to the $\hat{B}$ direction is:

$$\mathrm{A} - (A \cdot \hat{B})\ \hat{B}$$

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Dot Products are Commutative

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$$A \bullet B = B \bullet A$$

Dot Products are Distributive

$$A \bullet (B + C) = (A \bullet B) + (A \bullet C)$$

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The Perpendicular to a 2D Vector

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If $V = (x, y)$

then $V_\perp = (-y, x)$

You can tell that this is true because

$$V \bullet V_\perp = (x, y) \bullet (-y, x) = -xy + xy = 0 = cos90°$$

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Cross Product

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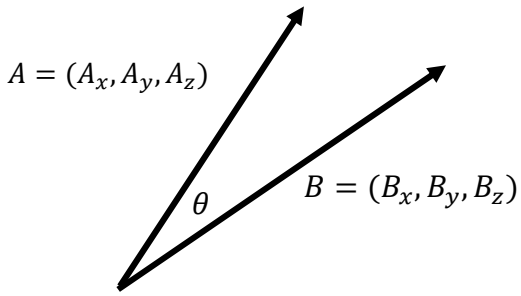

$$A \times B = (A_y B_z - A_z B_y, A_z B_x - A_x B_z, A_x B_y - A_y B_x)$$

$$\|A \times B\| = \|A\| \|B\| \sin\theta$$

Because it produces a vector result (i.e., three numbers), this is also called the *Vector Product*

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The Perpendicular Property of the Cross Product

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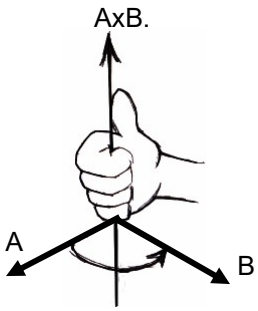
The vector $A \times B$ is both perpendicular to A and perpendicular to B



The Right-Hand-Rule Property of the Cross Product

Curl the fingers of your right hand in the direction that starts at A and heads towards B. Your thumb points in the direction of AxB.

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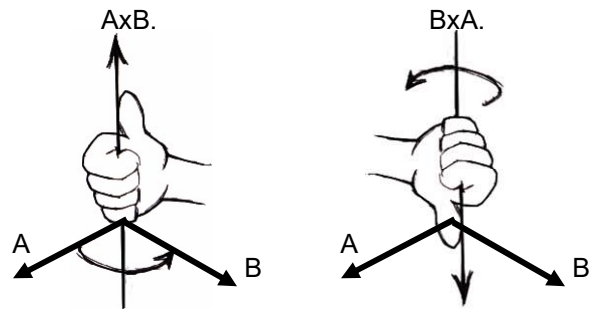
Cross Products are *Not* Commutative

$$A \times B = -B \times A$$



Cross Products are Distributive

$$A \times (B + C) = (A \times B) + (A \times C)$$

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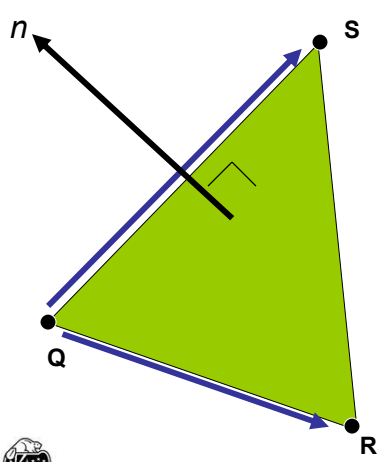
A CG Use for the Cross Product : Finding a Vector Perpendicular to a Plane (= the Surface Normal)



$$n = (R - Q) \times (S - Q)$$

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A Use for the Cross Product : Finding a Vector Perpendicular to a Plane (= the Surface Normal) – This is used in CG Lighting

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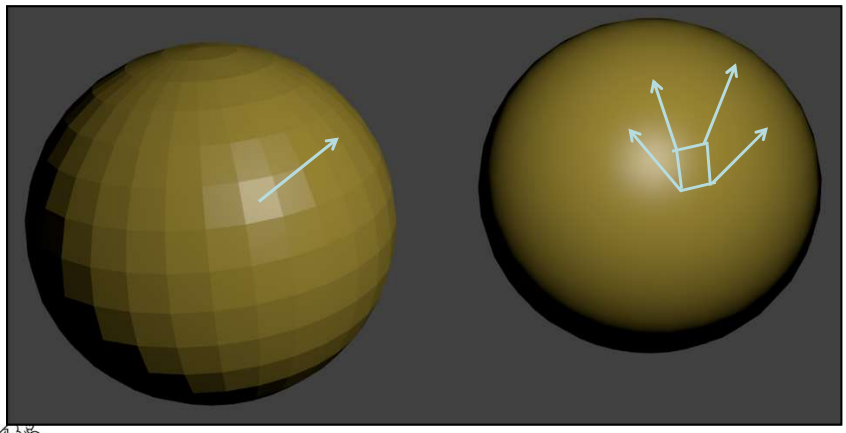

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A Use for the Cross and Dot Products : Is a Point Inside a Triangle? – 3D (X-Y-Z) Version

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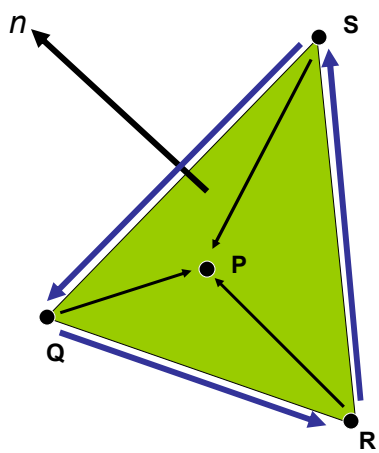

Let:

$$n = (R - Q) \times (S - Q)$$

$$n_q = (R - Q) \times (P - Q)$$

$$n_r = (S - R) \times (P - R)$$

$$n_s = (Q - S) \times (P - S)$$

If $(n \bullet n_q), (n \bullet n_r), and (n \bullet n_s)$

are all positive, then P is inside the triangle QRS

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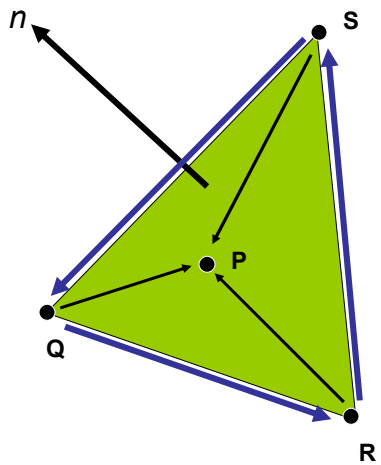
Is a Point Inside a Triangle?
This can be simplified if you are in 2D (X-Y)



$$E_{RS} = (P - R) \bullet (RS)_{\perp}$$

where: $RS = (S_x - R_x, S_y - R_y)$

and: $(RS)_{\perp} = (R_y - S_y, S_x - R_x)$

Similarly,

$$E_{SQ} = (P - S) \bullet (SQ)_{\perp}$$
$$E_{QR} = (P - Q) \bullet (QR)_{\perp}$$

If E_{RS}, E_{SQ}, E_{QR} are all positive, then P is inside the triangle QRS

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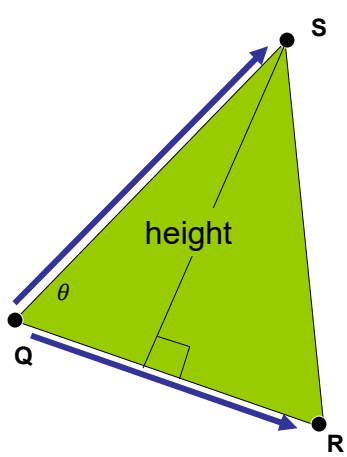
A Use for the Cross Product :
Finding the Area of a 3D Triangle



$$Area = \frac{1}{2} \cdot Base \cdot Height$$

$$Base = \|QR\|$$

$$Height = \|QS\| \sin\theta$$

$$Area = \frac{1}{2} \cdot \|QR\| \cdot \|QS\| \cdot \sin\theta = \frac{1}{2} \cdot \|(R - Q) \times (S - Q)\|$$

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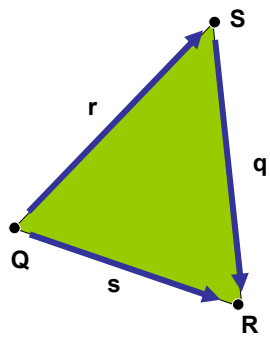
Derivation of the Law of Sines



$$2 * Area(\Delta QRS) = \|(S - Q) \times (R - Q)\|$$

$$= rs \sin Q$$

But, the area is the same regardless of which two sides we use to compute it, so:

$$rs \sin Q = qs \sin R = qr \sin S$$

Dividing by *(qrs)* gives:

$$\frac{\sin Q}{q} = \frac{\sin R}{r} = \frac{\sin S}{s}$$

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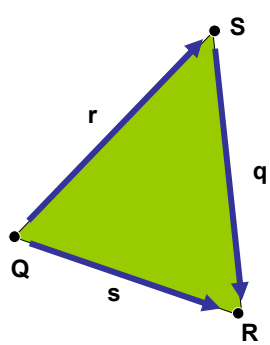
Derivation of the Law of Cosines



$$s = R - Q$$
$$s^2 = \|R - Q\|^2$$

$$s^2 = (R - Q) \bullet (R - Q)$$

$$s^2 = [(R - S) + (S - Q)] \bullet [(R - S) + (S - Q)]$$

$$s^2 = [(R - S)(R - S)] + [(S - Q)(S - Q)] - 2(R - S) \bullet (S - Q)$$

$$s^2 = q^2 + r^2 - 2qr \cos S$$

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Distance from a Point to a Plane

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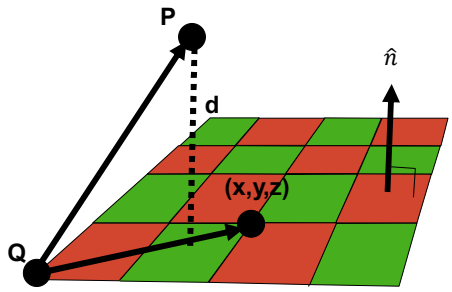

In high school, you defined a plane by:
Ax + By + Cz + D = 0

It is more useful to define it by a point on the plane (Q) combined with the plane's normal vector (n_x, n_y, n_z)

If you want the familiar equation of the plane, it is:

$$\left((x,y,z) - (Q_x, Q_y, Q_z)\right) \bullet (n_x, n_y, n_z) = 0$$

which expands out to become the more familiar Ax + By + Cz + D = 0

The perpendicular distance from the point P to the plane is based on the plane equation:

$$d = (P - Q) \bullet \hat{n}$$

The dot product is answering the question "How much of (P-Q) is in the $\hat{n}$ direction?". Note that this gives a *signed distance*. If d > 0., then P is on the same side of the plane as the normal points. This is *very* useful.

Where does a line segment intersect an infinite plane?

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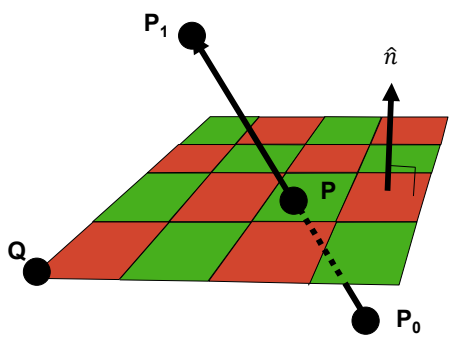

The equation of the line segment is:

$$P = (1 - t)P_0 + tP_1$$

If point P is in the plane, then:

$$\left((P_x, P_y, P_z) - (Q_x, Q_y, Q_z)\right) \bullet (n_x, n_y, n_z) = 0$$

If we substitute the parametric expression for P into the plane equation, then the only thing we don't know in that equation is t. Solve it for t*. Knowing t* will let us compute the (x,y,z) of the actual intersection using the line equation. If t* has a zero in the denominator, then that tells us that t*=∞, and the line must be parallel to the plane.

This gives us the point of intersection with the infinite plane. We could now use the method covered a few slides ago to see if P lies inside a particular triangle.

Minimal Distance Between Two 3D Lines

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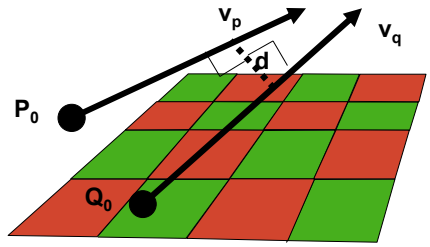

The equation of the lines are : $P = P_0 + t \cdot v_p$ $\quad Q = Q_0 + t \cdot v_q$

The minimal distance vector between the two lines must be perpendicular to both

A vector between them that is perpendicular to both is: $v_\perp = v_p \times v_q$

We need to answer the question "How much of (Q_0-P_0) is in the $v_\perp$direction?".
To do this, we once again use the dot product:

$$d = (P_0 - Q_0) \bullet \hat{v}_\perp$$

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Another use for Dot Products : Force One Vector to be Perpendicular to Another Vector

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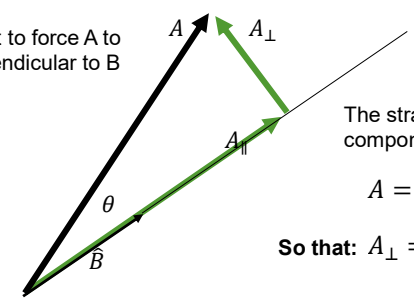
Here, we want to force A to become perpendicular to B



The strategy is to get rid of the parallel component, leaving just the perpendicular

$$A = A_\parallel + A_\perp$$

So that: $A_\perp = A - A_\parallel$

But, $A_\parallel = (A \bullet \hat{B})\hat{B}$

So that $A' = A_\perp = A - (A \bullet \hat{B})\hat{B}$

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This is known as ***Gram-Schmidt orthogonalization***

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