V BMU

Vectors

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Vectors have Direction and Magnitude

A Vector Can Also Be Defined as the Positional Difference Between Two Points

Unit Vectors have a Magnitude = 1.0

The circumflex (^) tells us this is a unit vector
The dot product of two vectors \(\mathbf{A} = (A_x, A_y, A_z)\) and \(\mathbf{B} = (B_x, B_y, B_z)\) is given by:

\[
\mathbf{A} \cdot \mathbf{B} = (A_x B_x + A_y B_y + A_z B_z) = \|\mathbf{A}\| \|\mathbf{B}\| \cos \theta
\]

Because it produces a scalar result (i.e., a single number), this is also called the **Scalar Product**.

**A Physical Interpretation of the Dot Product**

The dot product can be interpreted as the projection of \(\mathbf{A}\) onto \(\mathbf{B}\):

\[
\mathbf{A} \cdot \mathbf{\hat{B}} = \|\mathbf{A}\| \cos \theta
\]

This is how much of \(\mathbf{A}\) lives in the \(\mathbf{B}\) direction.

**A Physical Interpretation of the Dot Product**

The amount of force \(\mathbf{F}\) accelerating the car along the road is:

\[
F_{\text{road}} = F \cos \theta
\]

This is easy to see in 2D, but a 3D version of the same problem is trickier.
A Physical Interpretation of the Dot Product

\[ F_{\text{road}} = F \cos \theta = F \cdot \hat{R} \]

Dot Products are Commutative

\[ A \cdot B = B \cdot A \]

Dot Products are Distributive

\[ A \cdot (B + C) = (A \cdot B) + (A \cdot C) \]

The Perpendicular to a 2D Vector

If \( V = (x, y) \)

then \( V_{\perp} = (-y, x) \)

You can tell that this is true because

\[ V \cdot V_{\perp} = (x, y) \cdot (-y, x) = -xy + xy = 0 = \cos 90^\circ \]

Cross Product

\[ A = (A_x, A_y, A_z) \]
\[ B = (B_x, B_y, B_z) \]

\[ A \times B = (A_y B_z - A_z B_y, A_z B_x - A_x B_z, A_x B_y - A_y B_x) \]

\[ \|A \times B\| = \|A\| \|B\| \sin \theta \]

Because it produces a vector result (i.e., three numbers), this is also called the Vector Product
A Physical Interpretation of the Cross Product

\[ \| \mathbf{A} \times \mathbf{B} \| = \| \mathbf{A} \| \sin \theta \]

= How much of \( \mathbf{A} \) lives perpendicular to the \( \mathbf{B} \) direction

This is important — memorize this phrase!

The Perpendicular Property of the Cross Product

The vector \( \mathbf{A} \times \mathbf{B} \) is both perpendicular to \( \mathbf{A} \) and perpendicular to \( \mathbf{B} \)

The Right-Hand-Rule Property of the Cross Product

Curl the fingers of your right hand in the direction that starts at \( \mathbf{A} \) and heads towards \( \mathbf{B} \). Your thumb points in the direction of \( \mathbf{A} \times \mathbf{B} \).

Cross Products are Not Commutative

\[ \mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \]

A Use for the Cross Product: Finding a Vector Perpendicular to a Plane (= the Surface Normal)

\[ n = (\mathbf{R} - \mathbf{Q}) \times (\mathbf{S} - \mathbf{Q}) \]

Cross Products are Distributive

\[ \mathbf{A} \times (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C}) \]
A Use for the Cross Product:
Finding a Vector Perpendicular to a Plane (= the Surface Normal) –
This is used in CG Lighting

A Use for the Cross and Dot Products:
Is a Point Inside a Triangle? – 3D (X-Y-Z) Version

Let:
\[ n = (R - Q) \times (S - Q) \]
\[ n_q = (R - Q) \times (P - Q) \]
\[ n_r = (S - R) \times (P - R) \]
\[ n_s = (Q - S) \times (P - S) \]

\( (n \cdot n_q), (n \cdot n_r), \text{and} (n \cdot n_s) \)
are all positive, then P is inside the triangle QRS

Is a Point Inside a Triangle?
This can be simplified if you are in 2D (X-Y)

If \( E_{RS}, E_{SQ}, E_{QR} \) are all positive, then P is inside the triangle QRS

A Use for the Cross Product:
Finding the Area of a 3D Triangle

\[ \text{Area} = \frac{1}{2} \cdot \text{Base} \cdot \text{Height} \]
\[ \text{Base} = \|QR\| \]
\[ \text{Height} = \|QS\| \sin \theta \]

\[ \text{Area} = \frac{1}{2} \left( \|QR\| \cdot \|QS\| \cdot \sin \theta = \frac{1}{2} \| (R - Q) \times (S - Q) \| \right) \]
Derivation of the Law of Cosines

\[ s = R - Q \]
\[ s^2 = \|R - Q\|^2 \]
\[ s^2 = (R - Q) \cdot (R - Q) \]
\[ s^2 = [(R - S) + (S - Q)] \cdot [(R - S) + (S - Q)] \]
\[ s^2 = [(R - S)(R - S)] + [(S - Q)(S - Q)] - 2(R - S) \cdot (S - Q) \]
\[ s^2 = q^2 + r^2 - 2qr \cos \theta \]

Derivation of the Law of Sines

\[ 2 \cdot \text{Area}(\Delta QRS) = \|S - Q\| \times \|R - Q\| = rs \sin \theta \]

But, the area is the same regardless of which two sides we use to compute it, so:

\[ rs \sin \theta = qs \sin \alpha = qr \sin \gamma \]

Dividing by \( qrs \) gives:

\[ \frac{\sin \theta}{q} = \frac{\sin \alpha}{r} = \frac{\sin \gamma}{s} \]

Distance from a Point to a Plane

In high school, you defined a plane by:

\[ Ax + By + Cz + D = 0 \]

It is more useful to define it by a point on the plane combined with the plane’s normal vector.

If you want the familiar equation of the plane, it is:

\[ (x, y, z) - (Q_x, Q_y, Q_z) \cdot (n_x, n_y, n_z) = 0 \]

which expands out to become the more familiar \( Ax + By + Cz + D = 0 \)

The perpendicular distance from the point \( P \) to the plane is based on the plane equation:

\[ d = (P - Q) \cdot \hat{n} \]

The dot product is answering the question “How much of \( P - Q \) is in the \( \hat{n} \) direction?”.

Note that this gives a signed distance. If \( d > 0 \), then \( P \) is on the same side of the plane as the normal points. This is very useful.

Where does a line segment intersect an infinite plane?

The equation of the line segment is:

\[ P = (1 - t)P_0 + tP_1 \]

If point \( P \) is in the plane, then:

\[ (P_x, P_y, P_z) - (Q_x, Q_y, Q_z) \cdot (n_x, n_y, n_z) = 0 \]

If we substitute the parametric expression for \( P \) into the plane equation, then the only thing we don’t know in that equation is \( t \). Solve it for \( t \). Knowing \( t \) will let us compute the \((x, y, z)\) of the actual intersection using the line equation. If \( t \) has a zero in the denominator, then that tells us that \( t=\text{ INF} \) and the line must be parallel to the plane.

This gives us the point of intersection with the infinite plane. We could now use the method covered a few slides ago to see if \( P \) lies inside a particular triangle.
Minimal Distance Between Two 3D Lines

The equation of the lines are: \( P = P_0 + t \cdot v_p \), \( Q = Q_0 + t \cdot v_q \).

The minimal distance vector between the two lines must be perpendicular to both.

A vector between them that is perpendicular to both is: \( v_{\perp} = v_p \times v_q \).

We need to answer the question “How much of \( (Q_0 - P_0) \) is in the \( v_{\perp} \) direction?”.

To do this, we once again use the dot product:

\[
d = \left( P_0 - Q_0 \right) \cdot \hat{v}_{\perp}
\]

Another use for Dot Products:
Force One Vector to be Perpendicular to Another Vector

Here, we want to force \( A \) to become perpendicular to \( B \).

The strategy is to get rid of the parallel component, leaving just the perpendicular:

\[
A = A_\parallel + A_{\perp}
\]

\[
A_{\perp} = A - A_\parallel
\]

But, \( A_{\perp} = (A \cdot \hat{B})\hat{B} \)

So that

\[
A' = A_{\perp} = A - (A \cdot \hat{B})\hat{B}
\]

This is known as Gram-Schmidt orthogonalization.