Vectors

A Vector Can Also Be Defined as the Positional Difference Between Two Points

\[(V_x, V_y, V_z) = (Q_x - P_x, Q_y - P_y, Q_z - P_z)\]

Unit Vectors have a Magnitude = 1.0

\[\hat{V} = \frac{\mathbf{V}}{\|\mathbf{V}\|} = \frac{V_x}{\sqrt{V_x^2 + V_y^2 + V_z^2}}\]

The circumflex (^) tells us this is a unit vector

Dot Product

\[\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z = \|\mathbf{A}\| \|\mathbf{B}\| \cos \theta\]

Because it produces a scalar result (i.e., a single number), this is also called the Scalar Product

A Physical Interpretation of the Dot Product

\[\mathbf{A} \cdot \hat{\mathbf{B}} = \|\mathbf{A}\| \cos \theta = \text{How much of } \mathbf{A} \text{ lives in the } \mathbf{B} \text{ direction}\]
A Physical Interpretation of the Dot Product

The amount of the force accelerating the car along the road is “how much of F is in the horizontal direction?”

\[ F_{\text{road}} = F \cos \theta \]

This is easy to see in 2D, but a 3D version of the same problem is trickier.

The amount of the force accelerating the car along the road is “how much of F is in the R direction?”

\[ F_{\text{road}} = F \cos \theta = F \hat{R} \]

Dot Products are Commutative

\[ A \cdot B = B \cdot A \]

Dot Products are Distributive

\[ A \cdot (B + C) = (A \cdot B) + (A \cdot C) \]

The Perpendicular to a 2D Vector

\[ V = (x, y) \]
then \[ V_\perp = (-y, x) \]

You can tell that this is true because

\[ V \cdot V_\perp = (x, y) \cdot (-y, x) = -xy + xy = 0 = \cos 90^\circ \]

Cross Product

\[ A = (A_x, A_y, A_z) \]
\[ B = (B_x, B_y, B_z) \]

\[ A \times B = (A_y B_z - A_z B_y, A_z B_x - A_x B_z, A_x B_y - A_y B_x) \]

\[ |A \times B| = |A||B|\sin \theta \]

Because it produces a vector result (i.e., three numbers), this is also called the Vector Product.
A Physical Interpretation of the Cross Product

\[ \| \mathbf{A} \times \mathbf{B} \| = \| \mathbf{A} \| \sin \theta \]

This is important — memorize this phrase!

= How much of \( \mathbf{A} \) lives perpendicular to the \( \mathbf{B} \) direction

The Perpendicular Property of the Cross Product

\[ \mathbf{A} \times \mathbf{B} \]

The vector is both perpendicular to \( \mathbf{A} \) and perpendicular to \( \mathbf{B} \)

The Right-Hand-Rule Property of the Cross Product

Curl the fingers of your right hand in the direction that starts at \( \mathbf{A} \) and heads towards \( \mathbf{B} \). Your thumb points in the direction of \( \mathbf{A} \times \mathbf{B} \).

Cross Products are Not Commutative

\[ \mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \]

Cross Products are Distributive

\[ \mathbf{A} \times (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C}) \]

A Use for the Cross Product:

Finding a Vector Perpendicular to a Plane (= the Surface Normal)

\[ \mathbf{n} = (\mathbf{R} - \mathbf{Q}) \times (\mathbf{S} - \mathbf{Q}) \]

A Use for the Cross Product:

Finding a Vector Perpendicular to a Plane (= the Surface Normal) – This is used in CG Lighting

\[ \mathbf{n} \cdot \mathbf{n}_1, (\mathbf{n} \cdot \mathbf{n}_2), \text{ and } (\mathbf{n} \cdot \mathbf{n}_3) \]

are all positive, then \( \mathbf{P} \) is inside the triangle \( \mathbf{QRS} \)
Is a Point Inside a Triangle?
This can be simplified if you are in 2D (X-Y).

If all are positive, then P is inside the triangle QRS.

A Use for the Cross Product:
Finding the Area of a 3D Triangle

Using the plane equation, it is:
which expands out to become the more familiar Ax + By + Cz + D = 0.

The distance from a point to a plane is:

The cross product is answering the question “How much of (P-Q) is in the \( \vec{n} \) direction?”

Note that this gives a signed distance. If \( d > 0 \), then P is on the same side of the plane \( \vec{n} \) as the normal points. This is very useful.

Derivation of the Law of Sines

Dividing by \( qrs \) gives:

Derivation of the Law of Cosines

The substitution of the parametric expression for P into the plane equation, then the only thing we don’t know in that equation is \( t \). Solve it for \( t^* \). Knowing \( t^* \) will let us compute the \( x,y,z \) of the actual intersection using the line equation. If \( t^* \) has a zero in the denominator, then that tells us that \( t=\infty \) and the line must be parallel to the plane.

This gives us the point of intersection with the infinite plane. We could now use the method covered a few slides ago to see if P lies inside a particular triangle.
The equation of the lines are: \( P = P_0 + t \cdot v_p \) \( Q = Q_0 + t \cdot v_q \).

The minimal distance vector between the two lines must be perpendicular to both.

A vector between them that is perpendicular to both is: \( v = v_p \times v_q \).

We need to answer the question “How much of \((Q_0 - P_0)\) is in the \( v \) direction?”.

To do this, we once again use the dot product:

\[ d = (P_0 - Q_0) \cdot v \]

Another use for Dot Products:

**Force One Vector to be Perpendicular to Another Vector**

Here, we want to force \( A \) to become perpendicular to \( B \).

The strategy is to get rid of the parallel component, leaving just the perpendicular:

\[ A = A + A_\perp \]

\[ A_\perp = A - A_i \]

So that \( A' = A_\perp = A - (A \bullet \hat{B})\hat{B} \)

This is known as **Gram-Schmidt orthogonalization**.