Vectors

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Vectors have Direction and Magnitude

Magnitude: \[ \| \mathbf{V} \| = \sqrt{V_x^2 + V_y^2 + V_z^2} \]

A Vector Can Also Be Defined as the Positional Difference Between Two Points

\[(\mathbf{x}_p, \mathbf{y}_p, \mathbf{z}_p) \rightarrow (\mathbf{x}_q, \mathbf{y}_q, \mathbf{z}_q) \]

Unit Vectors have a Magnitude = 1.0

\[
\hat{\mathbf{V}} = \frac{\mathbf{V}}{\| \mathbf{V} \|}
\]

The circumflex (^) tells us this is a unit vector

Dot Product

\[
\mathbf{A} \cdot \mathbf{B} = \mathbf{A}_x \mathbf{B}_x + \mathbf{A}_y \mathbf{B}_y + \mathbf{A}_z \mathbf{B}_z = \| \mathbf{A} \| \| \mathbf{B} \| \cos \theta
\]

A Physical Interpretation of the Dot Product

This is important – memorize this phrase!

Because it produces a scalar result (i.e., a single number), this is also called the Scalar Product

\[
\mathbf{A} \cdot \hat{\mathbf{B}} = \| \mathbf{A} \| \cos \theta
\]

= How much of A lives in the B direction
A Physical Interpretation of the Dot Product

The amount of the force accelerating the car along the road is “how much of F is in the horizontal direction?”

\[ F_{\text{road}} = F \cos \theta \]

This is easy to see in 2D, but a 3D version of the same problem is trickier.

A Physical Interpretation of the Dot Product

The amount of the force accelerating the car along the road is “how much of F is in the R direction?”

\[ F_{\text{road}} = F \cos \theta = F \cdot \hat{R} \]

Dot Products are Commutative

\[ A \cdot B = B \cdot A \]

Dot Products are Distributive

\[ A \cdot (B + C) = (A \cdot B) + (A \cdot C) \]

The Perpendicular to a 2D Vector

If \( \mathbf{V} = (x, y) \)

then \( \mathbf{V}_\perp = (-y, x) \)

You can tell that this is true because

\[ \mathbf{V} \cdot \mathbf{V}_\perp = (x, y) \cdot (-y, x) = -xy + xy = 0 = \cos 90^\circ \]

Cross Product

\[ A = (A_x, A_y, A_z) \]
\[ B = (B_x, B_y, B_z) \]

\[ A \times B = (A_y B_z - A_z B_y, A_z B_x - A_x B_z, A_x B_y - A_y B_x) \]

\[ \|A \times B\| = \|A\| \|B\| \sin \theta \]

Because it produces a vector result (i.e., three numbers), this is also called the Vector Product.
A Physical Interpretation of the Cross Product

\[ \|A \times \hat{B}\| = \|A\| \sin \theta \]

This is important — memorize this phrase!

= How much of A lives perpendicular to the \( \hat{B} \) direction

The Perpendicular Property of the Cross Product

The vector \( A \times \hat{B} \) is both perpendicular to A and perpendicular to B

The Right-Hand-Rule Property of the Cross Product

Curl the fingers of your right hand in the direction that starts at A and heads towards B. Your thumb points in the direction of \( A \times \hat{B} \).

Cross Products are Not Commutative

\[ A \times B = -B \times A \]

Cross Products are Distributive

\[ A \times (B + C) = (A \times B) + (A \times C) \]

A Use for the Cross Product:
Finding a Vector Perpendicular to a Plane (= the Surface Normal)

\[ n = (R - Q) \times (S - Q) \]

A Use for the Cross and Dot Products:
Is a Point Inside a Triangle? – 3D (X-Y-Z) Version

Let:

\[ n = (R - Q) \times (S - Q) \]
\[ n_y = (R - Q) \times (P - Q) \]
\[ n_z = (S - R) \times (P - R) \]
\[ n_x = (Q - S) \times (P - S) \]

If \( (n \cdot n_y), (n \cdot n_z), \) and \( (n \cdot n_x) \) are all positive, then P is inside the triangleQRS.
Is a Point Inside a Triangle?
This can be simplified if you are in 2D (X-Y)

If \( P \) is inside the triangle \( QRS \)

\( E_{RS} = (P - R) \cdot (RS) \)
where \( RS = (R_x - R_x, R_y - R_y) \)
and \( (RS)_y = (R_x - S_x, S_y - R_y) \)

Similarly, \( E_{SQ} = (P - S) \cdot (SQ) \)
\( E_{QR} = (P - Q) \cdot (QR) \)

If \( E_{RS}, E_{SQ}, E_{QR} \) are all positive, then \( P \) is inside the triangle \( QRS \)

A Use for the Cross Product:
Finding the Area of a 3D Triangle

Area of \( QRQS \) triangle:

\[
\text{Area} = \frac{1}{2} \|QR\| |QS| \sin \theta
\]

Derivation of the Law of Cosines

\[
s^2 = (R - Q)^2 = (R - Q)^2
\]

\[
s^2 = (R - S) + (S - Q) = [(R - S) + (S - Q)]
\]

\[
s^2 = [(R - S) + (S - Q)] = (R - Q) \cdot (R - Q)
\]

\[
s^2 = q^2 + r^2 - 2qr \cos S
\]

Derivation of the Law of Sines

\[
sqrs = rs \sin Q
\]

Dividing by \( qrs \) gives:

\[
\frac{|S - Q|}{|R - Q|} \cdot \sin \theta = \frac{1}{2} \|R - Q\| \times (S - Q)
\]

Distance from a Point to a Plane

If you want the familiar equation of the plane, it is:

\[
Ax + By + Cz + D = 0
\]

It is more useful to define it by a point on the plane combined with the plane's normal vector.

The equation of the line segment is:

\[
P_{01}(1 - t) + P_1 t
\]

Where does a line segment intersect an infinite plane?

If \( P \) is in the plane, then:

\[
(P - P_0) \cdot \hat{n} = 0
\]

If we substitute the parametric expression for \( P \) into the plane equation, then the only thing we don't know in that equation is \( t \). Solve it for \( t \). Knowing \( t \) will let us compute the \( x,y,z \) of the actual intersection using the line equation. If \( t \) has a zero in the denominator, then that tells us that \( t = -\infty \) and the line must be parallel to the plane. This gives us the point of intersection with the infinite plane. We could now use the method covered a few slides ago to see if \( P \) lies inside a particular triangle.
The equation of the lines are: \( P = P_0 + t \cdot v_p \) \( Q = Q_0 + t \cdot v_q \).

The minimal distance vector between the two lines must be perpendicular to both:

\[ \mathbf{v}_\perp = \mathbf{v}_p \times \mathbf{v}_q \]

We need to answer the question “How much of \((Q_0-P_0)\) is in the \(\mathbf{v}_\perp\) direction?”

To do this, we once again use the dot product:

\[ d = (P_0 - Q_0) \cdot \mathbf{v}_\perp \]

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Another use for Dot Products: Force One Vector to be Perpendicular to Another Vector

Here, we want to force \( A \) to become perpendicular to \( B \).

The strategy is to get rid of the parallel component, leaving just the perpendicular:

\[ A = A + A_\perp \]
\[ A_\perp = A - A \]

But,

\[ A_\perp = (A \cdot \hat{B})\hat{B} \]

So that

\[ A' = A_\perp = A - (A \cdot \hat{B})\hat{B} \]

This is known as Gram-Schmidt orthogonalization.