Transformations

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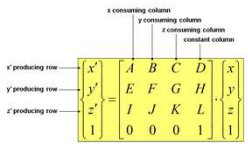
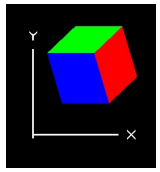
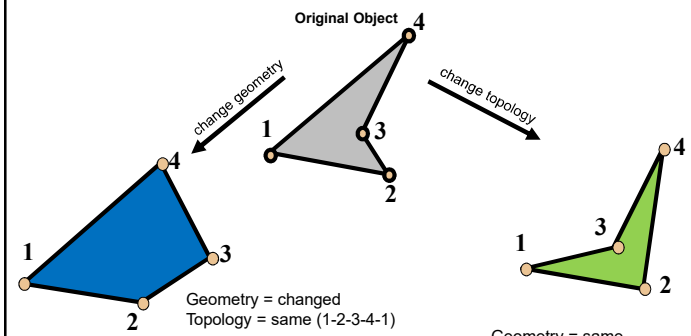
Mike Bailey

mjb@cs.oregonstate.edu





Geometry vs. Topology



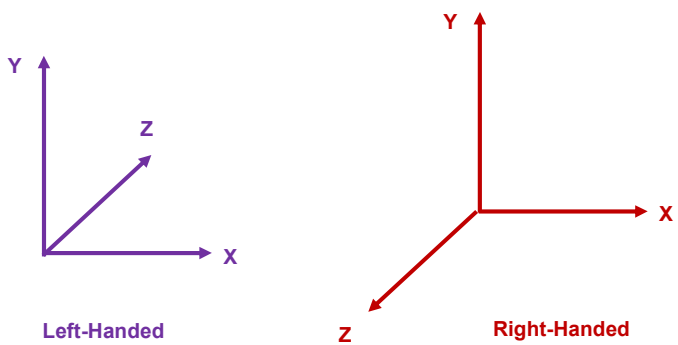
Geometry:

Where things are (e.g., coordinates)

Topology:

How things are connected

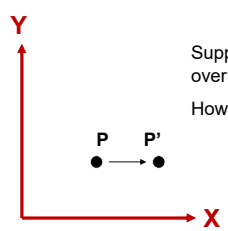
3D Coordinate Systems



Left-Handed

Right-Handed

Transformations



Suppose you have a point P and you want to move it over by 2 units in X.

How would you change P's coordinates?

$$P' = (P'_x, P'_y) = (P_x + 2., P_y)$$

This is known as a coordinate *transformation*

General Form of 3D Linear Transformations

5

$$x' = Ax + By + Cz + D$$
$$y' = Ex + Fy + Gz + H$$
$$z' = Ix + Jy + Kz + L$$

It's called a "Linear Transformation" because all of the coordinates are raised to the 1st power, that is, there are no x^2, x^3, etc. terms.

Transform the geometry – leave the topology as is

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Translation Equations

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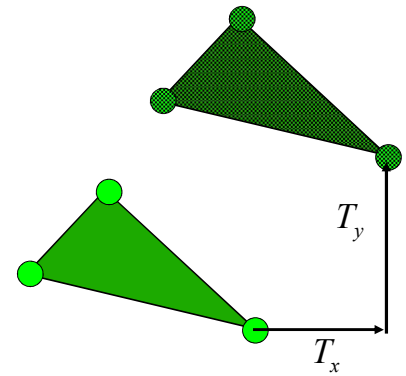
$$x' = x + T_x$$
$$y' = y + T_y$$
$$z' = z + T_z$$



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Scaling About the Origin

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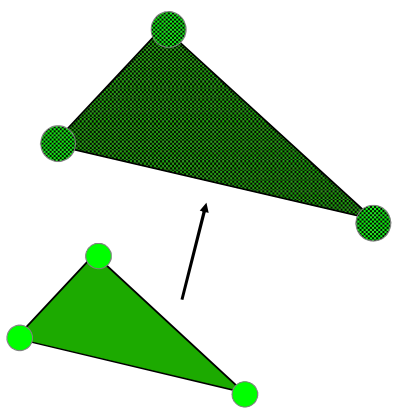
$$x' = x \cdot S_x$$
$$y' = y \cdot S_y$$
$$z' = z \cdot S_z$$



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2D Rotation About the Origin

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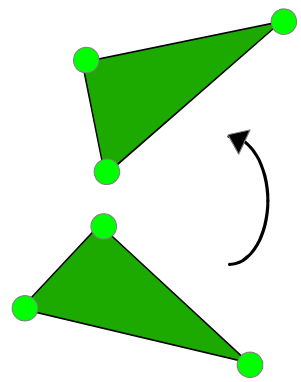
$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$



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Linear Equations in Matrix Form

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$$x' = Ax + By + Cz + D$$
$$y' = Ex + Fy + Gz + H$$
$$z' = Ix + Jy + Kz + L$$

x consuming column
y consuming column
z consuming column
constant column

x' producing row
y' producing row
z' producing row

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

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Identity Matrix ([I])

10

$$x' = Ax + By + Cz + D$$
$$y' = Ex + Fy + Gz + H$$
$$z' = Ix + Jy + Kz + L$$

x consuming column
y consuming column
z consuming column
constant column

x' producing row
y' producing row
z' producing row

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

$$x' = x$$
$$y' = y$$
$$z' = z$$

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

[I] signifies that "Nothing has changed"

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Matrix Inverse

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$$[M] \bullet [M]^{-1} = [I]$$

$$[M] \bullet [M]^{-1} = \text{"Nothing has changed"}$$

"Whatever [M] does, $[M]^{-1}$ undoes"

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Translation Matrix

12

$$x' = Ax + By + Cz + D$$
$$y' = Ex + Fy + Gz + H$$
$$z' = Ix + Jy + Kz + L$$

x consuming column
y consuming column
z consuming column
constant column

x' producing row
y' producing row
z' producing row

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

Quick! What is the inverse of this matrix?

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Scaling Matrix

13

$$x' = Ax + By + Cz + D$$
$$y' = Ex + Fy + Gz + H$$
$$z' = Ix + Jy + Kz + L$$

x consuming column, y consuming column, z consuming column, constant column

x' producing row, y' producing row, z' producing row

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

Quick! What is the inverse of this matrix?

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3D Rotation Matrix About Z

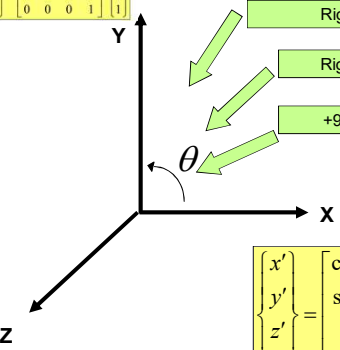
14

$$x' = Ax + By + Cz + D$$
$$y' = Ex + Fy + Gz + H$$
$$z' = Ix + Jy + Kz + L$$

x consuming column, y consuming column, z consuming column, constant column

x' producing row, y' producing row, z' producing row

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$



Right-handed coordinates

Right-handed positive rotation rule

+90º rotation gives: y' = x

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

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3D Rotation Matrix About Y

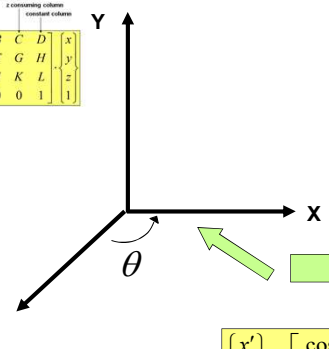
15

$$x' = Ax + By + Cz + D$$
$$y' = Ex + Fy + Gz + H$$
$$z' = Ix + Jy + Kz + L$$

x consuming column, y consuming column, z consuming column, constant column

x' producing row, y' producing row, z' producing row

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$



+90º rotation gives: x' = z

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

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3D Rotation Matrix About X

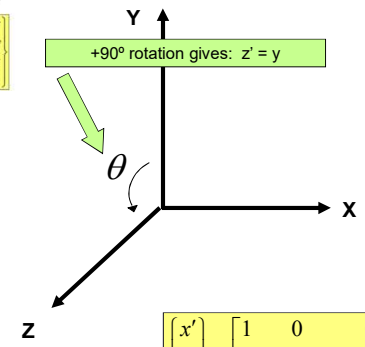
16

$$x' = Ax + By + Cz + D$$
$$y' = Ex + Fy + Gz + H$$
$$z' = Ix + Jy + Kz + L$$

x consuming column, y consuming column, z consuming column, constant column

x' producing row, y' producing row, z' producing row

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$



+90º rotation gives: z' = y

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$

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How it Really Works :-)

$$\begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90^\circ & \cos 90^\circ \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_2 & a_1 \end{bmatrix}$$

http://xkcd.com

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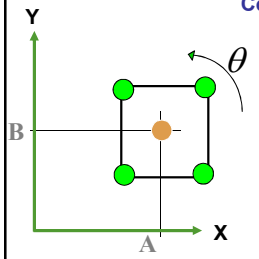
Compound Transformations



Q: Our rotation matrices only work around the origin. What if we want to rotate about an arbitrary point (A,B)?

A: Use more than one matrix.

Write it →

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \left(\overset{\boxed{3}}{\left[T_{+A,+B} \right]} \cdot \left(\overset{\boxed{2}}{\left[R_\theta \right]} \cdot \left(\overset{\boxed{1}}{\left[T_{-A,-B} \right]} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \right) \right) \right)$$

← Say it

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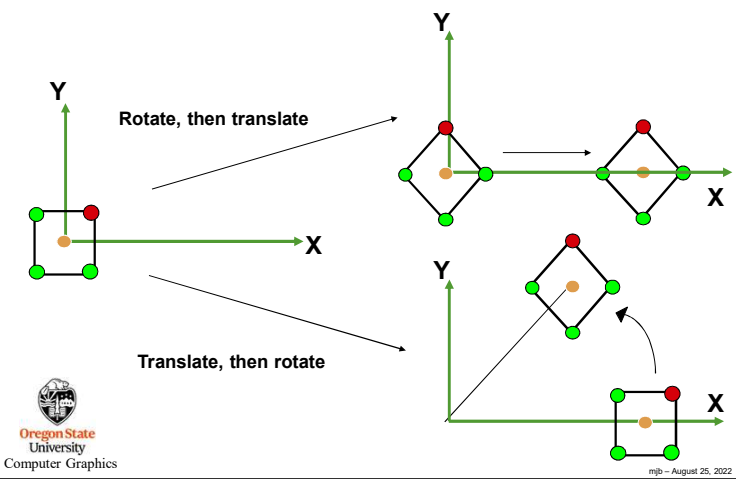
Matrix Multiplication *is not* Commutative



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Matrix Multiplication *is* Associative

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \left(\left[T_{+A,+B} \right] \cdot \left(\left[R_\theta \right] \cdot \left(\left[T_{-A,-B} \right] \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \right) \right) \right)$$

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \underbrace{\left(\left[T_{+A,+B} \right] \cdot \left[R_\theta \right] \cdot \left[T_{-A,-B} \right] \right)}_{\text{One matrix – the Current Transformation Matrix, or CTM}} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

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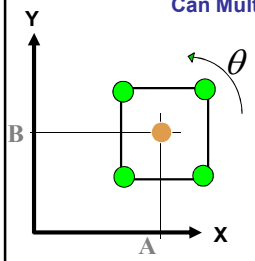
Can Multiply All Geometry by One Matrix !

21



$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = [M] \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Graphics hardware can do this very quickly!

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OpenGL Will Do the Transformation Compounding for You !

22

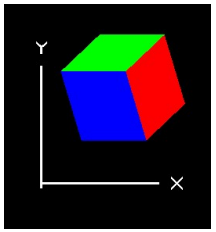
```
glTranslatef( A, B, C );

glRotatef(  (GLfloat)Yrot, 0., 1., 0. );
glRotatef(  (GLfloat)Xrot, 1., 0., 0. );

glScalef(  (GLfloat)Scale, (GLfloat)Scale, (GLfloat)Scale  );

glCallList( BoxList );
```

Typically objects are modeled around their own local origin, so the
glTranslate(-A, -B, -C)
step is unnecessary.




```
glTranslatef( A, B, C );
glRotatef(  (GLfloat)Yrot, 0., 1., 0. );
glRotatef(  (GLfloat)Xrot, 1., 0., 0. );
glScalef(  (GLfloat)Scale, (GLfloat)Scale, (GLfloat)Scale  );
glCallList( BoxList );
```

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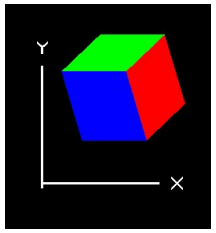
OpenGL Will Do the Transformation Compounding for You !

23

```
for( ; ; )
{
        << Turn mouse position into Xrot and Yrot >>

        glLoadIdentity( );

        glTranslatef( A, B, C );

        glRotatef(  (GLfloat)Yrot, 0., 1., 0. );
        glRotatef(  (GLfloat)Xrot, 1., 0., 0. );

        glScalef(  (GLfloat)Scale, (GLfloat)Scale, (GLfloat)Scale  );

        glCallList( BoxList );
}
```



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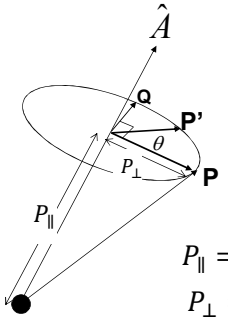
The Funky Rotation Matrix for an Arbitrary Axis and Angle

24



$$P_{\parallel} = \hat{A}(\hat{A} \bullet P)$$

$$P_{\perp} = P - P_{\parallel} = P - \hat{A}(\hat{A} \bullet P)$$

$$Q = \hat{A} \times P_{\perp} = \hat{A} \times P_{\perp} + 0 = \hat{A} \times P_{\perp} + \hat{A} \times P_{\parallel} = \hat{A} \times (P_{\perp} + P_{\parallel}) = \hat{A} \times P$$

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The Funky Rotation Matrix for an Arbitrary Axis and Angle

$$P_{\parallel}{}' = P_{\parallel} \qquad P_{\perp}{}' = P_{\perp}\cos\theta + Q\sin\theta$$

$$P' = P'_{\parallel} + P'_{\perp}$$

$$P' = \left[\hat{A}(\hat{A}\bullet P)\right] + \cos\theta\left[P - \hat{A}(\hat{A}\bullet P)\right] + \sin\theta\left[\hat{A}\times P\right]$$

The Funky Rotation Matrix for an Arbitrary Axis and Angle

$$\hat{A}(\hat{A}\bullet P) = \begin{bmatrix} A_xA_x & A_xA_y & A_xA_z \\ A_yA_x & A_yA_y & A_yA_z \\ A_zA_x & A_zA_y & A_zA_z \end{bmatrix}\{P\}$$

$$\hat{A}\times P = \begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix}\{P\}$$

The Funky Rotation Matrix for an Arbitrary Axis and Angle

$$P' = \left[\hat{A}(\hat{A}\bullet P)\right] + \cos\theta\left[P - \hat{A}(\hat{A}\bullet P)\right] + \sin\theta\left[\hat{A}\times P\right]$$

$$P' = \left(\begin{bmatrix} A_xA_x & A_xA_y & A_xA_z \\ A_yA_x & A_yA_y & A_yA_z \\ A_zA_x & A_zA_y & A_zA_z \end{bmatrix} + \cos\theta\left[[I] - \begin{bmatrix} A_xA_x & A_xA_y & A_xA_z \\ A_yA_x & A_yA_y & A_yA_z \\ A_zA_x & A_zA_y & A_zA_z \end{bmatrix}\right] + \sin\theta\begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix}\right)\{P\}$$

$$[M] = \left(\begin{bmatrix} A_xA_x & A_xA_y & A_xA_z \\ A_yA_x & A_yA_y & A_yA_z \\ A_zA_x & A_zA_y & A_zA_z \end{bmatrix} + \cos\theta\begin{bmatrix} 1-A_xA_x & -A_xA_y & -A_xA_z \\ -A_yA_x & 1-A_yA_y & -A_yA_z \\ -A_zA_x & -A_zA_y & 1-A_zA_z \end{bmatrix} + \sin\theta\begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix}\right)$$

The Funky Rotation Matrix for an Arbitrary Axis and Angle

$$[M] = \left(\begin{bmatrix} A_xA_x & A_xA_y & A_xA_z \\ A_yA_x & A_yA_y & A_yA_z \\ A_zA_x & A_zA_y & A_zA_z \end{bmatrix} + \cos\theta\begin{bmatrix} 1-A_xA_x & -A_xA_y & -A_xA_z \\ -A_yA_x & 1-A_yA_y & -A_yA_z \\ -A_zA_x & -A_zA_y & 1-A_zA_z \end{bmatrix} + \sin\theta\begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix}\right)$$

$$[M] = \begin{bmatrix} A_xA_x + \cos\theta(1-A_xA_x) & A_xA_y - \cos\theta(A_xA_y) - \sin\theta A_z & A_xA_z - \cos\theta(A_xA_z) + \sin\theta A_y \\ A_yA_x - \cos\theta(A_yA_x) + \sin\theta A_z & A_yA_y + \cos\theta(1-A_yA_y) & A_yA_z - \cos\theta(A_yA_z) - \sin\theta A_x \\ A_zA_x - \cos\theta(A_zA_x) - \sin\theta A_y & A_zA_y - \cos\theta(A_zA_y) + \sin\theta A_x & A_zA_z + \cos\theta(1-A_zA_z) \end{bmatrix}$$

For this to work, A must be a unit vector