Vectors

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Vectors have Direction and Magnitude

Magnitude: \[ \|V\| = \sqrt{V_x^2 + V_y^2 + V_z^2} \]
A Vector Can Also Be Defined as the Positional Difference Between Two Points

\[
\begin{align*}
(V_x, V_y, V_z) &= (Q_x - P_x, Q_y - P_y, Q_z - P_z)
\end{align*}
\]
Unit Vectors have a Magnitude = 1.0

\[ \left( V_x, V_y, V_z \right) \]

\[ \hat{V} = \frac{V}{\|V\|} \]

The circumflex (^) tells us this is a unit vector

\[ \|V\| = \sqrt{V_x^2 + V_y^2 + V_z^2} \]
Dot Product

\[ A = (A_x, A_y, A_z) \]
\[ B = (B_x, B_y, B_z) \]

\[ A \cdot B = (A_x B_x + A_y B_y + A_z B_z) = \|A\| \|B\| \cos \theta \]

Because it produces a scalar result (i.e., a single number), this is also called the Scalar Product
The amount of the force accelerating the car along the road is “how much of F is in the horizontal direction?”

\[ F_{road} = F \cos \theta \]

This is easy to see in 2D, but a 3D version of the same problem is trickier.
The amount of the force accelerating the car along the road is “how much of $F$ is in the $R$ direction?”

$$F_{road} = F \cos \theta = F \cdot \hat{R}$$
A Physical Interpretation of the Dot Product

\[ F_{\text{road}} = F \cos \theta = F \cdot \hat{R} \]
Generalizing How Much of A Lives in the B Direction

\[ A \cdot B = \|A\| \|B\| \cos \theta \]

\[ A \cdot \hat{B} = \|A\| \cos \theta \]

which is the length of the projection of A onto the B line

So, how much of A lives in the B direction is that magnitude times the B unit vector:

\[ \hat{B} (A \cdot \hat{B}) \]
From the previous slide, how much of $A$ lives in the $\hat{B}$ direction is:

$$\hat{B} (A \cdot \hat{B})$$

That, plus the perpendicular vector equals $A$, so that how much of $A$ is perpendicular to the $\hat{B}$ direction is:

$$A - \hat{B} (A \cdot \hat{B})$$
Dot Products are Commutative

\[ A \cdot B = B \cdot A \]

Dot Products are Distributive

\[ A \cdot (B + C) = (A \cdot B) + (A \cdot C) \]
The Perpendicular to a 2D Vector

If \( V = (x, y) \)

then \( V_\perp = (-y, x) \)

You can tell that this is true because
\[
V \cdot V_\perp = (x, y) \cdot (-y, x) = -xy + xy = 0 = \cos 90^\circ
\]
Cross Product

\[ A = (A_x, A_y, A_z) \]

\[ B = (B_x, B_y, B_z) \]

\[ A \times B = (A_y B_z - A_z B_y, A_z B_x - A_x B_z, A_x B_y - A_y B_x) \]

\[ \|A \times B\| = \|A\| \|B\| \sin \theta \]

Because it produces a vector result (i.e., three numbers), this is also called the Vector Product.
The Perpendicular Property of the Cross Product

The vector $A \times B$ is both perpendicular to A and perpendicular to B.

The Right-Hand-Rule Property of the Cross Product

Curl the fingers of your right hand in the direction that starts at A and heads towards B. Your thumb points in the direction of $A \times B$. 
Cross Products are Not Commutative

\[ A \times B = -B \times A \]

Cross Products are Distributive

\[ A \times (B + C) = (A \times B) + (A \times C) \]
A Use for the Cross Product:
Finding a Vector Perpendicular to a Plane (= the Surface Normal)

\[ n = (R - Q) \times (S - Q) \]
A Use for the Cross Product:
Finding a Vector Perpendicular to a Plane (= the Surface Normal) –
This is used in CG Lighting
A Use for the Cross and Dot Products: Is a Point Inside a Triangle? – 3D (X-Y-Z) Version

Let:

\[ n = (R - Q) \times (S - Q) \]

\[ n_q = (R - Q) \times (P - Q) \]

\[ n_r = (S - R) \times (P - R) \]

\[ n_s = (Q - S) \times (P - S) \]

If \((n \cdot n_q), (n \cdot n_r), \text{ and } (n \cdot n_s)\)

are all positive, then \(P\) is inside the triangle QRS
Is a Point Inside a Triangle?
This can be simplified if you are in 2D (X-Y)

If \( E_{RS}, E_{SQ}, E_{QR} \) are all positive, then \( P \) is inside the triangle QRS.

\[
E_{RS} = (P - R) \cdot (RS)_{\perp}
\]

where:
\[
RS = (S_x - R_x, S_y - R_y)
\]

and:
\[
(RS)_{\perp} = (R_y - S_y, S_x - R_x)
\]

Similarly,
\[
E_{SQ} = (P - S) \cdot (SQ)_{\perp}
\]
\[
E_{QR} = (P - Q) \cdot (QR)_{\perp}
\]
A Use for the Cross Product: Finding the Area of a 3D Triangle

\[ \text{Area} = \frac{1}{2} \cdot \text{Base} \cdot \text{Height} \]

\[ \text{Base} = \|QR\| \]

\[ \text{Height} = \|QS\| \sin \theta \]

\[ \text{Area} = \frac{1}{2} \cdot \|QR\| \cdot \|QS\| \cdot \sin \theta = \frac{1}{2} \cdot \|(R - Q) \times (S - Q)\| \]
Derivation of the Law of Cosines

\[ s = R - Q \]
\[ s^2 = \|R - Q\|^2 \]
\[ s^2 = (R - Q) \cdot (R - Q) \]

\[ s^2 = [(R - S) + (S - Q)] \cdot [(R - S) + (S - Q)] \]
\[ s^2 = [(R - S)(R - S)] + [(S - Q)(S - Q)] - 2(R - S) \cdot (S - Q) \]
\[ s^2 = q^2 + r^2 - 2qr \cos S \]
Derivation of the Law of Sines

\[ 2 \cdot \text{Area}(\Delta QRS) = \| (S - Q) \times (R - Q) \| \]
\[ = rs \sin Q \]

But, the area is the same regardless of which two sides we use to compute it, so:

\[ rs \sin Q = qs \sin R = qr \sin S \]

Dividing by \((qrs)\) gives:

\[ \frac{\sin Q}{q} = \frac{\sin R}{r} = \frac{\sin S}{s} \]
Distance from a Point to a Plane

In high school, you defined a plane by:
Ax + By + Cz + D = 0

It is more useful to define it by a point on the plane combined with the plane’s normal vector.

If you want the familiar equation of the plane, it is:

$$\left( (x, y, z) - (Q_x, Q_y, Q_z) \right) \cdot (n_x, n_y, n_z) = 0$$

which expands out to become the more familiar Ax + By + Cz + D = 0

The perpendicular distance from the point P to the plane is based on the plane equation:

$$d = (P - Q) \cdot \hat{n}$$

The dot product is answering the question “How much of (P-Q) is in the \( \hat{n} \) direction?”. Note that this gives a signed distance. If \( d > 0 \), then P is on the same side of the plane as the normal points. This is very useful.
Where does a line segment intersect an infinite plane?

If point \( P \) is in the plane, then:
\[
\left( (P_x, P_y, P_z) - (Q_x, Q_y, Q_z) \right) \cdot (n_x, n_y, n_z) = 0
\]

If we substitute the parametric expression for \( P \) into the plane equation, then the only thing we don’t know in that equation is \( t \). Solve it for \( t^* \). Knowing \( t^* \) will let us compute the \((x,y,z)\) of the actual intersection using the line equation. If \( t^* \) has a zero in the denominator, then that tells us that \( t^* = \infty \), and the line must be parallel to the plane.

This gives us the point of intersection with the infinite plane. We could now use the method covered a few slides ago to see if \( P \) lies inside a particular triangle.
Minimal Distance Between Two 3D Lines

The equation of the lines are: \( P = P_0 + t \cdot v_p \quad Q = Q_0 + t \cdot v_q \)

The minimal distance vector between the two lines must be perpendicular to both

A vector between them that is perpendicular to both is: \( \mathbf{v}_\perp = v_p \times v_q \)

We need to answer the question “How much of \((Q_0-P_0)\) is in the \( \mathbf{v}_\perp \) direction?”. To do this, we once again use the dot product:

\[
d = (P_0 - Q_0) \cdot \hat{v}_\perp
\]
Another use for Dot Products: 
Force One Vector to be Perpendicular to Another Vector

Here, we want to force $A$ to become perpendicular to $B$

The strategy is to get rid of the parallel component, leaving just the perpendicular

\[ A = A_{\parallel} + A_{\perp} \]
\[ A_{\perp} = A - A_{\parallel} \]

But, \[ A_{\parallel} = (A \cdot \hat{B})\hat{B} \]

So that \[ A' = A_{\perp} = A - (A \cdot \hat{B})\hat{B} \]

This is known as **Gram-Schmidt orthogonalization**