Vectors

Vectors have Direction and Magnitude

\[ \mathbf{V} = \mathbf{Q} - \mathbf{P} = (Q_x - P_x, Q_y - P_y, Q_z - P_z) \]

Magnitude:

\[ \|\mathbf{V}\| = \sqrt{V_x^2 + V_y^2 + V_z^2} \]

A Vector Can Also Be Defined as the Positional Difference Between Two Points

Unit Vectors have a Magnitude = 1.0

The circumflex (\(^\wedge\)) tells us this is a unit vector
Dot Product

\[ A = (A_x, A_y, A_z) \]

\[ B = (B_x, B_y, B_z) \]

\[ A \cdot B = (A_xB_x + A_yB_y + A_zB_z) = \| A \| \| B \| \cos \theta \]

Because it produces a scalar result (i.e., a single number), this is also called the \textit{Scalar Product}.

A Physical Interpretation of the Dot Product

The amount of the force accelerating the car along the road is “how much of F is in the horizontal direction?”

\[ F_{\text{road}} = F \cos \theta = F \cdot \hat{R} \]

This is easy to see in 2D, but a 3D version of the same problem is trickier.
So, how much of \( A \) lives in the \( \hat{B} \) direction is that magnitude times the \( \hat{B} \) unit vector:

\[ \hat{B} (A \cdot \hat{B}) \]

which is the length of the projection of \( A \) onto the \( B \) line.

From the previous slide, how much of \( A \) lives in the \( \hat{B} \) direction is:

\[ \hat{B} (A \cdot \hat{B}) \]

That, plus the perpendicular vector equals \( A \), so that how much of \( A \) is perpendicular to the \( B \) direction is:

\[ A - \hat{B} (A \cdot \hat{B}) \]

Dot Products are Commutative

\[ A \cdot B = B \cdot A \]

Dot Products are Distributive

\[ A \cdot (B + C) = (A \cdot B) + (A \cdot C) \]

The Perpendicular to a 2D Vector

If \( V = (x, y) \)

then \( V_\perp = (y, x) \)

You can tell that this is true because

\[ V \cdot V_\perp = (x, y) \cdot (y, x) = -xy + xy = 0 = \cos 90^\circ \]
Cross Product

\[ A = (A_x, A_y, A_z) \]
\[ B = (B_x, B_y, B_z) \]
\[ A \times B = (A_yB_z - A_zB_y, A_zB_x - A_xB_z, A_xB_y - A_yB_x) \]
\[ \| A \times B \| = \| A \| \| B \| \sin \theta \]

Because it produces a vector result (i.e., three numbers), this is also called the Vector Product.

The Perpendicular Property of the Cross Product

The vector \( A \times B \) is both perpendicular to \( A \) and perpendicular to \( B \).

The Right-Hand-Rule Property of the Cross Product

Curl the fingers of your right hand in the direction that starts at \( A \) and heads towards \( B \). Your thumb points in the direction of \( A \times B \).

Cross Products are Not Commutative

\[ A \times B = -B \times A \]

Cross Products are Distributive

\[ A \times (B + C) = (A \times B) + (A \times C) \]

A Use for the Cross Product: Finding a Vector Perpendicular to a Plane (= the Surface Normal)

\[ n = (R - Q) \times (S - Q) \]
A Use for the Cross Product: Finding a Vector Perpendicular to a Plane (= the Surface Normal) – This is used in CG Lighting

A Use for the Cross and Dot Products: Is a Point Inside a Triangle? – 3D (X-Y-Z) Version

Let:

\[ n = (R - Q) \times (S - Q) \]
\[ n_q = (R - Q) \times (P - Q) \]
\[ n_r = (S - R) \times (P - R) \]
\[ n_s = (Q - S) \times (P - S) \]

are all positive, then P is inside the triangle QRS

Is a Point Inside a Triangle? This can be simplified if you are in 2D (X-Y)

Similarly,

\[ E_{QS} = (P - S) \bullet (SQ) \perp \]
\[ E_{QR} = (P - Q) \bullet (QR) \perp \]

If \( E_{RS}, E_{SQ}, E_{QR} \) are all positive, then P is inside the triangle QRS

A Use for the Cross Product: Finding the Area of a 3D Triangle

\[ \text{Area} = \frac{1}{2} \cdot \text{Base} \cdot \text{Height} \]
\[ \text{Base} = \|QR\| \]
\[ \text{Height} = \|QS\| \sin \theta \]

\[ \text{Area} = \frac{1}{2} \|QR\| \cdot \|QS\| \cdot \sin \theta = \frac{1}{2} \| (R - Q) \times (S - Q) \| \]
Derivation of the Law of Cosines

\[ s = R - Q \]
\[ s^2 = \|R - Q\|^2 \]
\[ s^2 = (R - Q) \cdot (R - Q) \]
\[ s^2 = [(R - S) + (S - Q)] \cdot [(R - S) + (S - Q)] \]
\[ s^2 = [(R - S)(R - S)] + [(S - Q)(S - Q)] - 2(R - S) \cdot (S - Q) \]
\[ s^2 = q^2 + r^2 - 2qr \cos S \]

Derivation of the Law of Sines

\[ 2 \cdot \text{Area}(\Delta QRS) = \|S - Q\| \times \|R - Q\| \]
\[ = rs \sin Q \]

But, the area is the same regardless of which two sides we use to compute it, so:

\[ rs \sin Q = qs \sin R = qr \sin S \]

Dividing by \( qrs \) gives:

\[ \frac{\sin Q}{q} = \frac{\sin R}{r} = \frac{\sin S}{s} \]

Distance from a Point to a Plane

In high school, you defined a plane by:

\[ Ax + By + Cz + D = 0 \]

It is more useful to define it by a point on the plane combined with the plane’s normal vector

If you want the familiar equation of the plane, it is:

\[ \left( (x, y, z) - (Q_x, Q_y, Q_z) \right) \cdot (n_x, n_y, n_z) = 0 \]

which expands out to become the more familiar \( Ax + By + Cz + D = 0 \)

The perpendicular distance from the point \( P \) to the plane is based on the plane equation:

\[ d = (P - Q) \cdot \hat{n} \]

The dot product is answering the question “How much of (P-Q) is in the \( \hat{n} \) direction?”. Note that this gives a signed distance. If \( d > 0 \), then \( P \) is on the same side of the plane as the normal points. This is very useful.

Where does a line segment intersect an infinite plane?

The equation of the line segment is:

\[ P = (1-t)P_0 + tP_1 \]

If point \( P \) is in the plane, then:

\[ \left( (P_x, P_y, P_z) - (Q_x, Q_y, Q_z) \right) \cdot (n_x, n_y, n_z) = 0 \]

If we substitute the parametric expression for \( P \) into the plane equation, then the only thing we don’t know in that equation is \( t \). Solve it for \( t \). Knowing \( t \) will let us compute the \((x, y, z)\) of the actual intersection using the line equation. If \( t \) has a zero in the denominator, then that tells us that \( t=\infty \), and the line must be parallel to the plane.

This gives us the point of intersection with the infinite plane. We could now use the method covered a few slides ago to see if \( P \) lies inside a particular triangle.
Minimal Distance Between Two 3D Lines

The equation of the lines are: 

\[
P = P_0 + t \cdot v_p \quad Q = Q_0 + t \cdot v_q
\]

The minimal distance vector between the two lines must be perpendicular to both

A vector between them that is perpendicular to both is: 

\[
v_{p0} = v_p \times v_q
\]

We need to answer the question “How much of \((Q_0-P_0)\) is in the \(v_{p0}\) direction?”. To do this, we once again use the dot product:

\[
d = (P_0 - Q_0) \cdot \hat{v}_{p0}
\]

Another use for Dot Products:

Force One Vector to be Perpendicular to Another Vector

Here, we want to force \(A\) to become perpendicular to \(B\)

The strategy is to get rid of the parallel component, leaving just the perpendicular

\[
A = A_\parallel + A_\perp
\]

\[
A_\perp = A - A_\parallel
\]

But, 

\[
A_\perp = (A \cdot \hat{B}) \hat{B}
\]

So that

\[
A' = A_\perp = A - (A \cdot \hat{B}) \hat{B}
\]

This is known as **Gram-Schmidt orthogonalization**