Vectors

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A Vector Can Also Be Defined as the Positional Difference Between Two Points

Vector have Direction and Magnitude

Unit Vectors have a Magnitude = 1.0

Dot Product

A Physical Interpretation of the Dot Product

Because it produces a scalar result (i.e., a single number), this is also called the Scalar Product.
A Physical Interpretation of the Dot Product

The amount of the force accelerating the car along the road is "how much of \( F \) is in the \( R \) direction?"

\[
F_{\text{road}} = F \cos \theta = F \cdot \hat{R}
\]

Even Trickier...

\[
F_{\text{road}} = F \cos \theta = F \cdot \hat{R}
\]

Generalizing How Much of A Lives in the B Direction

\[
A \cdot B = \|A\|\|B\| \cos \theta
\]

\[
A \cdot \hat{B} = \|A\| \cos \theta
\]

which is the length of the projection of \( A \) onto the \( B \) line

So, how much of \( A \) lives in the \( B \) direction is that magnitude times the \( B \) unit vector:

\[
(A \cdot \hat{B}) \hat{B}
\]

Generalizing How Much of A Lives Perpendicular to the B Direction

From the previous slide, how much of \( A \) lives in the \( B \) direction is:

\[
(A \cdot \hat{B}) \hat{B}
\]

That, plus the perpendicular vector equals \( A \), so that how much of \( A \) is perpendicular to the \( B \) direction is:

\[
A - (A \cdot \hat{B}) \hat{B}
\]

Dot Products are Commutative

\[
A \cdot B = B \cdot A
\]

Dot Products are Distributive

\[
A \cdot (B + C) = (A \cdot B) + (A \cdot C)
\]

The Perpendicular to a 2D Vector

If \( V = (x, y) \)

then \( V_\perp = (-y, x) \)

You can tell that this is true because \( V \cdot V_\perp = (x, y) \cdot (-y, x) = -xy + xy = 0 = \cos 90^\circ \)
Cross Product

\[ A = (A_x, A_y, A_z) \]
\[ B = (B_x, B_y, B_z) \]
\[ A \times B = (A_y B_z - A_z B_y, A_z B_x - A_x B_z, A_x B_y - A_y B_x) \]
\[ \|A \times B\| = \|A\|\|B\| \sin \theta \]

Because it produces a vector result (i.e., three numbers), this is also called the Vector Product.

The Perpendicular Property of the Cross Product

The vector \( A \times B \) is both perpendicular to \( A \) and perpendicular to \( B \).

The Right-Hand-Rule Property of the Cross Product

Curl the fingers of your right hand in the direction that starts at \( A \) and heads towards \( B \). Your thumb points in the direction of \( A \times B \).

Cross Products are Not Commutative

\[ A \times B = -B \times A \]

Cross Products are Distributive

\[ A \times (B + C) = (A \times B) + (A \times C) \]

A Use for the Cross Product:

Finding a Vector Perpendicular to a Plane (= the Surface Normal)

Let:
\[ n = (R - Q) \times (S - Q) \]
\[ n_P = (R - Q) \times (P - Q) \]
\[ n_S = (S - R) \times (P - R) \]
\[ n_Q = (Q - S) \times (P - S) \]

If \( n \cdot n_P, n \cdot n_S, \text{ and } n \cdot n_Q \) are all positive, then \( P \) is inside the triangle QRS.

A Use for the Cross and Dot Products:

Is a Point Inside a Triangle? – 3D (X-Y-Z) Version
Is a Point Inside a Triangle?
This can be simplified if you are in 2D (X-Y)

If \( E_{QR}, E_{QS}, E_{RQ} \) are all positive, then P is inside the triangle QRS.

A Use for the Cross Product: Finding the Area of a 3D Triangle

\[
\text{Area} = \frac{1}{2} \cdot \text{Base} \cdot \text{Height} = \frac{1}{2} \cdot \|QR\| \cdot \|QS\| \cdot \sin \theta
\]

\[
\frac{1}{2} \cdot (R - Q) \times (S - Q) = QR \times QS \cdot \sin \theta = \frac{1}{2} \cdot \text{Base} \cdot \text{Height}
\]

Derivation of the Law of Sines

\[
\frac{\sin Q}{q} = \frac{\sin R}{r} = \frac{\sin S}{s}
\]

Derivation of the Law of Cosines

\[
s^2 = (R - Q)^2 = (R - Q) \cdot (R - Q) = (S - Q)^2 + (S - Q) \cdot (S - Q) - 2(R - S) \cdot (S - Q)
\]

Distance from a Point to a Plane

In high school, you defined a plane by:

\[
Ax + By + Cz + D = 0
\]

If you want the familiar equation of the plane, it is:

\[
\left( (x-x_0, y-y_0, z-z_0) \right) \cdot (n_x, n_y, n_z) = 0
\]

which expands out to become the more familiar \(Ax + By + Cz + D = 0\).

The perpendicular distance from the point \( P \) to the plane is based on the plane equation:

\[
d = \|P - R\| \cdot \hat{n}
\]

The dot product is answering the question "How much of (P-Q) is in the \( \hat{n} \) direction?". Note that this gives a signed distance. If \( d > 0 \), then \( P \) is on the same side of the plane \( n_x, n_y, n_z \) as the normal points. This is very useful.

Where does a line segment intersect an infinite plane?

The equation of the line segment is:

\[
P = (1 - t)P_0 + tP_1
\]

If point \( P \) is in the plane, then:

\[
(P - P_0, P - Q_0, P - Q_1) \cdot (n_x, n_y, n_z) = 0
\]

If we substitute the parametric expression for \( P \) into the plane equation, then the only thing we don’t know in that equation is \( t \). Solve it for \( t \). Knowing \( t \) will let us compute the \( (n_x, n_y, n_z) \) of the actual intersection using the line equation. If \( t \) has a zero in the denominator, then that tells us that \( t = \infty \) and the line must be parallel to the plane. This gives us the point of intersection with the infinite plane. We could now use the \( t \) method covered a few slides ago to see if \( P \) lies inside a particular triangle.
Minimal Distance Between Two 3D Lines

The equation of the lines are: \( P = P_0 + t \cdot v_0 \) \( Q = Q_0 + t \cdot v_0 \)

The minimal distance between the two lines must be perpendicular to both \( v_0 \) and \( v_1 \).

We need to answer the question "How much of \((Q_0 - P_0)\) is in the \( v_1 \) direction?".

To do this, we once again use the dot product:

\[
d = (Q_0 - P_0) \cdot \hat{v}_1
\]

Another use for Dot Products:

Force One Vector to be Perpendicular to Another Vector

This is known as Gram-Schmidt orthogonalization