Vectors have Direction and Magnitude

Vectors can also be defined as the positional difference between two points:

\[(V_x, V_y, V_z) = (Q_x - P_x, Q_y - P_y, Q_z - P_z)\]

Unit Vectors have a Magnitude = 1.0

Unit vectors have a magnitude of 1.0:

\[\hat{V} = \frac{V}{|V|} = \frac{V_x}{\sqrt{V_x^2 + V_y^2 + V_z^2}}\]

The circumflex (\(^\hat{\}\)) tells us this is a unit vector.

Dot Product

The dot product of two vectors is given by:

\[A \cdot B = (A_x B_x + A_y B_y + A_z B_z) = |A||B| \cos \theta\]

A Physical Interpretation of the Dot Product

The amount of the force accelerating the car along the road is "how much of F is in the horizontal direction?"

\[F_{\text{road}} = F \cos \theta\]

This is easy to see in 2D, but a 3D version of the same problem is trickier.
A Physical Interpretation of the Dot Product

The amount of the force accelerating the car along the road is “how much of F is in the R direction?”

\[ F_{\text{road}} = F \cos \theta = F \cdot \hat{R} \]

Generalizing How Much of A Lives in the B Direction

\[ A \cdot B = \|A\|\|B\|\cos \theta \]
\[ A \cdot \hat{B} = \|A\|\cos \theta \]

which is the length of the projection of A onto the B line

So, how much of A lives in the B direction is that magnitude times the B unit vector:

\[ \hat{B}(A \cdot \hat{B}) \]

The Perpendicular to a 2D Vector

If \( V = (x, y) \)

then \( V_\perp = (-y, x) \)

You can tell that this is true because

\[ V \cdot V_\perp = (x, y) \cdot (-y, x) = -xy + xy = 0 = \cos 90^\circ \]
Cross Product

\[ A = (A_x, A_y, A_z) \]
\[ B = (B_x, B_y, B_z) \]
\[ A \times B = (A_yB_z - A_zB_y, A_zB_x - A_xB_z, A_xB_y - A_yB_x) \]

\[ \|A \times B\| = \|A\|\|B\|\sin\theta \]

Because it produces a vector result (i.e., three numbers), this is also called the Vector Product.

The Perpendicular Property of the Cross Product

The vector \( A \times B \) is both perpendicular to A and perpendicular to B.

The Right-Hand-Rule Property of the Cross Product

Curl the fingers of your right hand in the direction that starts at \( A \) and heads towards \( B \). Your thumb points in the direction of \( A \times B \).

Cross Products are Not Commutative

\[ A \times B = -B \times A \]

Cross Products are Distributive

\[ A \times (B + C) = (A \times B) + (A \times C) \]

A Use for the Cross Product:

Finding a Vector Perpendicular to a Plane (= the Surface Normal)

\[ n = (R - Q) \times (S - Q) \]

A Use for the Cross and Dot Products:

Is a Point Inside a Triangle? – 3D (X-Y-Z) Version

If \((n \bullet n_x), (n \bullet n_y), \text{and} (n \bullet n_z)\) are all positive, then \( P \) is inside the triangle QRS.

A Use for the Cross Product:

Finding a Vector Perpendicular to a Plane (= the Surface Normal)

This is used in CG Lighting.
Is a Point Inside a Triangle?

This can be simplified if you are in 2D (X-Y):

$$E_{RS} = (P - R) \cdot (RS)_p$$

where:

$$RS = (S_x - R_x, S_y - R_y)$$

and:

$$(RS)_p = (R_x - S_x, S_y - R_y)$$

Similarly:

$$E_{SQ} = (P - S) \cdot (SQ)_p$$

$$E_{QR} = (P - Q) \cdot (QR)_p$$

If $E_{RS}, E_{SQ}, E_{QR}$ are all positive, then $P$ is inside the triangle $QRS$.

A Use for the Cross Product: Finding the Area of a 3D Triangle

Area = \frac{1}{2} Base \cdot Height

Base = \|QR\|

Height = \|QS\| \sin \theta

Area = \frac{1}{2} \|QR\| \|QS\| \sin \theta = \frac{1}{2} \|R - Q\| \times (S - Q)\\n
Derivation of the Law of Cosines

$$s^2 = (R - Q)^2 = |R - Q|^2$$

$$s^2 = (R - S) \times (S - Q)$$

$$s^2 = [(R - S) + (S - Q)] \times [(R - S) + (S - Q)]$$

$$s^2 = [(R - S)(R - S)] + [(S - Q)(S - Q)] - 2(R - S) \times (S - Q)$$

$$s^2 = q^2 + r^2 - 2qr \cos \theta$$

Derivation of the Law of Sines

$$2 \cdot \text{Area} = \|S - Q\| \times (R - Q)$$

$$= rs \sin Q$$

But, the area is the same regardless of which two sides we use to compute it, so:

$$\sin Q = \frac{rs \sin R}{q} = \frac{qr \sin S}{r}$$

Dividing by (qrs) gives:

$$\sin Q = \frac{r}{s} \sin R = \frac{q \sin S}{s}$$

Distance from a Point to a Plane

The equation of the line segment is:

$$\hat{P} = (1-t)P_0 + tP_1$$

Where does a line segment intersect an infinite plane?

The equation of the plane is:

$$Ax + By + Cz + D = 0$$

It is more useful to define it by a point on the plane $Q$ combined with the plane's normal vector $(n_x, n_y, n_z)$:

$$(x, y, z) \cdot (Q_x, Q_y, Q_z, n_x, n_y, n_z) = 0$$

which expands out to become the more familiar $Ax + By + Cz + D = 0$.

If $d = \langle P - Q \rangle \cdot \hat{n}$

The dot product is answering the question "How much of $(P - Q)$ is in the $\hat{n}$ direction?". Note that this gives a signed distance. If $d > 0$, then $P$ is on the same side of the plane $\hat{n}$ as the normal points. This is very useful.

If point $P$ is in the plane, then:

$$\langle (P_1 - P_0, P_1 - Q_0, Q_0) \rangle \cdot (n_x, n_y, n_z) = 0$$

If we substitute the parametric expression for $P$ into the plane equation, then the only thing we don’t know is $t$. Solve it for $t$. Knowing $t$ will let us compute the $(x, y, z)$ of the actual intersection using the line equation. If $t$ has a zero in the denominator, then that tells us that $t = \infty$ and the line must be parallel to the plane.

This gives us the point of intersection with the infinite plane. We could now use the method covered a few slides ago to see if $P$ lies inside a particular triangle.
The equation of the lines are: 

\[ P = P_0 + t \cdot v_p \quad \text{and} \quad Q = Q_0 + t \cdot v_q \]

The minimal distance vector between the two lines must be perpendicular to both 

A vector between them that is perpendicular to both is: 

\[ \hat{v} = v_p \times v_q \]

We need to answer the question “How much of \((Q_0 - P_0)\) is in the \(v_r\) direction?”. To do this, we once again use the dot product:

\[ d = (P_0 - Q_0) \cdot \hat{v} \]

Another use for Dot Products:

**Force One Vector to be Perpendicular to Another Vector**

Here, we want to force \(A\) to become perpendicular to \(B\)

The strategy is to get rid of the parallel component, leaving just the perpendicular

\[ A = A + A_\perp \]

\[ A_\perp = A - A_\perp \]

But, 

\[ A_\perp = (A \cdot \hat{B}) \hat{B} \]

So that

\[ A' = A_\perp = A - (A \cdot \hat{B}) \hat{B} \]

This is known as **Gram-Schmidt orthogonalization**