Quaternions

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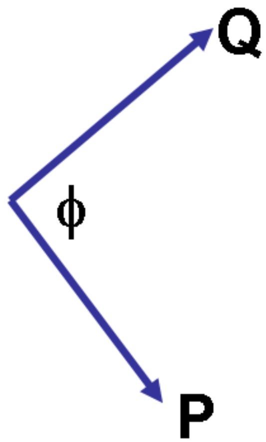
A Useful Concept: Spherical Linear Interpolation



$$Q'(t) = \frac{\sin(1-t)\phi}{\sin\phi} P + \frac{\sin t\phi}{\sin\phi} Q$$

$$0. \le t \le 1.$$

where:

$$cos\phi = P \cdot Q = (p_x q_x + p_y q_y + p_z q_z)$$

$$sin\phi = \sqrt{1 - cos^2\phi}$$

A Review of Complex Numbers

$$z = x + iy = r[\cos\theta + i\sin\theta] = re^{i\theta}$$

$$z_1 z_2 = (x_x + i\,y_1)(x_2 + i\,y_2) = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

(i.e., multiply the r’s and add the θ’s)

Since we are adding the θ’s, we see that complex multiplication is really just rotation by θ_2 if $r_2 = 1$.

Complex conjugate:

$$z^* = r\,e^{-i\theta} = x - iy$$

Complex inverse:

$$z^{-1} = \frac{1}{r} e^{-i\theta}$$

Note:
- $(z)(z^*) = r^2$
- $(z)(z^{-1}) = 1.$
- If r = 1., then $z^* = z^{-1}$

Quaternion Background

Discovered by Sir William Hamilton, 1843, while on a walk in Dublin.

Legend says that he was so excited that he took out a knife and carved the equation into the stone of a bridge.

Good thing spray-paint hadn't been invented yet…

Quaternions have 4 elements, one real and three complex:

$$Q = q_0 + iq_1 + jq_2 + kq_3 = (q_0, q_1, q_2, q_3) = (q_0, \overline{q})$$

By definition,

1. $ii = jj = kk = -1$
2. $ij = k,\ jk = i, ki = j$
3. $ji = -k, kj = -i, ik = -j$
4. $ijk = -1$

And, by definition, we always force $\|Q\| = 1.$ by making $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$

Quaternion Multiplication

$$PQ = (p_0 + i p_1 + j p_2 + k p_3)(q_0 + i q_1 + j q_2 + k q_3)$$

$$-\bar{p} \cdot \bar{q}$$

$$PQ = \begin{pmatrix} p_0 q_0 - p_1 q_1 - p_2 q_2 - p_3 q_3 & \\ p_1 q_0 + p_0 q_1 + p_2 q_3 - p_3 q_2 & i \\ p_2 q_0 + p_0 q_2 + p_3 q_1 - p_1 q_3 & j \\ p_3 q_0 + p_0 q_3 + p_1 q_2 - p_2 q_1 & k \end{pmatrix}$$

$$q_0 \bar{p} \qquad p_0 \bar{q} \qquad \bar{p} \times \bar{q}$$

Performing Rotations with Quaternions

A Quaternion can record a rotation transformation by an angle θ about an axis $\hat{n}$ like this:

$$[R(\theta,\hat{n})] = [Q(q_0,\bar{q})]$$

where:

$$q_0 = \cos(\theta/2)$$

$$\bar{q} = \sin(\theta/2)\hat{n}$$

Concatenated Quaternion Rotations are handled like this:

$$R'(\theta,\hat{n}) = R_2(\theta_2,\hat{n}_2) * R_1(\theta_1,\hat{n}_1)$$

R_1's rotation takes effect first, followed by R_2's

A Quaternion can represent a point P like this:

$$P = Q(0,\bar{p}) = (0, p_x, p_y, p_z)$$

A rotated point, P' by one rotation is:

$$P' = [R] * \{P\} * [R]^{-1}$$

A rotated point, P' by multiple rotations is:

$$P' = [R_2][R_1] * \{P\} * [R_1]^{-1}[R_2]^{-1}$$

Converting from a Quaternion to a Matrix

$$R(Q)=\begin{bmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2q_1q_2-2q_0q_3 & 2q_1q_3+2q_0q_2 \\ 2q_1q_2+2q_0q_3 & q_0^2-q_1^2+q_2^2-q_3^2 & 2q_2q_3-2q_0q_1 \\ 2q_1q_3-2q_0q_2 & 2q_2q_3+2q_0q_1 & q_0^2-q_1^2-q_2^2+q_3^2 \end{bmatrix}$$

From the GLM Notes:
The Rotation Matrix for an Angle (θ) about an Arbitrary Axis (Ax, Ay, Az)

$$[M]=\begin{bmatrix} A_xA_x+\cos\theta(1-A_xA_x) & A_xA_y-\cos\theta(A_xA_y)-\sin\theta A_z & A_xA_z-\cos\theta(A_xA_z)+\sin\theta A_y \\ A_yA_x-\cos\theta(A_yA_x)+\sin\theta A_z & A_yA_y+\cos\theta(1-A_yA_y) & A_yA_z-\cos\theta(A_yA_z)-\sin\theta A_x \\ A_zA_x-\cos\theta(A_zA_x)-\sin\theta A_y & A_zA_y-\cos\theta(A_zA_y)+\sin\theta A_x & A_zA_z+\cos\theta(1-A_zA_z) \end{bmatrix}$$

For this to be correct, A must be a unit vector

Converting from a Quaternion to a Matrix

$$R(Q)=\begin{bmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2q_1q_2-2q_0q_3 & 2q_1q_3+2q_0q_2 \\ 2q_1q_2+2q_0q_3 & q_0^2-q_1^2+q_2^2-q_3^2 & 2q_2q_3-2q_0q_1 \\ 2q_1q_3-2q_0q_2 & 2q_2q_3+2q_0q_1 & q_0^2-q_1^2-q_2^2+q_3^2 \end{bmatrix}$$

$$R=\begin{bmatrix} c+n_x^2(1-c) & n_xn_y(1-c)-sn_z & n_xn_y(1-c)+sn_y \\ n_yn_x(1-c)+sn_z & c+n_y^2(1-c) & n_yn_z(1-c)-sn_x \\ n_zn_x(1-c)-sn_y & n_zn_y(1-c)+sn_x & c+n_z^2(1-c) \end{bmatrix}$$

where: $c = cos\theta$
$s = sin\theta$

Note that the sum of the Trace (diagonal) elements is: $3c+(1-c)=1+2cos\theta$

Converting from a Matrix to a Quaternion

$$R(Q) = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_1q_2 - 2q_0q_3 & 2q_1q_3 + 2q_0q_2 \\ 2q_1q_2 + 2q_0q_3 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_2q_3 - 2q_0q_1 \\ 2q_1q_3 - 2q_0q_2 & 2q_2q_3 + 2q_0q_1 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

Note that the sum of the Trace (diagonal) elements is: $3c + (1 - c) = 1 + 2cos\theta$

Letting $t = Trace(R) = 1 + 2\cos\theta$, then:

$$\cos\theta = \frac{1}{2}(t-1)$$

$$\sin\theta = \sqrt{1-\cos^2\theta}$$

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Notes from Converting from a Quaternion to a Matrix

$$\boldsymbol{R} = \begin{bmatrix} c + n_x^2(1-c) & n_x n_y(1-c) - s n_z & n_x n_y(1-c) + s n_y \\ n_y n_x(1-c) + s n_z & c + n_y^2(1-c) & n_y n_z(1-c) - s n_x \\ n_z n_x(1-c) - s n_y & n_z n_y(1-c) + s n_x & c + n_z^2(1-c) \end{bmatrix}$$

$$\boldsymbol{R}^T = \begin{bmatrix} c + n_x^2(1-c) & n_y n_x(1-c) + s n_z & n_z n_x(1-c) - s n_y \\ n_x n_y(1-c) - s n_z & c + n_y^2(1-c) & n_z n_y(1-c) + s n_x \\ n_x n_y(1-c) + s n_y & n_y n_z(1-c) - s n_x & c + n_z^2(1-c) \end{bmatrix}$$

$$\boldsymbol{R}\text{-}\boldsymbol{R}^T = \begin{bmatrix} 0 & -2sn_z & +2sn_y \\ +2sn_z & 0 & -2sn_x \\ -2sn_y & +2sn_x & 0 \end{bmatrix}$$

$$c = cos\left(\frac{\theta}{2}\right)$$

$$s = sin\left(\frac{\theta}{2}\right)$$

Converting from a Matrix to a Quaternion

$$R - R^T = \begin{bmatrix} 0 & -2n_z \sin\theta & +2n_y \sin\theta \\ +2n_z \sin\theta & 0 & -2n_x \sin\theta \\ -2n_y \sin\theta & +2n_x \sin\theta & 0 \end{bmatrix} = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$$

If we let

$$d = \sqrt{a^2 + b^2 + c^2}$$, then $$\hat{n} = \left(\frac{a}{d}, \frac{b}{d}, \frac{c}{d} \right)$$

$$q_0 = \cos(\theta/2)$$

$$\bar{q} = \sin(\theta/2)\hat{n}$$

Quaternions in GLM

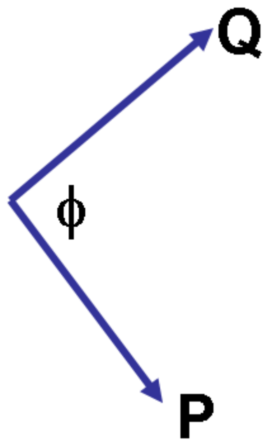

```
#include "glm/vec2.hpp"
#include "glm/vec3.hpp"
#include "glm/mat4x4.hpp"
#include "glm/gtc/matrix_transform.hpp"
#include "glm/gtc/matrix_inverse.hpp"
#include “gtc/quaternion.hpp”
#include “glm/gtx/quaternion.hpp”



glm::quat rot1 = glm::angleAxis(  glm::radians(45.f), glm::vec3( 0.707f, 0.707f, 0. )   );
glm::quat rot2 = glm::angleAxis(  glm::radians(90.f), glm::vec3( 1.       , 0.       , 0.,)   );

glm::quat  combinedRots = rot2 * rot1;
glm::vec4  v    = glm::vec4( 1., 1., 1., 1. );
glm::vec4  vp  = combinedRots * v;

glm::mat4  rotMatrix =  glm::toMat4( combinedRots );
glm::vec4  vpp          = rotMatrix * v;        // same result as vp
```

Converting from a Quaternion to OpenGL

glRotate3f(θ°, n_x, n_y, n_z);

glm::rotate(glm::quat const & q, θ^R, glm::vec3(n_x, n_y, n_z));

A Useful Concept: Spherical Linear Interpolation, where P and Q are Quaternions



$$Q'(t) = \frac{\sin(1-t)\phi}{\sin\phi} P + \frac{\sin t\phi}{\sin\phi} Q$$

$$0. \le t \le 1.$$

where:

$$cos\phi = P \cdot Q = (p_0 q_0 + p_x q_x + p_y q_y + p_z q_z)$$

$$sin\phi = \sqrt{1 - cos^2\phi}$$

My Code (Quat.h, Quat.cpp)

```
Rotate
Slerp( float t, const Quat& p, const Quat& q )
{
        float angr;             // angle between p and q in radians
        float c, s;             // cosine and sine of the angle between p and q
        float cp, cq;           // coefficients to multiply quaternions p and q
        Rotate r;

        // dot product to get the angle between p and q:
        c = p.s*q.s + p.vx*q.vx + p.vy*q.vy + p.vz*q.vz;
        angr = acos( c );

        // sine of that angle:
        s = sin( angr );

        // if the sine is 0., then p == q:
        if( s == 0. )
        {
                r = p;
                return r;
        }

        // do spherical interpolation:
        cp = sin( (1.-t)*angr ) / s;
        cq = sin(     t *angr  ) / s;

        r = cp*p  +  cq*q;
        return r;
}
```

My Code (Quat.h, Quat.cpp)

```
int
main( int argc, char *argv[ ] )
{
	Rotate r1 = Rotate( 45.*D2R, 0., 0., 1. );
	Rotate r2 = Rotate( 90.*D2R, 0., 0., 1. );
	Rotate r3 = Rotate( 90.*D2R, 1., 0., 0. );
	Rotate r4 = r2 * r1;
	Rotate r5 = r3 * r2 * r1;

	fprintf( stderr, "r1       = %s\n", r1.toString() );
	fprintf( stderr, "r2       = %s\n", r2.toString() );
	fprintf( stderr, "r2*r1    = %s\n", r4.toString() );

	fprintf( stderr, "r3       = %s\n", r3.toString() );
	fprintf( stderr, "r3*r2*r1 = %s\n", r5.toString() );
	fprintf( stderr, "\nr3*r2*r1 matrix =\n" );
	r5.printMatrix();

	Point p1 = Point( 1., 1., 0. );
	Point p2 = r4 * p1;

	fprintf( stderr, "Original point    = %s\n", p1.toString() );
	fprintf( stderr, "Transformed point = %s\n", p2.toString() );

	// try interpolating from r1 to r5:
	const int N = 10;
	float dt = 1. / (float)( N - 1 );
	float t = 0.;
	for( int i = 0; i < N; i++, t += dt )
	{
		Rotate r15 = Slerp( t, r1, r5 );
		fprintf( stderr, "%2d   %5.3f   %s\n", i, t, r15.toString() );
	}
}
```

My Code (Quat.h, Quat.cpp)

```
r1       =    45.000 degrees about (   0.000    0.000    1.000 )
r2       =    90.000 degrees about (   0.000    0.000    1.000 )
r2*r1    =   135.000 degrees about (   0.000    0.000    1.000 )
r3       =    90.000 degrees about (   1.000    0.000    0.000 )
r3*r2*r1 =   148.606 degrees about (   0.281   -0.678    0.679 )

r3*r2*r1 matrix =
        -0.707   -0.707    0.000    0.000
         0.000    0.000   -1.000    0.000
         0.707   -0.707    0.000    0.000
         0.000    0.000    0.000    1.000

Original point    =   1.000    1.000    0.000
Transformed point =  -1.414    0.001    0.000

 0  0.000    45.000 degrees about (   0.000    0.000    1.000 )
 1  0.111    53.754 degrees about (   0.080   -0.194    0.978 )
 2  0.222    64.007 degrees about (   0.136   -0.328    0.935 )
 3  0.333    75.145 degrees about (   0.175   -0.423    0.889 )
 4  0.444    86.824 degrees about (   0.204   -0.493    0.846 )
 5  0.556    98.848 degrees about (   0.226   -0.546    0.807 )
 6  0.667   111.101 degrees about (   0.244   -0.588    0.771 )
 7  0.778   123.508 degrees about (   0.258   -0.623    0.739 )
 8  0.889   136.021 degrees about (   0.270   -0.652    0.708 )
 9  1.000   148.606 degrees about (   0.281   -0.678    0.679 )
```

A Really Good (i.e., Complete) Reference

Andrew Hanson, *Visualizing Quaternions*, Morgan-Kaufmann, 2006.