CS515: Algorithms and Data Structures, Fall 2017

Homework 3*

Due: Tue, 11/21/17

Homework Policy:

1. Students should work on homework assignments in groups of preferably three people. Each group submits to TEACH one set of typeset solutions, and hands in a printed hard copy in class or slides the hard copy under my door before the midnight of the due day. The hard copy will be graded.
2. The goal of the homework assignments is for you to learn solving algorithmic problems. So, I recommend spending sufficient time thinking about problems individually before discussing them with your friends.
3. You are allowed to discuss the problems with other groups, and you are allowed to use other resources, but you *must* cite them. Also, you must write everything in your own words, copying verbatim is plagiarism.
4. *I don't know policy:* you may write "I don't know" *and nothing else* to answer a question and receive 25 percent of the total points for that problem whereas a completely wrong answer will receive zero.
5. Algorithms should be explained in plain english. Of course, you can use pseudocodes if it helps your explanation, but the grader will not try to understand a complicated pseudocode.
6. More items might be added to this list. ☺

Problem 1. **[30 pts]** Consider the following algorithm for finding the smallest element in an unsorted array:

RandomMin $A[1..n]$

$min \leftarrow \infty$
for $i \leftarrow 1$ to n in random order **do**
  if $A[i] < min$ **then**
    $min \leftarrow A[i]$ $(*)$
  end if
end for
return min

(a) In the worst case, how many times does RandomMin execute line $(*)$?

(b) What is the probability that line $(*)$ is executed during the nth iteration of the for loop?

(c) What is the *exact* expected number of executions of line $(*)$?

*The problems are from Jeff Erickson's lecture notes.

Problem 2. **[30 pts]** Consider the following randomized algorithm for choosing the largest bolt. Draw a bolt uniformly at random from the set of n bolts, and draw a nut uniformly at random from the set of n nuts. If the bolt is smaller than the nut, discard the bolt, draw a new bolt uniformly at random from the unchosen bolts, and repeat. Otherwise, discard the nut, draw a new nut uniformly at random from the unchosen nuts, and repeat. Stop either when every nut has been discarded, or every bolt except the one in your hand has been discarded. What is the exact expected number of nut-bolt tests performed by this algorithm? Prove your answer is correct. [Hint: What is the expected number of unchosen nuts and bolts when the algorithm terminates?]

Problem 3. **[40 pts]** A flow f is acyclic if the subgraph of directed edges with positive flow contains no directed cycles.

(a) Prove that for any flow f, there is an acyclic flow with the same value as f. (In particular, this implies that some maximum flow is acyclic.)

(b) A path flow assigns positive values only to the edges of one simple directed path from s to t. Prove that every acyclic flow can be written as the sum of $O(E)$ path flows.

(c) Describe a flow in a directed graph that cannot be written as the sum of path flows.

(d) A cycle flow assigns positive values only to the edges of one simple directed cycle. Prove that every flow can be written as the sum of $O(E)$ path flows and cycle flows.

(e) Prove that every flow with value 0 can be written as the sum of $O(E)$ cycle flows. (Zero-value flows are also called circulations.)