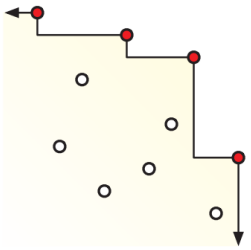
CS515: Algorithms and Data Structures, Fall 2015

Midterm (November 2, 2015)

Problem 1. [25 pts] Suppose you are given a set $S = \{s_1, s_2, \ldots, s_n\}$ of n integers, and a target integer B. Design and analyze an algorithm that returns "yes" if there is a subset $S' \subseteq S$ such that $\sum_{s \in S'} s = B$, and returns "no" otherwise. Justify the correctness of you algorithm. The running time must be polynomial in B and n.

For example, if $S = \{1, 2, 4, 5\}$ and $B = 8$, the algorithm should return "yes" because $B = 8 = 1 + 2 + 5$. If $S = \{1, 2, 6, 10\}$ and $B = 5$, the algorithm should return "no" because the elements of no subset of S add up to 5.

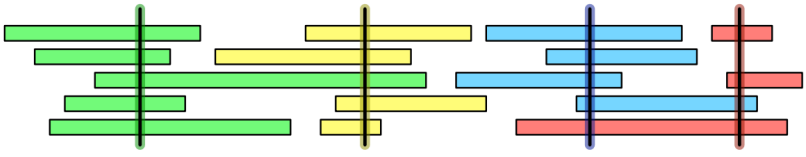
Problem 2. [25 pts] Suppose you are given a set P of n points in the plane. A point $p \in P$ is maximal in P if no other point in P is both above and to the right of p. Intuitively, the maximal points define a "staircase" with all the other points of P below it.



A set of ten points, four of which are maximal.

Describe and analyze an algorithm to compute the number of maximal points in P in $O(n \log n)$ time. Justify the correctness of your algorithm. A correct $O(n^2)$ time algorithm receives 8 points.

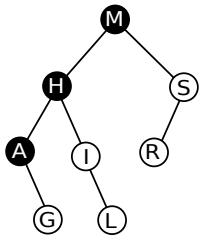
Problem 3. [25 pts] Let X be a set of n intervals on the real line. We say that a set P of points stabs X if every interval in X contains at least one point in P. Describe and analyze an efficient algorithm to compute the smallest set of points that stabs X. Assume that your input consists of two arrays $X_L = [1..n]$ and $X_R = [1..n]$, representing the left and right endpoints of the intervals in X. Prove that your algorithm is correct.



A set of intervals stabbed by four points (shown here as vertical segments)

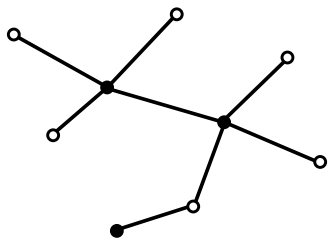
Problem 4. Consider a randomized treap T with n vertices.

(a) [13 pts] The left spine of a binary tree is a path starting at the root and following left-child pointers down to the first node with no left child pointer. What is the expected number of nodes in the left spine of T? For example, the number of nodes on the left spine of the following tree is 3, all black nodes. [Hint: What is the probability that a node is on the left spine.]



(b) [12 pts] What is the expected number of leaves in T? [Hint: What is the probability that a node is a leaf?]

Problem 5. (extra credit) [15 pts] Let $T = (V, E)$ be a tree. A dominating set of T is a subset of its vertices $D \subseteq V$ such that every vertex not in D is adjacent to at least one vertex in D. For example, the solid black vertices in the following tree form a dominating set.



Design and analyze a polynomial time algorithm to compute a dominating set of minimum cardinality for T.