Problem 1. Let $G = (X \cup Y, E)$ be a regular bipartite graph, and let $M$ be a matching in $G$. Let $X_M \cup Y_M$ and $X_U \cup Y_U$ be the set of matched and the set of unmatched vertices with respect to $M$, respectively. For any vertex $v \in X \cup Y$ let $b(v)$ be the expected length of a minimal alternating random walk from $v$ to $Y_U$ as defined in the class and lecture notes.

In class, we proved an upper bound for $b(v)$ by solving a set of equations that relate $b(v)$ for different $v$’s. As you might have noticed, this method is only valid under the assumption that all $b(v)$’s are finite. In this exercise, we prove that this condition holds.

(a) For any vertex $v \in X \cup Y$, prove that there exists an $M$-alternating path from $v$ to $Y_U$ if $M$ is not perfect.

(b) What is the probability that a random walk starting at $v$ follows the path of part (a)?

(c) Use (b) to give an upper bound for $b(v)$.

Problem 2. Let $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ be a permutation. Let $S$ be a set of swaps that transform $\pi$ into the identity permutation. Recall that the sign of $\pi$ is $(-1)^{|S|}$. Prove that the sign of a permutation is well-defined. Precisely, let $S'$ be another set of swaps that transforms $\pi$ into the identity permutation, and show that $|S|$ and $|S'|$ have the same parity.

Problem 3. Let the lattice $\mathbb{Z}^2 = \{(x, y) | x, y \in \mathbb{Z}\}$ be the set of all points in the plane with integer coordinates. A triangle is elementary if its vertices lie on the lattice $\mathbb{Z}^2$, but it does not intersect any other lattice point (See Figure 1, left.)
Figure 1: Left: Four elementary triangles, right: a polygon with area $6 + 13/2 - 1 = 11.5$ ($N_{int} = 6$, $N_{bdr} = 13$.)

(a) **(Extra credit)** Prove that the area of any elementary triangle is exactly $\frac{1}{2}$.

(b) Let $P$ be a polygon with vertices in $\mathbb{Z}^2$. Suppose, there are $N_{int}$ and $N_{bdr}$ lattice points in the interior and on the boundary of $P$, respectively. Use part (a) and Euler’s formula to show that the area of $P$ is:

$$N_{int} + \frac{1}{2}N_{bdr} - 1.$$ 

For an example, see Figure 1, right.

**Problem 4.** Define the density of a graph $G = (V, E)$ to be:

$$\rho(G) = \frac{|E|}{|V|} = \frac{m}{n}.$$ 

(a) Let $\mathcal{F}$ be a minor closed family of graphs, and let $\rho_0$ be the maximum density of all simple graphs in $\mathcal{F}$. Show that the Boruvka algorithm can be adapted to work in $O(\rho_0 \cdot n)$ time for graphs in $\mathcal{F}$.

(b) What is $\rho_0$ if $\mathcal{F}$ is the family of all planar graphs? Conclude that your adaptation of the Boruvka algorithm works in $O(n)$ for planar graphs.