CS420/520: Graph Theory with Applications to CS, Winter 2014

Homework 3

Due: Tue, March/4/14

Homework Policy: Each student should submit his/her own set of solutions, independently. You are allowed to discuss the homework with other students, however, you need to indicate their names in your submission. Also, you are allowed to use other sources, but you must cite every source that you use.

Problem 1. Let $G = (X \cup Y, E)$ be a regular bipartite graph, and let M be a matching in G. Let $X_M \cup Y_M$ and $X_U \cup Y_U$ be the set of matched and the set of unmatched vertices with respect to M, respectively. For any vertex $v \in X \cup Y$ let $b(v)$ be the expected length of a minimal alternating random walk from v to Y_U as defined in the class and lecture notes.

In class, we proved an upper bound for $b(v)$ by solving a set of equations that relate $b(v)$ for different v's. As you might have noticed, this method is only valid under the assumption that all $b(v)$'s are finite. In this exercise, we prove that this condition holds.

(a) For any vertex $v \in X \cup Y$, prove that there exists an M-alternating path from v to Y_U if M is not perfect.

(b) What is the probability that a random walk starting at v follows the path of part (a)?

(c) Use (b) to give an upper bound for $b(v)$.

problem 2. Let $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ be a permutation. Let S be a set of swaps that transform π into the identity permutation. Recall that the sign of π is $(-1)^{|S|}$. Prove that the sign of a permutation is well-defined. Precisely, let S' be another set of swaps that transforms π into the identity permutation, and show that $|S|$ and $|S'|$ have the same parity.

problem 3. Let the lattice $\mathbb{Z}^2 = \{(x, y) | x, y \in \mathbb{Z}\}$ be the set of all points in the plane with integer coordinates. A triangle is *elementary* if its vertices lie on the lattice $\mathbb{Z}^2$, but it does not intersect any other lattice point (See Figure 1, left.)

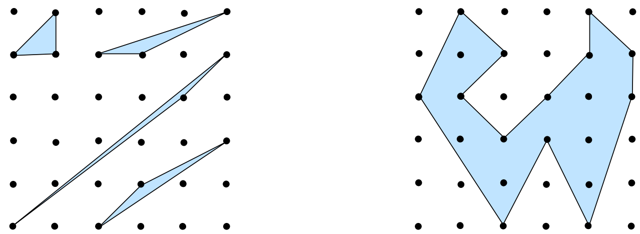

Figure 1: Left: Four elementary triangles, right: a polygon with area $6+13/2-1 = 11.5$ ($N_{int} = 6$, $N_{bdr} = 13$.)

(a) (**Extra credit**) Prove that the area of any elementary triangle is exactly $\frac{1}{2}$.

(b) Let P be a polygon with vertices in $\mathbb{Z}^2$. Suppose, there are N_{int} and N_{bdr} lattice points in the interior and on the boundary of P, respectively. Use part (a) and Euler's formula to show that the area of P is:

$$N_{int} + \frac{1}{2}N_{bdr} - 1.$$

For and example, see Figure 1, right.

Problem 4. Define the density of a graph $G = (V, E)$ to be:

$$\rho(G) = \frac{|E|}{|V|} = \frac{m}{n}.$$

(a) Let $\mathcal{F}$ be a minor closed family of graphs, and let ρ_0 be the maximum density of all *simple* graphs in $\mathcal{F}$. Show that the Boruvka algorithm can be adapted to work in $O(\rho_0 \cdot n)$ time for graphs in $\mathcal{F}$.

(b) What is ρ_0 if $\mathcal{F}$ is the family of all planar graphs? Conclude that your adaptation of the Boruvka algorithm works in $O(n)$ for planar graphs.