

CS420/520: Graph Theory with Applications to CS, Winter 2015

Homework 2

Due: Tue, March/03/15

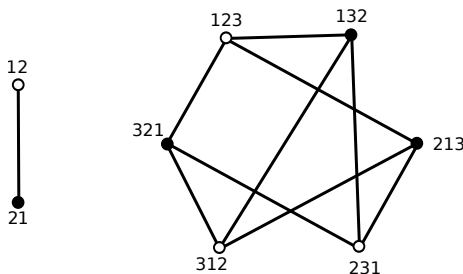
Homework Policy: Each student should submit his/her own set of solutions, independently. You are allowed to discuss the homework with other students, however, you need to indicate their names in your submission. Also, you are allowed to use other sources, but you must cite every source that you use.

Problem 1. Let $G = (X \cup Y, E)$ be a connected regular bipartite graph, and let M be a matching in G . Let $X_M \cup Y_M$ and $X_U \cup Y_U$ be the set of matched and the set of unmatched vertices with respect to M , respectively. For any vertex $v \in X \cup Y$ let $b(v)$ be the expected length of a minimal alternating random walk from v to Y_U as defined in the class and lecture notes.

In class, we proved an upper bound for $b(v)$ by solving a set of equations that relate $b(v)$ for different v 's. As you might have noticed, this method is only valid under the assumption that all $b(v)$'s are finite. In this exercise, we prove that this condition holds.

- (a) For any vertex $v \in X \cup Y$, prove that there exists an M -alternating path from v to Y_U if M is not perfect.
- (b) What is the probability that a random walk starting at v follows the path of part (a)?
- (c) Use (b) to give an upper bound for $b(v)$.

Problem 2. Let $G_n = (\Pi_n, E_n)$ be a graph, whose vertex set is the set of all permutations of $\{1, 2, \dots, n\}$. Two vertices $\pi_1, \pi_2 \in \Pi$ are adjacent, $(\pi_1, \pi_2) \in E_n$, if and only if π_2 can be obtained from π_1 by one swap operations. The figure illustrates G_2 and G_3 .



Prove that G_n is bipartite for every n . Conclude, that sign of a permutation, defined in class, is well-defined.

Problem 3. Let $T = (V, E)$ be a tree with n vertices. Prove there exists a vertex $v \in V$ such that each connected component of $T \setminus \{v\}$ has at most $n/2$ vertices.

Problem 4. A coloring of a graph G is an assignment of colors to the vertices G . A coloring is valid if it assigns different colors to all adjacent pair of vertices. A graph G is k -colorable if there is a valid coloring of G with k colors. For example, any bipartite graph is 2-colorable, and any cycle of odd length is 3-colorable.

1. Use Euler's formula to show that any planar graph has a vertex of degree at most 5.
2. Prove that any planar graph is 6-colorable.

Problem 5. Let $G = (V, E)$ be an undirected unweighted graph, and let $s \in V$.

- (a) Design and analyze an $O(m)$ time algorithm to compute the shortest cycle of G that contains s .
- (b) Use part (a) to find an algorithm to compute the shortest cycle of G in $O(mn)$ time.
- (c) Argue that the algorithm of part (b) has running time $O(n^2)$ if G is planar.
- (d) Design and analyze a faster algorithm for the case that G is planar using the separator theorem.