CS420/520: Graph Theory with Applications to CS, Winter 2016

Homework 1

Due: Tue, Jan/24/17

Homework Policy:

1. Students should work on homework assignments in groups of preferably three people. Each group submits to TEACH one set of typeset solutions, and hands in a printed hard copy in class or slides the hard copy under my door before the midnight of the due day. The hard copy will be graded.
2. The goal of the homework assignments is for you to learn solving algorithmic problems. So, I recommend spending sufficient time thinking about problems individually before discussing them with your friends.
3. You are allowed to discuss the problems with other groups, and you are allowed to use other resources, but you *must* cite them. Also, you must write everything in your own words, copying verbatim is plagiarism.
4. *I don't know policy:* you may write "I don't know" *and nothing else* to answer a question and receive 25 percent of the total points for that problem whereas a completely wrong answer will receive zero.
5. Algorithms should be explained in plain english. Of course, you can use pseudocodes if it helps your explanation, but the grader will not try to understand a complicated pseudocode.

Readings:

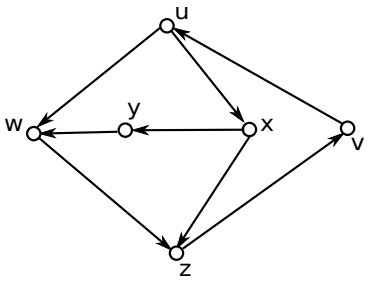
(A) Jeff lecture notes on basic graph algorithms: http://jeffe.cs.illinois.edu/teaching/algorithms/notes/18-graphs.pdf.

(B) Jeff lecture notes on graph search: http://jeffe.cs.illinois.edu/teaching/algorithms/notes/19-dfs.pdf.

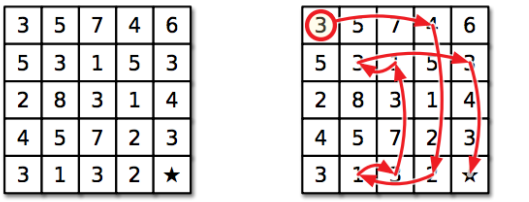
Problems for practice.

- Problems (1), (3), (7), (12) from (A).
- Problems (7), (8) from (B).

Problem 1. Draw all possible DFS trees rooted at x for the following graph. Mark, tree, forward, backward, and cross edges.



Problem 2. A number maze is an $n \times n$ grid of positive integers. A token starts in the upper left corner; your goal is to move the token to the lower-right corner. On each turn, you are allowed to move the token up, down, left, or right; the distance you may move the token is determined by the number on its current square. For example, if the token is on a square labeled 3, then you may move the token three steps up, three steps down, three steps left, or three steps right. However, you are never allowed to move the token off the edge of the board. Describe and analyze an efficient algorithm that either returns the minimum number of moves required to solve a given number maze, or correctly reports that the maze has no solution.



A 5 × 5 number maze that can be solved in eight moves.

Problem 3. Let $G = (V, E)$ be an undirected unweighted graph, and let $s, t \in V$. Show that at least one of the following conditions hold.

1. The distance between s and t is at most $\sqrt{V} + 1$.
2. There is a subset $S \subset V$ of cardinality at most $\sqrt{V}$ whose removal disconnects s and t.

Problem 4. The edges if a directed graph can be categorized into (i) tree edges, (ii) forward edges, (iii) backward edges, and (iv) cross edges based on a DFS search. In this problem, we look at edges of a connected *undirected* graph with respect to a DFS search. (What is a forward/backward edge? The graph is not directed anymore.)

1. Show no cross edge can exist.
2. Design an algorithm that finds a pair of disjoint cycles in an undirected graph if it exists. Ideally, your algorithm should run in $O(V)$ time (Note that E can be much larger than V).