Computing the Fréchet distance between polygons with holes

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Abstract

We study the problem of computing the Fréchet distance between subsets of Euclidean space. Even though the problem has been studied extensively for 1-dimensional curves, very little is known for d-dimensional spaces, for any $d \geq 2$. For general polygons in $\mathbb{R}^2$, it has been shown to be NP-hard, and the best known polynomial-time algorithm works only for polygons with at most a single puncture [Buchin *et al.*, 2010]. Generalizing [Buchin *et al.*, 2008] we give a polynomial-time algorithm for the case of arbitrary polygons with a constant number of punctures. Moreover, we show that approximating the Fréchet distance between polyhedral domains in $\mathbb{R}^3$ to within a factor of $n^{1/\log\log n}$ is NP-hard.

1 Introduction

Computing the similarity between two geometric objects is a fundamental problem that arises in several application scenarios, such as computer vision, and graphics (see [8,12,17,24] and the references therein for a more detail account of the various applications). A classical way for estimating such a similarity is the notion of the Hausdorff distance between two subsets of a metric space. However, when the objects under consideration are endowed with topological information, it is desirable to use a similarity function that takes this additional structure into account.

One of the most well-studied similarity functions that combines topological and geometric information is the *Fréchet distance*, which we define here for subsets of Euclidean space. Let $X \subset \mathbb{R}^d$ be a *parameter space*, for some $d \geq 1$, and let $h_P : X \to \mathbb{R}^d$, $h_Q : X \to \mathbb{R}^d$ be embeddings[1]. Let $P = h_P(X)$, and $Q = h_Q(X)$. Then, for a homeomorphism $f : P \to Q$, we define its *Fréchet length* to be $\delta_F(f) = \sup_{x \in P} \|x - f(x)\|_2$, and $\delta_F(P,Q) = \inf_f \delta_F(f)$, where $f : P \to Q$ ranges over all orientation-preserving homeomorphisms[2]. We remark that the Fréchet distance can also be defined for more general ambient spaces, e.g. for surfaces (see, e.g. [19]), but we will restrict our attention to Euclidean space.

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[1] In the most general setting, the parameterizations h_P, and h_Q may not be required to be embeddings. However, we restrict our attention here to *simple* polygons with holes, in which case the maps h_P and h_Q are embeddings.

[2] More generally, one can consider homeomorphisms $g : X \to X$, i.e. from the parameter space into itself, and define their Fréchet length to be $\delta_F(g) = \sup_{x \in X} \|h_P(x) - g(h_Q(x))\|_2$. However, we are dealing with *simple* polygonal/polyhedral domains, and therefore the maps h_P, and h_Q are homeomorphisms, which implies that our simpler definition is equivalent.

1.1 Previous work on computing Fréchet distance

Most of the work on computing the Fréchet distance between two subsets P and Q of some ambient space has been focused on the case where P and Q are one-dimensional curves [1, 3, 4, 7, 9, 14, 15, 19]. In contrast, very little is known for computing the Fréchet distance between two-dimensional spaces P and Q. Buchin *et al.* [6] describe the first polynomial-time algorithm to compute the Fréchet distance between two simple polygons in $\mathbb{R}^2$. Buchin *et al.* [5] prove that the problem becomes NP-hard for arbitrary polygons in $\mathbb{R}^2$, and they give a polynomial-time algorithm for polygons with a single puncture. They also prove that computing the Fréchet distance between two terrains in $\mathbb{R}^3$ is NP-hard. It has also been shown by Godau [18] that computing the Fréchet distance between two surfaces is NP-hard, and it is known that this problem is upper semi-computable [2]. Finally, Cook *et al.* [10] gave exact and approximation algorithms for special classes of simply-connected 2-dimensional polygons in $\mathbb{R}^3$.

1.2 Our contribution

We focus on the problems of computing and approximating the Fréchet distance between two subsets of d-dimensional Euclidean space, and we obtain both upper and lower bounds. Our main contributions are described below.

We present a polynomial-time algorithm for computing the Fréchet distance between two polygons $P, Q \subset \mathbb{R}^2$, with a constant number of punctures. This answers a question of Buchin *et al.* [5], and resolves the main open problem from their paper. Our algorithm uses tools and ideas developed in the context of computing shortest non-crossing walks in the plane [16]. The following summarizes our algorithm for arbitrary polygons in $\mathbb{R}^2$.

Theorem 1.1. *Let P and Q be simple polygons in $\mathbb{R}^2$, with h punctures. There exists a $2^{O(h^2)} n^{O(h)}$ time algorithm for computing $\delta_F(P, Q)$.*

The NP-hardness result of [5] leaves open the possibility of an approximation algorithm for the problem. We show that such a result is unlikely in $\mathbb{R}^3$. The formal statement of our result follows.

Theorem 1.2. *Approximating the Fréchet distance between two polyhedral domains in $\mathbb{R}^3$ within a factor of $n^{1/\log\log n}$ is NP-hard.*

We remark that it is not known if there always exists an optimal (or even near-optimal) piecewise linear homeomorphism of polynomial complexity. In fact, it is not known whether any of the problems considered is in NP. The proof of [5] only shows that computing the Fréchet distance between polygons in $\mathbb{R}^2$ is NP-hard, and it is open whether it is in NP.

1.3 Our techniques

The exact algorithm in $\mathbb{R}^2$. Our exact algorithm for computing the Fréchet distance between two polygonal domains starts by picking a small set of diagonals in P and guessing their image in Q. Then, our algorithm cuts P along these diagonals, and Q along their maps, thus reducing the problem to computing the Fréchet distance between two simple polygons. In order to bound the number of possible images of a diagonal, we need to bound the number of possible choices for its endpoints and its homotopy class. To achieve the former, we look into the refined free space diagram [3] and prove that there is a quadratic number of possibilities for each endpoint. For the latter purpose, we exploit ideas from the problem of computing non-crossing walks in a planar arrangement [16]. More specifically, we consider a collection of segments that cut Q into a topological disk. We observe that if the number of crossings of the diagonal maps with any of these segments is sufficiently large, then one of the diagonal maps can be shortcut along a straight line segment without introducing crossings among diagonal maps. Following Buchin *et al.* [6], we observe that such a shortcutting does not increase the Fréchet distance between the diagonals of P and their maps in Q. This shows that the number of homotopy classes is bounded by a function of the number of punctures, and by the above discussion, implies the algorithm.

Inapproximability in $\mathbb{R}^3$. Our inapproximability result is obtained by reducing the Closest Vector Problem under ℓ_∞ norm (CVP_∞) to the problem of computing the Fréchet distance between two polyhedral domains in $\mathbb{R}^3$. Our inapproximability factor follows by a result due to Dinur [13] who showed that CVP_∞ is NP-hard to approximate within a factor of $n^{1/\log\log n}$.

1.4 Organization

The rest of the paper is organized as follows. Section 2 introduces background and notation. Section 3 introduces skeleton maps and explains their use in the computation of Fréchet distance. Section 4 presents the exact algorithm for $\mathbb{R}^2$. Finally, Section 5 presents the inapproximability result for $\mathbb{R}^3$.

2 Background and notation

Given two points $p, q \in \mathbb{R}^2$, we use $\overline{(p,q)}$ to refer to the line segment with endpoints p and q. We say that a path with endpoints p and q is a (p,q)*-path*. In particular, $\overline{(p,q)}$ is a (p,q)-path. For a simple path $p \in \mathbb{R}^2$ and for points $x, y \in p$, we use $p[x,y]$ to refer to the subpath of p with endpoints x and y. We say that two paths p and q *cross* if for any path p' that is obtained from p by an infinitesimal perturbation, we have $p' \cap q \neq \emptyset$.

Free space diagrams. Let us recall the notion of a free space diagram, introduced by Alt and Godau [3]. Let p, q be two closed curves in $\mathbb{R}^2$, and let $\delta > 0$. The *double free space diagram* F_δ (also denoted as F when δ is clear from the context), is a data structure that is represented by a $[0, 2|p|] \times [0, |q|]$ rectangle, where $|\cdot|$ denotes the curve length. Each point of the double free space diagram corresponds to a pair (x, y) where $x \in p$ and $y \in q$. A pair (x, y) is *feasible* if and only if $\|x - y\|_2 \leq \delta$. The collection of all feasible pairs is called the *feasible subspace* of the free space diagram. An orientation preserving homeomorphism $f : p \to q$ corresponds to a monotonically increasing path ρ in F that has endpoints $(x_0, 0)$ and $(x_0 + |p|, |q|)$, for some $x_0 \in [0, |p|]$. The homeomorphism f has Fréchet length at most δ if it resides within the feasible subspace of F.

Suppose that p and q are piecewise linear curves, and let $p_0, \ldots, p_n$, and $q_1, \ldots, q_m$ be the vertices of p and q, respectively. The vertical lines with x-coordinates corresponding to p_i's and the horizontal lines with y-coordinates corresponding to q_i's partition F into a collection of *cells*. A cell represents all the pairs of points from a specific pair of segments of p and q. The feasible subset of a cell is the intersection of a certain ellipse with the cell, and so it is convex [3].

Consider all points on the vertical segments of F that are also on the boundary of the feasible region and add horizontal lines with their y-coordinates to further refine F. By the convexity of the feasible region inside a cell, it follows that there are at most four such points in each cell and so $O(nm)$ such points overall. Perform the same refinement by adding similar vertical lines as well to obtain a $O(nm) \times O(nm)$ refined grid. Following Alt and Godau we call this diagram the *refined free space diagram* and we denote it by $\mathcal{F}_\delta$ (or just $\mathcal{F}$ when δ is clear from the context). We refer

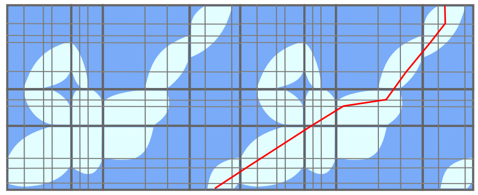


Figure 1: A path in a refined free space diagram.

to the cells and segments of this refined diagram as *refined cells* and *refined segments*, respectively. Each vertical refined segment of $\mathcal{F}$ at a vertex of P and each horizontal refined segment of $\mathcal{F}$ at a vertex of Q is either completely feasible or completely infeasible (see Figure 1).

3 Skeletons and skeleton maps

Let us now briefly describe the algorithm of [6] for computing the Fréchet distance between two simple polygons in $\mathbb{R}^2$. They first show that a map f_s between the boundaries of a convex polygon and a simple polygon can be extended to a map between the polygons with Fréchet length arbitrarily close to $\delta_F(f_s)$.

Lemma 3.1 (Buchin *et al.* [6]). *Let P be a convex polygon, let Q be a simple polygon in $\mathbb{R}^2$, and let $f_s : \partial P \to \partial Q$ be a homeomorphism. Then, for any $\varepsilon > 0$, there exists a homeomorphism $f : P \to Q$, extending f_s, and such that $\delta_F(f) \leq \delta_F(f_s) + \varepsilon$.*

Given two simple polygons $P, Q \subset \mathbb{R}^2$, Buchin et al. partition P into convex regions. They also obtain a combinatorially equivalent partition of Q into simple regions. Then, they use Lemma 3.1 to find a collection of maps from each convex region of P to its corresponding simple region of Q. This collection induces the desired homeomorphism between P and Q. Following their idea, we define skeletons and skeleton maps for polygonal domains.

3.1 Skeletons

Let P and Q be polygonal domains with boundaries $\partial P = b_0 \cup \cdots \cup b_h$ and $\partial Q = c_0 \cup \ldots \cup c_h$, respectively, where each b_i, c_j is a closed polygonal curve. Suppose, without loss of generality, that b_0 and c_0 are the outer boundary components.

Let $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ be any set of pairwise interior-disjoint straight line segments that partitions P into convex polygons, and let s_i, t_i be the endpoints of σ_i. Σ, in particular, can be the set of diagonals of any triangulation of P. We refer to the segments σ_i as *diagonals*. We refer to $\mathcal{S}(P) = \partial P \cup \bigcup_{\sigma \in \Sigma} \sigma$ as the *skeleton* of P (see Figure 2).

A continuous map $f_s : \mathcal{S}(P) \to Q$ is called a *skeleton map*. For each $i \in \{1, \ldots, k\}$, let $\gamma_i = f_s(\sigma_i)$. We refer to the paths $\gamma_1, \ldots, \gamma_k$ as the diagonals of Q. We also refer to $\mathcal{S}(Q) = f_s(\mathcal{S}(P))$ as the skeleton of Q (w.r.to f_s). A skeleton map f_s is called *admissible* if the following conditions hold:

(A1) There exists a permutation $\pi : \{0, \ldots, h\} \to \{0, \ldots, h\}$, with $\pi(0) = 0$, and such that for any $i \in \{0, \ldots, h\}$, the map $f_s|_{b_i}$ is an orientation-preserving homeomorphism between the cycles b_i and $c_{\pi(i)}$.

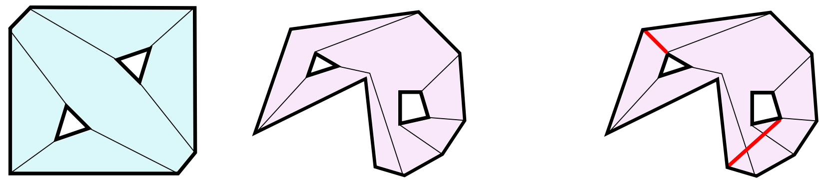


Figure 2: P and Q, with their diagonals Σ and Γ (left), and the segments that cut Q into a disk drawn in red (right).

(A2) For each $i \in \{1, \ldots, k\}$, we have that $f_s|\sigma_i$ is a homeomorphism between σ_i and γ_i. Moreover, the collection of paths $\gamma_1, \ldots, \gamma_k$ is pairwise non-crossing.

(A3) Intuitively, we require that f_s induces a combinatorially equivalent drawing of the planar map corresponding to $\mathcal{S}(P)$. Formally, let $i \in \{1, \ldots, k\}$, and let $v \in \{s_i, t_i\}$ be an endpoint of σ_i. Note that the neighborhood of v in P intersects two segments from some boundary component b_j, and at least one segment from the diagonals of P. Let $\ell_1, \ldots, \ell_t$ be these segments. Then, the circular ordering of $\ell_1, \ldots, \ell_t$ around v is the same as the circular ordering of $f_s(\ell_1), \ldots, f_s(\ell_t)$ around $f_s(v)$ (see Figure 2).

The following lemma, which is similar in spirit to Lemma 3.1, guarantees that a skeleton map of small Fréchet length implies an actual map of small Fréchet length.

Lemma 3.2. *Let $f_s : \mathcal{S}(P) \to \mathcal{S}(Q)$ be an admissible skeleton map, and $\delta_F(f_s)$ be its Fréchet length. Then, for any $\varepsilon > 0$, there exists a homeomorphism $f : P \to Q$, such that $\delta_F(f) \leq \delta_F(f_s) + \varepsilon$.*

Proof. Cutting P along the set of diagonals in Σ we obtain a partition into convex polygons $P_1, \ldots, P_\ell$. By infinitesimally perturbing f_s we can obtain a skeleton map f'_s such that the images of the diagonals $f'_s(\sigma_1), \ldots, f'_s(\sigma_k)$ are interior disjoint and $\delta_F(f'_s) \leq \delta_F(f_s) + \varepsilon/2$. Consider $f'_s|_{\partial P_i}$ for each i, and observe that the range of this map is the boundary of some simple polygon $Q_i \subseteq Q$. Lemma 3.1 implies that $f'_s|_{\partial P_i}$ can be extended to a homeomorphism between P_i and Q_i of Fréchet length at most $\delta_F(f'_s|_{\partial P_i}) + \varepsilon/2 \leq \delta_F(f'_s) + \varepsilon/2 \leq \delta_F(f_s) + \varepsilon$. Taking the union of all such maps we obtain the desired f, concluding the proof. $\square$

3.2 Shortcutting diagonals

Buchin *et al.* [6] observe that when P and Q are simply-connected polygons, without loss of generality, each diagonal of P can be mapped to a shortest geodesic path within Q. Unfortunately, this is not true in the case of punctured polygons. We now derive a generalization of this property in our setting that takes into account the homotopy class of the diagonals and their images. We first recall the following auxiliary lemma from Buchin *et al.* [6].

Definition 3.3 (Shortcutting)**.** *Let $\gamma : [0,1] \to \mathbb{R}^2$ be a path, and let $0 \leq t_1 < t_2 \leq 1$. Let γ' be the path obtained from γ by replacing $\gamma[t_1, t_2]$ with the line segment $\overline{(\gamma(t_1), \gamma(t_2))}$. That is, $\gamma' = \gamma[0, t_1] \circ \overline{(\gamma(t_1), \gamma(t_2))} \circ \gamma[t_2, 1]$, where $\circ$ denotes path composition. Then, we say that γ' is obtained from γ via a* shortcutting *operation.*

Lemma 3.4 (Buchin *et al.* [6])**.** *Let $\ell \subset \mathbb{R}^2$ be any line segment, and let γ, γ' be paths, such that γ' is obtained from γ via a shortcutting operation. Then, we have $\delta_F(\ell, \gamma') \leq \delta_F(\ell, \gamma)$.*

We will use the following result, implicit in the work of Colin de Verdière and Erickson [11], which in turn follows by a result of Scott and Hass [20].

Lemma 3.5 (Colin de Verdière and Erickson [11], Scott and Hass [20])**.** *Let γ_1, γ_2 be simple non-homotopic paths in Q. For any $i \in \{1, 2\}$, let γ'_i be a shortest path in the homotopy class of γ_i. If γ_1 and γ_2 do not cross, then γ'_1 and γ'_2 do not cross, either.*

Let γ, γ' be two homotopic simple paths. Recall that two homotopic paths have common endpoints. A component of $\mathbb{R}^2 \setminus (\gamma \cup \gamma')$ is called a *bigon* if it is simply connected and its boundary is composed by one subpath of γ and one subpath of γ'. The following lemma is a special case of Lemma 3.1 of Hass and Scott [20].

Lemma 3.6 (Hass and Scott [20], Lemma 3.1)**.** *Let γ and γ' be distinct homotopic simple paths in a polygonal domain Q. Then, one of the connected components of $\mathbb{R}^2 \setminus (\gamma \cup \gamma')$ is a bigon.*

We use Lemma 3.6 and Lemma 3.4 to obtain the following result, which allows us to assume w.l.o.g. that the diagonals of P are mapped into shortest homotopic paths in Q. The high-level technique is similar to the proof of Lemma 4 of Buchin *et al.* [6]. Due to lack of space the proof of Lemma 3.7 is given in Section **??**.

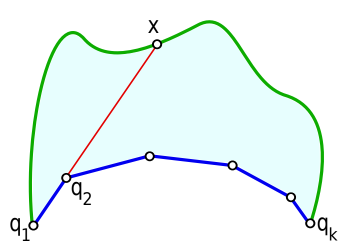
Lemma 3.7. *Let γ' be a simple path in Q, and let γ be a shortest path in the homotopy class of γ'. Then, there exists a sequence of paths $\gamma^0, \ldots, \gamma^t$, with $\gamma^0 = \gamma'$, $\gamma^t = \gamma$, such that:*

(1) For any $i \in \{0, \ldots, t\}$, γ^i is homotopic to γ'.

(2) For any $i \in \{0, \ldots, t-1\}$, γ^{i+1} is obtained from γ^i via a shortcutting operation.

Proof. If $\gamma = \gamma'$, then the assertion holds trivially. Otherwise, Lemma 3.6 implies there exists a bigon B in $\mathbb{R}^2 \setminus (\gamma \cup \gamma')$. Let $\beta \subseteq \gamma$ and $\beta' \subseteq \gamma'$ be the subpaths such that $\partial B = \beta \cup \beta'$. In the rest of the proof we show that we can substitute β' with β in γ' via a sequence of shortcutting operations. Moreover, since any bigon is simply connected, it follows that the new curve $(\gamma' \setminus \beta') \cup \beta$ is homotopic to γ'. Therefore, we can repeatedly eliminate all bigons one at the time. Every time we eliminate a bigon we decrease the number of crossings between γ and γ', and therefore the process terminates after a finite number of eliminations.

It remains to show how to eliminate the bigon B via a sequence of shortcutting operations. Let the vertices of β be $q_1, \ldots, q_k$.



We use induction on k to show how to substitute β' with β via a sequence of shortcutting operations. If $k = 2$ then substituting β' with the line segment β is clearly a shortcutting operation on γ'. Otherwise, observe that β is a shortest (q_1, q_k)-path within B, and so the angle (q_1, q_2, q_3) in the interior of B is larger than π. Shoot a ray r along (q_1, q_2) and from q_1 to q_2, and let x be its first intersection point with $\gamma \cup \gamma'$. Because (q_1, q_2, q_3) is reflexive, the segment (q_1, x) must reside inside B. Moreover, since β is a shortest path, we have $x \notin \beta \subseteq \gamma$, and thus $x \in \beta' \subseteq \gamma'$. We can now replace the subpath $\beta'[q_1, x]$ by the segment $\overline{(q_1, x)}$ via a shortcutting operation. Let $\beta'' = (\beta' \setminus \beta'[q_1, x]) \cup \overline{(q_1, x)}$ be the new path. This gives rise to a new bigon B' bounded by $\beta'' \cup \beta[q_2, q_k]$. By induction hypothesis, B' can be eliminated via a sequence of shortcutting operations, which concludes the proof. $\square$

Corollary 3.8. *Let $\ell \subset \mathbb{R}^2$ be any line segment, and let γ, γ' be paths, such that γ' is the shortest path in the homotopy class of γ. Then, we have $\delta_F(\ell, \gamma') \leq \delta_F(\ell, \gamma)$.*

Proof. Follows by Lemma 3.4 and induction on the sequence given by Lemma 3.7. $\square$

Lemma 3.9. *If there exists an admissible skeleton map $g_s : \mathcal{S}(P) \to Q$, then there exists an admissible skeleton map $g'_s : \mathcal{S}(P) \to Q$ satisfying the following conditions.*

(1) $\delta_F(g'_s) \leq \delta_F(g_s)$.

(2) For any $i \in \{1, \ldots, k\}$, $g_s(\sigma_i)$ and $g'_s(\sigma_i)$ are in the same homotopy class. Further, $g'_s(\sigma_i)$ is a shortest path in its homotopy class.

Proof. For any $i \in \{1, \ldots, k\}$, let γ'_i be a shortest path in the homotopy class of $\gamma_i = g(\sigma_i)$, let $\Gamma' = \{\gamma'_1, \ldots, \gamma'_k\}$, and let $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$. Since the diagonals in Σ are pairwise non-crossing and g is a homeomorphism, we have that the paths in Γ are pairwise non-crossing. By Lemma 3.5 it follows that the paths in Γ' are also pairwise non-crossing.

For any $i \in \{1, \ldots, k\}$, we have by Corollary 3.8 that $\delta_F(\gamma'_i, \sigma_i) \leq \delta_F(\gamma_i, \sigma_i)$. It follows that there exists a skeleton map $g'_s : \mathcal{S}(P) \to Q$ that maps every diagonal σ_i onto the path γ'_i, with $\delta_F(g'_s) \leq \delta_F(g_s)$. We next argue that g'_s is admissible. Consider a vertex $v \in \mathcal{S}(P)$ of degree greater than two. Since the paths in Γ are pairwise non-crossing and are contained in the same homotopy classes as the paths in Γ', it is immediate that the clockwise orderings around $g_s(v)$ and $g'_s(v)$ of the images of the paths of the skeleton incident to v is the same under g_s and g'_s respectively. It follows that g'_s admissible, concluding the proof. $\square$

3.3 Bounding the number of possible homotopy classes of a diagonal

By the preceding discussion we may restrict our attention to skeleton maps that map every diagonal of P onto some shortest homotopic path in Q. Therefore, in order to determine the image of a diagonal it suffices to guess the endpoints and the homotopy class of its image. In a simply connected polygon, the endpoints of a curve completely determine its homotopy class. Unfortunately, there are infinitely many homotopy classes of curves with the same pair of endpoints in a non-simply connected polygonal domain.

We now derive a bound on the number of possible homotopy classes for images of diagonals of P. Our argument uses tools from the work of Erickson and Nayyeri [16] on computing non-crossing walks in a planar polygonal arrangement (for similar ideas on crossing patterns see Schaefer *et al.* [22, 23]).

Let $\{r_1, r_2, \ldots, r_h\}$ be a set of disjoint line segments that cut Q into a topological disk (see Figure 2). The following lemma is implicit in [16].

Lemma 3.10 (Erickson and Nayyeri [16])**.** *Let $\{\gamma_1, \ldots, \gamma_k\}$ be a collection of pairwise non-crossing paths in Q with endpoints on ∂Q. Then, there exists a collection of paths $\{\gamma'_1, \ldots, \gamma'_k\}$ satisfying the following conditions.*

(1) For any $i \in \{1, \ldots, k\}$, the path γ'_i is obtained from γ_i via a sequence of zero or more short-cutting operations.

(2) The collection of paths $\{\gamma'_1, \ldots, \gamma'_k\}$ is pairwise non-crossing.

(3) For any $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, h\}$, γ'_i crosses r_j at most 2^{2h-2} times.

We next show that we can shortcut the images of the diagonals without violating admissibility.

Lemma 3.11. *Let $\sigma_1, \ldots, \sigma_k$ be the diagonals of P. Let g_s be an admissible skeleton map. For any $i \in \{1, \ldots, k\}$, let γ'_i be a path in Q that is obtained by performing a sequence of zero or more shortcutting operations on $\gamma_i = g_s(\sigma_i)$. Suppose further that the collection of paths $\{\gamma'_1, \ldots, \gamma'_k\}$ is pairwise non-crossing. Then, there exists an admissible skeleton map $g'_s : \mathcal{S}(P) \to \partial Q \cup (\bigcup_{i=1}^k \gamma'_i)$, with $\delta_F(g'_s) \leq \delta_F(g_s)$, and such that for every $i \in \{1, \ldots, k\}$, $g'_s|_{\sigma_i}$ is a homeomorphism between σ_i and γ'_i.*

Proof. By Lemma 3.4 we have that for any $i \in \{1, \ldots, k\}$, $\delta_F(\sigma_i, \gamma'_i) \leq \delta_F(\sigma_i, \gamma_i)$. It follows that there exists a homeomorphism $\phi_i : \sigma_i \to \gamma'_i$, with $\delta_F(\phi_i) \leq \delta_F(\sigma_i, \gamma'_i) \leq \delta_F(\sigma_i, \gamma_i) \leq \delta_F(g_s)$. Setting $g'_s = g_s|_{\partial P} \cup (\bigcup_{i=1}^k \phi_i)$, we obtain a skeleton map $g'_s : \mathcal{S}(P) \to \partial Q \cup (\bigcup_{i=1}^k \gamma'_i)$, with $\delta_F(g'_s) \leq \delta_F(g_s)$, and such that for every $i \in \{1, \ldots, k\}$, $g'_s|_{\sigma_i}$ is a homeomorphism between σ_i and γ'_i. It remains to argue that g'_s is admissible. Conditions (A1) and (A2) in the definition of admissibility are clearly satisfied. It remains to verify condition (A3). Let $i \in \{1, \ldots, k\}$, let $v \in \partial P$ be and endpoint of the diagonal σ_i. Observe that performing a shortcutting operation on the image of any diagonal γ_j does not change the circular ordering of the images of the paths of the skeleton of Q around the image of v. Since the original skeleton map g_s is admissible, and therefore satisfies condition (A3), it follows that the resulting skeleton map g'_s also satisfies condition (A3). Therefore, g'_s is admissible, concluding the proof. $\square$

Lemma 3.12. *Let $\sigma_1, \ldots, \sigma_k$ be the diagonals of P. If there exists an admissible skeleton map g_s, then there exists an admissible skeleton map g'_s, with $\delta_F(g'_s) \leq \delta_F(g_s)$, such that for each $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, h\}$ the path $g'_s(\sigma_i)$ crosses r_j at most 2^{2h-2} times.*

Proof. For any $i \in \{1, \ldots, k\}$, let $\gamma_i = g_s(\sigma_i)$. If for any $i \in \{1, \ldots, k\}$ and for any $j \in \{1, \ldots, h\}$ the path γ_i intersects r_j at most 2^{2h-2} times, then we are done by setting $g'_s = g_s$. Otherwise, by Lemma 3.10 there exists a collection of pairwise non-crossing paths $\gamma'_1, \ldots, \gamma'_k$ such that for any $i \in \{1, \ldots, k\}$ the path γ'_i is obtained from γ_i via a sequence of zero or more shortcutting operations. By Lemma 3.11 it follows that there exists a skeleton map $g'_s : \mathcal{S}(P) \to \partial Q \cup (\bigcup_{i=1}^k \gamma'_i)$, with $\delta_F(g'_s) \leq \delta_F(g_s)$, that sends every diagonal σ_i onto the path γ'_i. Since any γ'_i intersects any r_j at most 2^{2h-2} times, the assertion follows. $\square$

We next derive an upper bound on the number of possible homotopy classes. The proof of the following lemma uses an argument from Erickson and Nayyeri [16].

Lemma 3.13. *Let $\sigma_{\iota_1}, \ldots, \sigma_{\iota_h} \in \Sigma$ be a collection of diagonals of P that cut P into a disk. Let g_s be an admissible skeleton map. Then, there exists an efficiently computable set of h-tuples $\mathcal{X} = \{\langle \chi_{i,1}, \ldots, \chi_{i,h} \rangle\}_{i \in I}$, with $|I| = 2^{O(h^2)}$, where each $\chi_{i,j}$ is a homotopy class of paths in Q, satisfying the following. There exists $i \in I$ and an admissible skeleton map g'_s, such that for each $j \in \{1, \ldots, h\}$ the path $g'_s(\sigma_{\iota_j})$ is in the homotopy class $\chi_{i,j}$, and $\delta_F(g'_s) \leq \delta_F(g_s)$.*

Proof. Let g'_s be the skeleton map obtained by applying Lemma 3.12 on g_s, and let $r_1, \ldots, r_h$ be a set of diagonals cutting Q to a topological disk. For any $i \in \{1, \ldots, h\}$, let $\gamma_{\iota_i} = g'_s(\sigma_{\iota_i})$. It suffices to bound the number of h-tuples of possible homotopy classes for the paths $\gamma_{\iota_1}, \ldots, \gamma_{\iota_k}$ in Q, provided that every γ_{ι_i} crosses every r_j at most 2^{2h-2} times. Cutting Q along the diagonals $r_1, \ldots, r_h$ we obtain a disk Q', with a collection of disjoint paths $a_1, \ldots, a_{2h} \subset \partial Q'$ corresponding to the sides of $r_1, \ldots, r_h$. Contracting each a_i into a point a'_i we obtain a disk Q''. Every path γ_{ι_i} gives rise to a collection of paths in Q'', each having both endpoints in the set $a'_1, \ldots, a'_{2h}$. Let $T = \{\tau_1, \ldots, \tau_t\}$ be the collection of all such paths. We obtain a weighted triangulation of Q'' as follows: For every pair of points a'_i, a'_j that is connected via a path in T, we add an edge in the triangulation with weight equal to the number of paths in T having endpoints a'_i and a'_j. We complete the resulting set of edges to a triangulation by greedily adding a set of edges of weight 0. Observe that since the collection of paths $\gamma_{\iota_1}, \ldots, \gamma_{\iota_k}$ is pairwise non-crossing, it follows that we can recover $\gamma_{\iota_1}, \ldots, \gamma_{\iota_k}$ from the weighted triangulation. Therefore, it suffices to bound the number of such weighted triangulations. The number of unweighted triangulations is equal to the $2h$-th Catalan number, which is $2^{O(h)}$. Since every γ_{ι_i} crosses every r_j at most 2^{2h-2} times, it follows that the maximum weight is at most $h2^{2h-2}$. Since every triangulation has $O(h)$ edges, the number of possible weight assignments for a fixed triangulation is at most $2^{O(h^2)}$. It follows that the total number of possible weighted triangulations is at most $2^{O(h)}2^{O(h^2)} = 2^{O(h^2)}$, concluding the proof. □

3.4 Bounding the number of possible endpoints of a diagonal

Lemma 3.9 implies that the image of a diagonal can be computed if its homotopy class and its endpoints are known. Lemma 3.12 bounds the possibilities for the homotopy class of a diagonal image. The following discretization technique bounds the number of possibilities for each endpoint of a diagonal image.

Lemma 3.14. *Let $f_s : \mathcal{S}(P) \to Q$ be an admissible skeleton map, and let $\pi : \{0, 1, \ldots, h\} \to \{0, 1, \ldots, h\}$ be the permutation such that for any $i \in \{0, \ldots, h\}$, the map f_s induces a homeomorphism between b_i and $c_{\pi(i)}$. Then, there exists an admissible skeleton map f'_s satisfying the following conditions.*

(1) $\delta_F(f'_s) \le \delta_F(f_s)$.

(2) For any $i \in \{0, \ldots, h\}$, the map f'_s induces a homeomorphism between b_i and $c_{\pi(i)}$.

(3) For any $i \in \{0, \ldots, h\}$, let $\mathcal{F}^i$ be the refined free space diagram for b_i and $c_{\pi(i)}$, and let ρ'_i be the path in $\mathcal{F}^i$ corresponding to the homeomorphism $f'_s|_{b_i}$. Then for any vertex $x \in b_i$, $\rho'_i(x)$ is an endpoint of a refined vertical segment in $\mathcal{F}^i$.

Proof. For any $i \in \{0, \ldots, h\}$, let ρ_i be the path in $\mathcal{F}^i$ corresponding to the homeomorphism $f_s|_{b_i}$.

If for all $i \in \{0, \ldots, h\}$, and for any vertex $x \in b_i$, we have that $\rho_i(x)$ is an endpoint of a refined vertical segment in $\mathcal{F}^i$, then the skeleton map $f'_s = f_s$ satisfies the assertion, and there is nothing to prove. Otherwise, suppose that there exists $i \in \{0, \ldots, h\}$, and a vertex $x \in b_i$ such that $\rho_i(x)$ is not an endpoint of a refined vertical segment in $\mathcal{F}^i$. Pick such a vertex $x \in b_i$ with minimum horizontal coordinate. Suppose that $\rho_i(x)$ is strictly on the interior of a refined vertical segment $((x, y), (x, y')) \in C$, where C is a non-refined cell of the free space diagram. Let $v = (v_x, v_y)$ and $w = (x, \rho_i(x))$ be the intersection of ρ_i with the boundary of C. Finally, let $\rho'_i = \rho_i[a, v] \circ \overline{(v, (x, y))} \circ \overline{((x, y), (x + \varepsilon, \rho_i(x + \varepsilon)))} \circ \rho_i[(x + \varepsilon, \rho_i(x + \varepsilon)), z]$, where a and z are (the equivalent) endpoints of ρ_i, and $\varepsilon > 0$ is sufficiently small so that the segment $\overline{((x, y), (x + \varepsilon, \rho_i(x + \varepsilon)))}$ is contained in the feasible region of $\mathcal{F}^i$ (note that such ε always exists due to the convexity of the feasible region inside each cell of the free space diagram).

If v is on the horizontal boundary of C then $v_y \le y$. Otherwise, if v is on the other vertical boundary of C then because its horizontal coordinate is smaller than x, it follows by the choice of x that it is an endpoint of a refined vertical segment. We again obtain $v_y \le y$. Therefore, in either case, the path ρ'_i is monotone. By the convexity of the feasible region inside C, it follows that $\overline{(v, (x, y))}$ is contained inside the feasible region. It follows that ρ'_i is feasible. Finally, let σ be a diagonal with an endpoint x that is mapped to a path γ with an endpoint $\rho_i(x)$. Observe that the Fréchet distance between the point x and the segment $\overline{(\rho_i(x), y)}$ is at most $\delta_F(f_s)$ because the entire segment is feasible. Thus, the Fréchet distance between σ and $\gamma \circ \overline{(\rho_i(x), y)}$ is at most

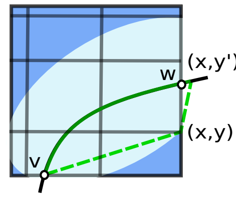


Figure 3: A subpath of ρ_i that intersects C, the solid green path is substituted by the dashed green path to obtain ρ_i'.

δ. Therefore, changing the endpoint of a diagonal within a refined segment does not affect the feasibility of any other diagonals.

We repeat the above process, each time modifying the homeomorphism between some pair b_i and $c_{\pi(i)}$, until there are no vertices v violating the assertion. Let f_s' be the resulting skeleton map. By the preceding discussion, it remains to show that f_s' is admissible. Condition (A1) from the definition of admissibility is clearly satisfied, since we only modify the images of the diagonals. Observe that at each step, we modify the current skeleton map by composing it with some homeomorphism $b_i \to c_{\pi(i)}$ for some $i \in \{1, \ldots, h\}$. It follows that the new collection of images of the diagonals remain pairwise non-crossing. Moreover, it is immediate that the clockwise ordering of the paths in the skeleton of Q around every vertex in ∂Q remain the same. Therefore, conditions (A2) and (A3) are also satisfied, which concludes the proof. □

4 An exact algorithm for polygonal domains in $\mathbb{R}^2$

In this section we describe an exact polynomial time algorithm to compute the Fréchet distance between two polygonal domains in $\mathbb{R}^2$ with a constant number of boundary components. The proof of the following lemma is essentially identical to the proof for the case of simply connected polygons given by Buchin *et al.* [6], so it is omitted. In light of Lemma 4.1, for the remainder of this section, we focus on obtaining an algorithm for the decision version of the problem.

Lemma 4.1. *Let P and Q be polygonal domains in $\mathbb{R}^2$. There is a polynomial size set S of real numbers that contains the value $\delta_F(P,Q)$. Moreover, S can be computed in polynomial time.*

4.1 An auxiliary algorithm

We now present an auxiliary algorithm that will be used as a subroutine in our exact algorithm for arbitrary polygons. The input is two simply connected polygons $\widetilde{P}, \widetilde{Q} \subset \mathbb{R}^2$, and a partial homeomorphism between their boundaries. The goal is to compute an orientation preserving homeomorphism between $\widetilde{P}$ and $\widetilde{Q}$ of minimum Fréchet length that extends the given partial homeomorphism of their boundaries.

We are now ready to obtain our auxiliary algorithm. We note that Theorem 15 of Buchin *et al.* [6] can be extended to obtain the same result. The precise statement follows.

Lemma 4.2. *Let $\widetilde{P}, \widetilde{Q} \subset \mathbb{R}^2$ be simply connected polygons, and let $\Sigma = \{\sigma_1, \ldots, \sigma_k\}$ be a set of diagonals of $\widetilde{P}$ that partition it into convex regions. Let $\alpha_1, \ldots, \alpha_t \subset \partial\widetilde{P}$ be pairwise disjoint subpaths of $\partial\widetilde{P}$ that appear in this order in a clockwise traversal of $\partial\widetilde{P}$, and that are internally disjoint from the endpoints of the diagonals in Σ. Similarly, let $\beta_1, \ldots, \beta_t \subset \partial\widetilde{Q}$ be pairwise disjoint subpaths of $\partial\widetilde{Q}$ that appear in this order in a clockwise traversal of $\partial\widetilde{Q}$. For any $i \in \{1, \ldots, t\}$, let $\phi_i : \alpha_i \to \beta_i$ be an orientation-preserving homeomorphism (where every path is considered to* *be oriented according to a clockwise traversal of $\partial\widetilde{P}$ and $\partial\widetilde{Q}$ respectively). Then, there exists a polynomial-time algorithm which given $\delta > 0$ decides whether there exists an orientation-preserving homeomorphism $f : \widetilde{P} \to \widetilde{Q}$ with $\delta_F(f) \leq \delta$, subject to the constraint that for any $i \in \{1, \ldots, t\}$, we have $f|_{\alpha_i} = \phi_i$. Moreover, if such a homeomorphism exists, the algorithm outputs a homeomorphism f' with $\delta_F(f') \leq \delta + \varepsilon$, for any $\varepsilon > 0$.*

Proof. Let $\mathcal{F}$ be the refined free space diagram of $\widetilde{P}$ and $\widetilde{Q}$ with respect to δ, and let $\mathcal{F}^\ell$ be its left half as defined above. Let $p, p' \in \partial\widetilde{P}$ and $q, q' \in \partial\widetilde{Q}$. We say that (p', q') is *reachable* from (p, q) if there exists a path ρ' in $\mathcal{F}^\ell$ from (p, q) to (p', q') with the following properties.

(1) The path ρ' is monotonically increasing, and it resides inside the feasible region of $\mathcal{F}^\ell$.

(2) Let $\sigma_i = (s_i, t_i) \in \Sigma$ be any diagonal such that $p \leq s_i, t_i \leq p'$, and let γ_i be the shortest $(\rho(s_i), \rho(t_i))$-path in $\widetilde{Q}$. Then, we have $\delta_F(\gamma_i, \sigma_i) \leq \delta$.

(3) Let α_i be any subpath with endpoints a and a'. If $p \leq a, a' \leq p'$ then $\rho'[a, a']$ and ϕ_i are equivalent. That is for any $p \leq x \leq p'$ we have $\rho'(x) = \phi(x)$ (interpreting $\rho'(x)$ as a point on $\partial\widetilde{Q}$).

As explained above, we can assume that if there is a homeomorphism between $\partial\widetilde{P}$ and $\partial\widetilde{Q}$ with Fréchet length δ then there is one with Fréchet length at most δ whose corresponding path in $\mathcal{F}^\ell$ contains $(0, 0)$ (and so $(|\partial\widetilde{P}|, |\partial\widetilde{Q}|)$. Based on the second property of reachability and Lemma 3.2 such a homeomorphism can be extended to a homeomorphism between $\widetilde{P}$ and $\widetilde{Q}$ with Fréchet length arbitrary close to δ. Our algorithm checks whether $(|\partial\widetilde{P}|, |\partial\widetilde{Q}|)$ is reachable from $(0, 0)$.

Let Y be the set of all vertical coordinates of all grid vertices in $\mathcal{F}^\ell$ (one can interpret Y as a set of points on ∂Q). For each pair of vertices $p, p' \in \partial\widetilde{P}$ we define the reachability directed graph $H_{p,p'}$ by its vertex set and edge set. The vertex set of $H_{p,p'}$ is $\{p, p'\} \times Y$. There is a directed edge $(p, y) \to (p', y')$ if and only if (p', y') is reachable from (p, y). Our algorithm checks whether $(0, 0) \to (|\partial\widetilde{P}|, |\partial\widetilde{Q}|)$ is an edge in $H_{0,|\partial\widetilde{P}|}$ using dynamic programming.

To this end, our algorithm constructs $H_{p,p'}$, recursively, for all pair of points $p, p' \in \partial\widetilde{P}$ with the property that there is no diagonal or fixed path with exactly one endpoint in $[p, p']$. We refer to the paths α_i's as fixed paths. We proceed through a case analysis of $[p, p']$.

Case I. If the interval $[p, p']$ does not contain any endpoint of the diagonals or the fixed paths then we compute $H_{p.p'}$ by looking at the $\mathcal{F}^\ell$ using standard techniques; see [3].

Case II. If p and p' are the endpoints of a fixed path α_i, then $H_{p,p'}$ contains only one edge that is $(p, \phi_i(p)) \to (p', \phi_i(p'))$.

Case III. The final case is when the interval $[p, p']$ contains a nonempty subset of the fixed path endpoints and diagonal endpoints.

We say that a diagonal $\sigma_i = (s_i, t_i)$ is exposed in $[p, p']$ if (i) $p \leq s_i, t_i \leq p'$, and (ii) there is no other diagonal $\sigma_j = (s_j, t_j)$ such that $p \leq s_j, t_j \leq p'$ and $s_j \leq s_i, t_i \leq t_j$. Similarly, we say that a fixed path α_i with endpoints a and a' is exposed in $[p, p']$ if (i) $p \leq a, a' \leq p'$, and (ii) there is no diagonal $\sigma_j = (s_j, t_j)$ such that $p \leq s_j, t_j \leq p'$ and $s_j \leq a, a' \leq t_j$.

Let $x_1, \ldots, x_r$ be the endpoints of all exposed diagonals and exposed fixed paths in $[p, p']$. We define $G_{p,p'}$ to be $H_{p,x_1} \cup H_{x_1,x_2} \cup \ldots \cup H_{x_r,p'}$.

Next we build $H_{p,p'}$ from $G_{p,p'}$. For all $y, y' \in Y$, we add $(p, y) \to (p', y')$ to $H_{p,p'}$ if there is a directed path from (p, y) to (p', y') in $G_{p,p'}$.

Constructing $H_{p,p'}$ requires a polynomial number of searches in a graph of polynomial size if the reachability graphs of all of the subproblems is already constructed. Since we need to consider a quadratic number of subproblems, the total running time of the algorithms is polynomial. $\square$

4.2 The main algorithm

Proof of Theorem 1.1. By Lemma 4.1 it suffices to obtain an algorithm which given some $\delta \geq 0$ decides whether $\delta_F(P, Q) \leq \delta$. Let $f : P \to Q$ be a homeomorphism with $\delta_F(f) = \delta_F(P, Q)$. Let f_s be the admissible skeleton map obtained by restricting f on the skeleton $\mathcal{S}(P) = \partial P \cup (\bigcup_{\sigma \in \Sigma} \sigma)$. There exists a permutation $\pi : \{0, \ldots, h\} \to \{0, \ldots, h\}$, such that f induces a homeomorphism between b_i and c_i. We guess the permutation π. That is, we run the following procedure for every possible permutation π, and output the best solution found, which results in a multiplicative factor of $O(h!) = 2^{O(h \log h)}$ in the running time.

By Lemma 3.14 there exists an admissible skeleton map f'_s with $\delta_F(f'_s) \leq \delta_F(f_s)$, and such that for any $i \in \{1, \ldots, h\}$, for every endpoint x of σ_i, with $x \in b_j$ for some $j \in \{1, \ldots, h\}$, we have that $f'_s(\sigma_i)$ is a vertex in the refined free space diagram $\mathcal{F}_j$ that corresponds to the pair of boundary components b_j and c_j. We guess all the endpoints of $f'_s(\sigma_i)$ for all $i \in \{1, \ldots, h\}$. There is a total of at most $n^{O(h)}$ possibilities.

Let Σ' be a subset of segments in Σ that cut P into a topological disc, with $|\Sigma'| = h$. We can compute Σ' by greedily cutting P along diagonals with endpoints on different boundary components (and updating the set of boundary components after each cut). We may assume, after permuting the indices, and without loss of generality, that $\Sigma' = \{\sigma_1, \sigma_2, \ldots, \sigma_h\}$.

By Lemma 3.13 there exists a collection $\mathcal{X} = \{\langle \chi_{i,1}, \ldots, \chi_{i,h} \rangle\}_{i \in I}$ of efficiently computable h-tuples of homotopy classes of paths in Q, with $|I| = 2^{O(h^2)}$, and an admissible skeleton map f''_s satisfying all the above conditions as f'_s, and such that there exists $i \in I$, such that for every $j \in \{1, \ldots, h\}$, the path $f''_s(\sigma_j)$ is in the homotopy class $\chi_{i,j}$. We compute the set $\mathcal{X}$, and we try all of the $2^{O(h^2)}$ tuples in $\mathcal{X}$, and return the best solution found.

By Lemma 3.9 there exists a skeleton map f'''_s satisfying all the above conditions as f''_s, and such that for every $i \in \{1, \ldots, h\}$, the path $\gamma'''_i = f'''_s(\sigma_i)$ is shortest in its homotopy class. We compute each path γ'''_i in linear time, using the algorithm of Hershberger and Snoeyink [21]. After computing $\Gamma''' = \{\gamma'''_1, \ldots, \gamma'''_h\}$ we check whether the paths in Γ''' are pairwise non-crossing, and whether cutting Q along Γ''' results in more than one connected component. In either of these cases the algorithm disregards Γ''', and proceeds to the next choice of homotopy classes.

If Γ''' passes the above test, then we compute the homeomorphism $f'''_s|_{\sigma_i}$ between σ_i and γ'''_i, using the algorithm of Alt and Godau [3] for the Fréchet distance between polygonal curves.

Let $\widetilde{P}, \widetilde{Q}$ be the simply connected polygons obtained by cutting P along $\sigma_1, \ldots, \sigma_h$, and along $\gamma'''_1, \ldots, \gamma'''_h$ respectively. The paths $\sigma_1, \ldots, \sigma_h$ correspond to pairwise disjoint paths $\alpha_1, \ldots, \alpha_{2h} \subset \partial\widetilde{P}$. Similarly, the paths $\gamma'''_1, \ldots, \gamma'''_h$ correspond to pairwise disjoint paths $\beta_1, \ldots, \beta_{2h} \subset \partial\widetilde{Q}$. Moreover, the maps $f'''_s|_{\sigma_1}, \ldots, f'''_s|_{\sigma_h}$ induce a collection of homeomoprhisms $\phi_1 : \alpha_1 \to \beta_1, \ldots, \phi_{2h} : \alpha_{2h} \to \beta_{2h}$. By Lemma 4.2 we can compute in polynomial time a homeomorphism $\widetilde{f} : \widetilde{P} \to \widetilde{Q}$ of minimum Fréchet length. By the above discussion, for the right choice of the permutation π, and the endpoints and homotopy classes of the paths $\gamma'''_1, \ldots, \gamma'''_h$, we have $\delta_F(\widetilde{f}) \leq \delta_F(\widetilde{P}, \widetilde{Q}) \leq \delta_F(P, Q) + \varepsilon$, for any $\varepsilon > 0$. By the construction of $\widetilde{P}$ and $\widetilde{Q}$ the map $\widetilde{f}$ induces a homeomorphism $f : P \to Q$, with $\delta_F(f) \leq \delta_F(\widetilde{f})$, which completes the description of the algorithm. The total running time for all the above steps $2^{O(h \log h)} n^{O(h)} 2^{O(h^2)} n^{O(1)} = 2^{O(h^2)} n^{O(h)}$, concluding the proof. □

5 Inapproximability for polyhedral domains in $\mathbb{R}^3$

In this section we present our inapproximability result for the Fréchet distance between polyhedral domains in $\mathbb{R}^3$. Our proof follows by a reduction from the Closest Vector Problem (CVP_∞), as defined below. As an intermediate step, we reduce to the problem of computing a minimum cost solution to a linear system of Diophantine equations.

Closest Vector Problem (CVP$_\infty$). An instance to CVP$_\infty$ is a pair (L, d), where $L \in \mathbb{Z}^{d\times n}$ and $b \in \mathbb{Z}^d$. The goal is to find a vector $x \in \mathbb{Z}^n$ minimizing $\|Lx - b\|_\infty$.

Mininimum Diophantine Problem (MDP$_\infty$). An instance to MDP$_\infty$ is a tuple (A, b, c), where $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^n$, and $c \in \{0, 1\}^m$. The goal is to find a vector $x \in \mathbb{Z}^m$, such that $Ax = b$, minimizing $\|c^T x\|_\infty$.

Lemma 5.1 (Dinur [13])**.** *Approximating CVP$_\infty$ within a factor of $n^{1/\log\log n}$ is NP-hard. Moreover, this holds even when all integer values of the instance are polynomially bounded.*

Lemma 5.2. *Let $\phi = (L, b)$ be an instance to CVP$_\infty$, where $L \in \mathbb{Z}^{d\times n}$, $b \in \mathbb{Z}^n$. Let $c \in \mathbb{Z}^{n+d}$ be the concatenation of n zeros and d ones, i.e. $c = 1^n 0^d$. Let $\psi = ([L\ -I], b, c)$ be an instance to MDP$_\infty$. Then, we have that $y \in \mathbb{Z}^d$ is in the lattice L if and only if there exists $x \in \mathbb{Z}^n$ such that $\begin{bmatrix} x \\ y-b \end{bmatrix}$ is a solution to ψ. Moreover, the CVP$_\infty$ cost of the solution y with respect to the instance ϕ is equal to the MDP$_\infty$ cost of the solution $\begin{bmatrix} x \\ y-b \end{bmatrix}$ with respect to the instance ψ.*

Proof. Suppose $y = Lx$ for some $x \in \mathbb{Z}^n$, and let $z = y - b = Lx - b$. It follows that

$$Lx - z = b \Rightarrow [L\ \ -I]\begin{bmatrix} x \\ z \end{bmatrix} = b \Rightarrow [L\ \ -I]\begin{bmatrix} x \\ y-b \end{bmatrix} = b.$$

For the other direction, suppose

$$[L\ \ -I]\begin{bmatrix} x \\ y-b \end{bmatrix} = b \Rightarrow Lx - y + b = b \Rightarrow y = Lx.$$

Finally, the cost of y with respect to CVP$_\infty$ is $||y-b||_\infty$, which is equal to the cost of $[x\ y-b]^T$ with respect to $[L\ \ -I]$ and c. ☐

Combining Lemmas 5.1 and 5.2 we immediately obtain the following.

Corollary 5.3. *Approximating MDP$_\infty$ within a factor of $n^{1/\log\log n}$ is NP-hard. Moreover, this holds even when all integer values of the instance are polynomially bounded.*

5.1 The gadgets

In the remainder we use a constant $\varepsilon = 1/100$.

Definition 5.4 (Spike)**.** *Let $h, t \in \mathbb{R}^3$, with $h = (h_x, h_y, 0)$ and $t = (t_x, t_y, 0)$. We say that the rectangle $\overline{(h, t)} \times \overline{((0, 0, 1), (0, 0, -1))}$ is a (h, t)-*spike.

Definition 5.5 (Gear)**.** *Let $c = (c_x, c_y, 0) \in \mathbb{R}^3$, $r \in \mathbb{R}^+$ and $\eta \in \mathbb{N}$. An η-*teeth gear of radius r centered at c *is a collection of spikes $\{s_1, \ldots, s_\eta\}$, where s_i is a $(c + \varepsilon r u_i, c + r u_i)$-spike, $u_i = (\cos i\theta, \sin i\theta, 0)$, and $\theta = 2\pi/\eta$. We say that the circle $\{(x, z, 0) \in \mathbb{R}^3 : \sqrt{(x - c_x)^2 + (y - c_y)^2} = r\}$ is the* enclosing circle *of the gear.*

Definition 5.6 (Point cloud)**.** *A* point cloud *of size $k > 0$ centered at a point $c \in \mathbb{R}^d$ is a collection of k distinct points inside $\mathsf{ball}(c, \varepsilon)$.*

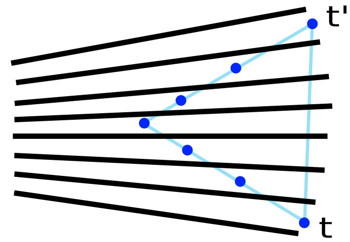


Figure 4: A connector with ports t and t'. Blue solid circles represent point clouds.

Definition 5.7 (Storage)**.** *Let $c = (c_x, c_y, 0) \in \mathbb{R}^3$, and let $k \in \mathbb{Z} \setminus \{0\}$. If $k > 0$, then a* storage of size k at c *is a white point cloud of size k centered at c. If $k < 0$, then it is a black point cloud of size $-k$ centered at c.*

Definition 5.8 (Connector)**.** *Let G be a gear of radius $r > 0$ centered at some point $c \in \mathbb{R}^3$. Let C be the enclosing circle of G. Let $t, t' \in C$ such that: (i) the angle tct' is less than $\pi/3$, and (ii) t and t' are not on any spikes. Let Δ be the equilateral triangle with all vertices inside the disk bounded by C that has $\overline{(t, t')}$ as one side. Let the other two sides of Δ be ℓ and ℓ'. The spikes of the gear G partition the polygonal curve $\ell \cup \ell'$ into a sequence of paths $p_0, \ldots, p_r$, where $t \in s_0$, $t' \in s_r$, and for every $i \in \{1, \ldots, r-1\}$, the path s_i has its endpoints in consecutive spikes of G. For any $i \in \{1, \ldots, r-1\}$, let t_i be the midpoint of p_i. A* connector *of order k with ports t and t' is a sequence of point clouds each of size k located at $t_1, \cdots, t_{r-1}$ and two point clouds of size $k-1$ located at t and t' (see Figure 4).*

Definition 5.9 (Chain)**.** *Let $p, q \in \mathbb{R}^3$. A* (p,q)-chain *is a sequence of points in $\mathbb{R}^3$ starting at p and ending at q. A chain is* α-dense, *for some $\alpha > 0$, if the maximum distance between any pair of consecutive points is at most α.*

5.2 Construction

We now describe the reduction from MDP_∞ to the problem of computing the Fréchet distance between two polyhedral domains in $\mathbb{R}^3$. Let (A, b, c) be an instance to MDP_∞, where $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^n$, and $c \in \{0,1\}^m$. Let $A = (a_{i,j})_{i,j}$, $b = (b_1, \ldots, b_n)^T$, and $c = (c_1, \ldots, c_m)^T$. Recall that the goal is to find a vector $x = (x_1, \ldots, x_m) \in \mathbb{Z}^m$, such that $Ax = b$, minimizing $\|c^T x\|_\infty$.

We will construct in polynomial time two sets $X_{\mathsf{black}}, X_{\mathsf{white}} \subset \mathbb{R}^3$. We refer to X_{black} and X_{white} as the *black* and *white* sets, respectively. Each set X_{black}, X_{white} is constructed by taking the union of appropriately chosen gears, storages, connectors, and chains, as defined above. We refer to each copy of such gadget as black, or white, depending on whether it belongs to the black or white set, respectively. We will show that the instance (A, b, c) has a solution of small cost if and only if the Fréchet distance between $\mathbb{R}^3 \setminus X_{\mathsf{black}}$ and $\mathbb{R}^3 \setminus X_{\mathsf{white}}$ is small.

Variables. For each variable x_i, $i \in \{1, \ldots, m\}$, we have a gear X_i of radius R centered at $(i \cdot (3R), 0)$, where R is a large enough value; setting $R = \Theta(n)$ is sufficient for our construction. If $c_i = 1$, then we set X_i to be an η-teeth gear, setting $\eta = 2\pi R$. Otherwise, if $c_i = 0$, then we set X_i to be an η'-teeth gear, setting $\eta' = n \cdot \eta$. In the former case, we say that X_i is *constrained*, and in the latter case we say that it is *free*. We introduce a copy of X_i in both the white and black sets.

Constants. For every constant b_i, $i \in \{1, \ldots, n\}$, we have a storage B_i^+ of size b_i located at $(i \cdot R, 10R, 0)$ and a storage B_i^- of size $-b_i$ located at $(i \cdot R, -10R, 0)$.

Monomials. For every $i \in \{1,\ldots,n\}$ we add the following connectors at the gear X_i. Let $t_1,\ldots,t_{2m}$ be points on the enclosing circle of X_i such that (i) their convex hall is a regular $2m$-gon, and (ii) they are disjoint from the spikes of X_i. For each $j \in \{1,\ldots,m\}$, there is a connector of order $a_{j,i}$ with ports t_{2j-1} and t_{2j}. Also, for each monomial there are two chains of density $1/|a_{j,i}|$. If $a_{j,i}$ is positive then one chain connects B_i^- to t_{2j-1}, and the other one connects t_{2j} to B_i^+. If $a_{j,i}$ is negative then one chain connects B_i^+ to t_{2j-1}, and the other one connects t_{2j} to B_i^-. A port is called an *input port* if it is connected to a B_i^-, and it is called an *output port* if it is connected to a B_i^+. We introduce copies of the above connectors and chains in both the white and black sets. We ensure in our construction that for any pair of distinct chains C, C' that we introduce, for any pair of points $p \in C$, $p' \in C'$, such that both p and p' are at distance at least $\Omega(\alpha)$ from all storages, we have $\|p-p'\|_2 = \Omega(n)$. Moreover, for any chain $C = p_1,\ldots,p_\ell$ of density $\chi > 0$, for any $i,j \in \{1,\ldots,\ell\}$, we have $\|p_i - p_j\|_2 = \Omega(\min\{R, |i-j|\cdot\chi\})$.

5.3 Small cost for MDP$_\infty$ implies small Fréchet distance

Lemma 5.10. *If there is a solution for the MDP$_\infty$ instance (A,b,c) with cost α, then $\delta_F(\mathbb{R}^3 \setminus X_{\mathsf{black}}, \mathbb{R}^3 \setminus X_{\mathsf{white}}) = O(\alpha)$.*

The proof of Lemma 5.10 is given in the rest of this subsection.

Suppose $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_m)$ is a solution for $Ax = b$. We show a homeomorphism from the black setting to the white setting of Frechet length $O(\max_j |\bar{x}_j|)$. We define three homoeomorphisms, r, p and d (which we call rotate, push, drag). The final homeomorphism that proves Lemma 5.10 is $d \circ p \circ r$.

Definition 5.11. *Let X_j be any gear, and let $C_{i,j}$ be a any chain connected to X_j. $C_{i,j}$ is incoming if it is connected to an input port and $\bar{x}_j$ is positive or it is connected to an output port and $\bar{x}_j$ is negative. $C_{i,j}$ is outgoing if it is connected to an output port and $\bar{x}_j$ is positive or it is connected to an input port and $\bar{x}_j$ is negative.*

Definition 5.12. *Let X_j be any gear, let $\mathcal{X}_j$ be the minimal convex set that contains X_j and its connectors, and let $\mathcal{X}_j^\epsilon$ the ϵ-neighborhood of $\mathcal{X}_j$. Let $p_{i,j}$ be a port attached to an incoming chain $I_{i,j}$. Let $\iota_{i,j}$ be the piecewise linear subpath of $I_{i,j}$ of length $a_{i,j}\bar{x}_j$ attached to $p_{i,j}$. Let $q_{i,j}$ be the output port of the same connector as $p_{i,j}$, and let $Q_{i,j}$ be a point set of size $a_{i,j}\bar{x}_j$ within $\mathcal{X}_j^\epsilon \setminus \mathcal{X}_j$ such that the distance between any point of $Q_{i,j}$ and $q_{i,j}$ is $O(\epsilon)$. Choose Z_j with the following properties.*

(1) Z_j is a topological disc in the $O(\epsilon)$ neighborhood of $\mathcal{X}_{i,j}^\epsilon \cup \iota_{i,j} \cup Q_{i,j}$.

(2) Z_j contains X_j, but $\partial Z_j \cup X_j = \emptyset$.

(3) For any port $p_{i,j}$ connected to an incoming chain Z_j contains the corresponding $Q_{i,j}$ and $\iota_{i,j}$.

(4) Z_j is disjoint from all other gears and all other points.

Lemma 5.13. *There is a continuous map $r : \mathbb{R}^3 \to \mathbb{R}^3$ with the following properties.*

(1)For any point $x \in \mathbb{R}^3 \setminus \bigcup_j Z_j$, we have $r(x) = x$.

(2)The restriction of r to the set of all white spikes is a homeomorphism onto the set of all black spikes.

(3)The restriction of r to the set of vertices of all $\iota_{i,j}$'s and the points of all connectors is a homeomorphism to a set composed of all points in all $Q_{i,j}$'s and the points of all connectors.

(4) $\delta_F(r) = O(\max_j \bar{x}_j)$.

Proof Sketch. Basically r rotates X_j for $\bar{x}_j$ clicks for all $1 \leq j \leq n$. We sketch the function r_j that rotates X_j for $\bar{x}_j$ clicks and does not affect any point outside Z_j. Then, r would be the composition of all r_j.

The function r_j is composed of a collection of functions, each describing one click of the gear. On a click each sheet is mapped to its next or previous sheet. Point clouds between the gears are mapped accordingly. The point clouds on the input port are moved slightly inside (to the following gap of the gear), and the point clouds on the output ports are moved out side (to $Q_{i,j}$'s).

Suppose that $p_{i,j}$ is a port of multiplicity k. Then, each click takes the remaining furthest points of $\iota_{i,j}$ and move them to $Q_{i,j}$. We remark that it is crucial r_j should be the identity function at $\mathbb{R}^3 \backslash Z_j$, and in particular at ∂Z_j, that can be achieved through an interpolation similar to what we described for moving one point within an ϵ-neighborhood of a curve in our approximation algorithm.

The Fréchet length of each click is $O(1)$, so $\delta_F(r_j) = O(|\bar{x}_j|)$. Because r's affect disjoint regions $\delta_F(r) = O(\max_j |\bar{x}_j|)$. □

Definition 5.14. *Let $O_{i,j}$ be an outgoing chain between the port $q_{i,j}$ and the storage B_i. Let $Q_{i,j}$ be the auxiliary point set as defined in Definition 5.12. Let $Q'_{i,j}$ be a point set of the same cardinality placed within B_i. Choose the set $U_{i,j}$ with the following properties.*

(1) *$U_{i,j}$ is a topological disc in the $O(\epsilon)$ neighborhood of $Q_{i,j} \cup O_{i,j} \cup Q'_{i,j}$.*

(2) *$U_{i,j}$ contains all points in $Q_{i,j} \cup O_{i,j} \cup Q'_{i,j}$ as internal (non-boundary) points.*

(3) *$U_{i,j}$ is disjoint from all gears, all other chains, and all other set of auxiliary points.*

Lemma 5.15. *There is a collection of $Q'_{i,j}$'s and a collection of disjoint $U_{i,j}$'s matching Definition 5.14.*

Proof Sketch. For each storage B_i add $\sum_j |Q'_{i,j}|$ points in general position within B_i. Let $Y_{i,j}$ and $Y'_{i,j}$ be arbitrary permutations of $Q_{i,j}$ and $Q'_{i,j}$, respectively. For each i, j, consider the piecewise linear paths $Y_{i,j} \circ O_{i,j} \circ Y'_{i,j}$ and let $U_{i,j}$ be the ϵ-neighborhood of the path. Perturb all $U_{i,j}$ so that they become disjoint. A perturbation of Fréchet length $O(\epsilon)$ exists since we are in three dimensions. □

Lemma 5.16. *There is a continuous map $p : \mathbb{R}^3 \to \mathbb{R}^3$ with the following properties.*

(1) *For any point $x \in \mathbb{R}^3 \backslash \bigcup_{i,j} U_{i,j}$, we have $r(x) = x$.*

(2) *Let $\mathcal{O}$ be the set of all vertices of all outgoing chains. The restriction of p into $\mathcal{O} \cup \bigcup Q_{i,j}$ is a homeomorphism onto $\mathcal{O} \cup \bigcup Q'_{i,j}$.*

(3) *$\delta_F(p) = O(\max_j \bar{x}_j)$.*

Proof Sketch. The map p is composed of a collection of maps on disjoint neighborhoods $U_{i,j}$'s, each essentially shifting a chain, and thereby, moving a $Q_{i,j}$ to $Q'_{i,j}$. Let these maps be $p_{i,j}$ following the indices of their chains. We show a sketch for an arbitrary $p_{i,j}$.

Recall that in our approximation algorithm we show how to move a point x to another point y along a curve γ by a map that is identity everywhere except the ϵ-neighborhood of γ. Call this map $u_\gamma[x, y]$ and recall that $\delta_F(u_\gamma[x, y]) = O(|\gamma|)$. Now let $x_1, x_2, \ldots, x_k$ be a chain of points, and let γ_s be a curve inside $U_{i,j}$ between x_s and x_{s+1} that is ϵ-close to a line segment and let u_s denote

$u_{\gamma_s}[x_s, x_{s+1}]$. Then $u_1 \circ \ldots \circ u_{k-1}$ is a homeomorphism that moves $\{x_1, \ldots, x_{k-1}\}$ to $\{x_2, \ldots, x_k\}$ and its Fréchet distance is $O(\max_s |\gamma_s|) = O(\max_s \overline{(x_s, x_{s+1})})$.

Let $Y_{i,j}$ and $Y'_{i,j}$ be arbitrary permutations of $Q_{i,j}$ and $Q'_{i,j}$, respectively. To obtain $p_{i,j}$ we apply the map explained above $|Q_{i,j}|$ times to proper subsequences of $Y_{i,j} \circ C_{i,j} \circ Y'_{i,j}$ to move $Q_{i,j}$ to $Q'_{i,j}$. Because the chain $C_{i,j}$ is $1/|a_{i,j}|$ dense, the Fréchet distance of each shift is $O(1/|a_{i,j}|)$. On the other hand, $|Q_{i,j}| = |a_{i,j}\bar{x}_j|$. It follows that the Fréchet distance of the overall shift is $O(|\bar{x}_j|)$. $\square$

Definition 5.17. *Let $I_{i,j}$ be an incoming chain between the storage B_i and the port $p_{i,j}$. Let $\iota_{i,j}$ be as defined in Definition 5.12 and let $P_{i,j}$ be all vertices of $\iota_{i,j}$. Choose $Q''_{i,j}$ so that*

(1) $Q''_{i,j} \subseteq B_i \cup \bigcup_k Q'_{i,k}$

(2) $|Q''_{i,j}| = |P_{i,j}|$.

Choose the set $W_{i,j}$ with the following properties.

(1) $W_{i,j}$ is a topological disc in the $O(\epsilon)$ neighborhood of $Q''_{i,j} \cup I_{i,j} \cup P_{i,j}$.

(2) $W_{i,j}$ contains all points in $Q''_{i,j} \cup I_{i,j} \cup P_{i,j}$ as internal (non-boundary) points.

(3) $W_{i,j}$ is disjoint from all gears, all other chains, and all other set of auxiliary points.

Lemma 5.18. *There is a collection of disjoint $W_{i,j}$'s (one for each outgoing chains) as defined in Definition 5.17.*

Proof Sketch. For each b_i partition $B_i \cup \bigcup_k Q'_{i,k}$ into sets $Q''_{i,j}$'s such that the cardinalities of $Q''_{i,j}$'s are as in Definition 5.17. This is doable because $\bar{x}$ is a Diophantine solution. Now, let $Y''_{i,j}$ be an arbitrary permutation of $Q''_{i,j}$. Consider ϵ-neighborhoods of the piecewise linear curves $Y''_{i,j} \circ I_{i,j}$ and perturb them to become disjoint. A perturbation of Fréchet length $O(\epsilon)$ exists since we are in three dimensions. $\square$

Lemma 5.19. *There is a continuous map $d : \mathbb{R}^3 \to \mathbb{R}^3$ with the following properties.*

(1) For any point $x \in \mathbb{R}^3 \setminus \bigcup_{i,j} W_{i,j}$, we have $r(x) = x$.

(2) Let $\mathcal{I}$ be the set of all vertices of all incoming chains. The restriction of p into $(\mathcal{I} \setminus \bigcup_{i,j} P_{i,j}) \cup \bigcup Q''_{i,j}$ is a homeomorphism onto $\mathcal{I}$.

(3) $\delta_F(d) = O(\max_j \bar{x}_j)$.

Proof Sketch. This proof is very similar to the proof of Lemma 5.16. It is omitted in this version. $\square$

5.4 Small Fréchet distance implies small cost for MDP$_\infty$

Fix $f : \mathbb{R}^3 \setminus X_{\mathsf{black}} \to \mathbb{R}^3 \setminus X_{\mathsf{white}}$ to be a homeomorphism with $\delta_F(f) \le \alpha$. We assume $\alpha \ll R$. Every such homeomorphism induces a bijection between the connected components of X_{black} and X_{white}. More precisely, for every connected component C of X_{black} there exists a unique connected component C' of X_{white} such that some neighborhood of C is mapped to some neighborhood of C'. In order to simplify the notation we denote this event by writing $f(C) = C'$.

Definition 5.20. *Let X_i be a k-teeth gear with spikes $(s_0, s_1, \cdots, s_{k-1})$ in cyclic order, and let $t \in \mathbb{Z}$. We say that f* rotates X by t clicks *if $f(s_i) = s_{i+t}$ for all $0 \le i < k$, where the summation is mod k.*

Let γ be a curve with endpoints x, y. We define the *breadth* of γ to be the real value $b \geq 0$, satisfying $b = \inf\{r \geq 0 : \gamma \subseteq \mathsf{ball}(x, r) \cap \mathsf{ball}(y, r)\}$.

Lemma 5.21. *f rotates all gears.*

Proof. For each spike of a gear X we define a normal vector that is parallel to the tangent of the enclosing counterclockwise circle of X. We call the side of the spike that the normal points to the left side of the spike and the other side of it the right side of the spike. Since $\delta_F(f) = \alpha < R$, f preserves the orientation of the neighborhood of each spike.

Suppose, for the purpose of contradiction, that f does not rotate some gear X. It follows that f swaps at least two consecutive spikes. Without loss of generality, suppose it swaps s_1 and s_2. That is, $f(s_1) = s_{j+k}$ and $f(s_2) = j$ for some $k \geq 1$.

Let c_1 and c_2 be the centers of s_1 and s_2, respectively. Consider the line segment ℓ that connects c_1 to c_2, and observe that ℓ and X are internally disjoint. Further, note that ℓ connects to s_1 and s_2 from left and right side respectively. Therefore, since f preserves the orientation of the neighborhoods of every spike, it follows that $f(\ell)$ is a path between $f(c_1)$ and $f(c_2)$, and it connects to $f(s_1)$ and $f(s_2)$ from left and right side respectively.

Since $\delta_F(f) \leq \alpha$, the distance between $f(c_1)$ (resp. $f(c_2)$) and the boundary of $f(s_1)$ (resp. $f(s_2)$) is at least $R - \alpha$. Because the left side of $f(s_1)$ and the right side of $f(s_2)$ are not consecutive, the breadth of $f(\ell)$ is at least $R - \alpha$. Therefore, $\delta_F(f) \geq (R - 2\alpha)/2$, which is a contradiction, concluding the proof. $\square$

Definition 5.22 (Shifting)**.** *Let $C = (p_1, p_2, \ldots, p_\ell)$ be a chain, and let $t \geq 0$. Let*

$$N^+ = \{i \leq \ell/2 : f(p_i) = p_j \textit{ for some } j > \ell/2\},$$

and

$$N^- = \{i > \ell/2 : f(p_i) = p_j \textit{ for some } j \leq \ell/2\}.$$

We say that f shifts C for t steps *if $|N^+| = |N^-| + t$. We say that f* shifts C for $-t$ steps *if $|N^-| = |N^+| + t$.*

Lemma 5.23. *Let X_i be a gear, and let C be a chain connected to a port of X_i via a port of size k. Suppose that f rotates X_i for h clicks. If C is connected to X_i via an input port then f shifts C for kh steps. If C is connected to X_i via an output port then f shifts C for $-kh$ steps.*

Proof. We consider only the case where C is connected to X_i via an input port; the other case is symmetric. Let Q be a connector of order k with input and output ports t and t', respectively. Also, let $(t_1, t_2, \cdots, t_r)$ be the internal point clouds locations of Q in order. First we prove that if f rotates X_i for h clicks then f maps t_j to t_{j+h} for all $1 \leq j \leq r - h$. To this end, consider the point cloud t_j and let its enclosing spikes be s and s'. Let x be any point of t_j. Also, let c and c' be the centers of s and s', let ℓ be the two segment path (c, x, c'). Observe that $f(c)$ and $f(c')$ are α-close to the center of $f(s)$ and $f(s')$, respectively. Thus, $f(\ell)$ is entirely between $f(s)$ and $f(s')$, otherwise its breadth is at least $(R - 2\alpha)/2$. In particular, x is between $f(s)$ and $f(s')$. Since f maps $(t_1, t_2, \cdots, t_h)$ to later point clouds, it must be the case that $f^{-1}(t_1 \cup t_2 \cup \cdots \cup t_h)$ is from the points of C or t the input port. On the other hand, for any point $x \in C$ that is far from the chain's endpoints we have $f^{-1}(x) \in C$. So, the set of points in the second half of C and the set of points in $t_1 \cup t_2 \cup \cdots \cup t_h$ must be covered by points in C. Therefore, C must shift for kh steps to cover $(t_1, t_2, \cdots, t_h)$. $\square$

Lemma 5.24. *If $\delta_F(\mathbb{R}^3 \setminus X_{\mathsf{black}}, \mathbb{R}^3 \setminus X_{\mathsf{white}}) \leq \alpha$, then there exists a solution to the MDP$_\infty$ instance (A, b, c) of cost $O(\alpha)$.*

Proof. Suppose f rotates X_j for x_j clicks, $1 \leq i \leq n$, and observe that $x_j = O(\alpha)$ if X_j is a constrained gear. Consider an arbitrary storage B_i^-. Let S denote the collection of the points in B_i and in half-chains attached to B_i. Let s_b and s_w be the number of black and white points of S, respectively, and observe that $b_i = s_w - s_b$. On the other hand, Lemma 5.23 implies that $s_w - s_b = a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,n}x_n$. Thus,

$$0 = b_i - a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,n}x_n,$$

as desired, and the proof is complete. □

Proof of Theorem 1.2. Follows immediately by Lemmas 5.10 and 5.24. □

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