On the Decidability of the Fréchet Distance between Surfaces

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Abstract

We show that the Fréchet distance between two piecewise linear surfaces can be decided in finite time, hence, the problem is decidable. For the special case that one of the surfaces is a triangle, we show that the problem is in PSAPCE. In both cases, our computational model is a Turing Machine, and our algorithms rely on Canny's result [STOC 1988] that the existential theory of the real numbers is decidable in PSPACE.

1 Introduction

Measuring the similarity between two geometric objects is a central problem that shows up in many engineering tasks, for example registration in image processing and function detection in protein modeling. Among many existing measures of similarity for geometric objects, the Fréchet distance is particularly informative as it takes into account the underlying topology of the objects. As a result, its computation produces an alignment of the objects that respects their topology.

The Fréchet distance between curves have been studied extensively since the seminal paper of Alt and Godau [AG95], and efficient exact and approximation algorithms have been discovered [EM94, AHPK$^+$06, DHW12] and used in different application [EM94, AHPK$^+$06, DHW12].

In contrast, computing the Fréchet distance between two surfaces appears to be a much harder problem. In fact, for computing the Fréchet distance between general piecewise linear surfaces of genus zero, only two positive results were known before this paper. Alt and Buchin [AB09] showed the upper semi-computability of the Fréchet distance, i.e. there is a non-halting Turing Machine that produces a decreasing sequence of rational numbers that converges to the Fréchet distance. In addition, Nayyeri and Xu [NX16] described a $(1 + \varepsilon)$-approximation algorithm for computing the Fréchet distance between two piecewise linear surfaces, whose running time is super-exponential in the size of the input and the total area of the surfaces. Nevertheless, none of these results implies an exact algorithm for computing or even deciding the Fréchet distance between two surfaces.

In this paper, we give the first exact algorithm to decide the Fréchet distance between two piecewise linear surfaces. Moreover, we show that the Fréchet distance between a piecewise linear surface and a triangle can be decided in PSPACE.

Let $\mathcal{R} = (\mathcal{R}_V, \mathcal{R}_E, \mathcal{R}_T)$ and $\mathcal{S} = (\mathcal{S}_V, \mathcal{S}_E, \mathcal{S}_T)$ be two homeomorphic piecewise linear surfaces of genus zero: each of $\mathcal{R}$ and $\mathcal{S}$ is constructed from a set of Euclidean triangles by identifying pairs of equal-length edges. Additionally, let $\varphi_{\mathcal{R}} : \mathcal{R} \to \mathbb{R}^3$ and $\varphi_{\mathcal{S}} : \mathcal{S} \to \mathbb{R}^3$ be immersions that map each triangle of the input to a congruent triangle in $\mathbb{R}^3$. The Fréchet length of a homeomorphism

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$f : \mathcal{R} \to \mathcal{S}$ under the ℓ_p norm is defined as $\delta_F(f) = \sup_{x\in\mathcal{R}} ||\varphi_{\mathcal{R}}(x) - \varphi_{\mathcal{S}}(f(x))||_p$. In turn, the Fréchet distance between $\mathcal{R}$ and $\mathcal{S}$ is defined as $\delta_F(\mathcal{R}, \mathcal{S}) = \inf_f \delta_F(f)$, where f goes over all homeomorphisms. Note that the Fréchet distance between $\mathcal{R}$ and $\mathcal{S}$ depends on the immersions $\varphi_{\mathcal{R}}$ and $\varphi_{\mathcal{S}}$, which are not explicitly mentioned here and throughout the paper to simplify the notation.

We study the decision Fréchet distance problem: given $\mathcal{R}$ and $\mathcal{S}$ and δ, we seek an algorithm that accepts if $\delta_F(\mathcal{R}, \mathcal{S}) \leq \delta$, and rejects otherwise.

1.1 Previous Work

In contrast to curves, very little is known about computing the Fréchet distance between two surfaces. Polynomial time algorithms are known only when the surfaces are flat polygons with constant numbers of holes [BBW08, BBS10, NS15], or folded polygons [CDHP$^+$11]. Moreover, the problem becomes NP-hard even for seemingly simple special cases, such as for a surface and a triangle [God99], for two terrains, and for two polygons with holes [BBS10]. Recently, Buchin et al. [BOS17, BOS15] studied two interesting cases: the Fréchet distance between real-valued surfaces and between moving curves. They show that computing the Fréchet distance between real-valued surfaces is in NP-complete via introducing a new notion of distance between contour trees. For moving curves, they show some variants that are polynomial time solvable, and other variants that are NP-complete.

1.2 Our Results

For two surfaces of genus zero, we describe the first algorithm for deciding their Fréchet distance.

Theorem 1. *Let $\mathcal{R}$ and $\mathcal{S}$ be piecewise linear surfaces with m and n vertices, respectively, and let $\delta \geq 0$. There is an algorithm to decide whether $\delta_F(\mathcal{R}, \mathcal{S}) \leq \delta$.*

If one of the surfaces is a triangle, we show that their Fréchet distance can be decided in PSPACE. Godau [God99] has shown that this special case is NP-hard.

Theorem 2. *Let $\mathcal{R}$ be a piecewise linear surface with m vertices, $\mathcal{S}$ be a triangle, and and $\delta \geq 0$. There is an algorithm to decide whether $\delta_F(\mathcal{R}, \mathcal{S}) \leq \delta$ in PSPACE.*

Our computational model is a Turing Machine, so, we measure the time and space complexity as a function of the size of input: the number of required bits to specify the combinatorial surfaces, the side lengths of triangles, and the images of vertices under $\varphi_{\mathcal{R}}$ and $\varphi_{\mathcal{S}}$.

1.3 Overview

Here, we give an overview of our algorithm that decides whether an $\mathcal{R}$-$\mathcal{S}$ homeomorphism of Fréchet length δ exists. We use many insights from previous works in particular from references [CDHP$^+$11, BBW08, NX16, NS15].

Let f be an $\mathcal{R}$-$\mathcal{S}$ homeomorphism of Fréchet length δ. Let $\widetilde{\mathcal{R}} = (\widetilde{\mathcal{R}}_V, \widetilde{\mathcal{R}}_E, \widetilde{\mathcal{R}}_T)$ be a refined triangulation of $\mathcal{R}$ with vertex set $\widetilde{\mathcal{R}}_V = \mathcal{R}_V \cup f^{-1}(\mathcal{S}_V)$. Similarly, let $\widetilde{\mathcal{S}} = (\widetilde{\mathcal{S}}_V, \widetilde{\mathcal{S}}_E, \widetilde{\mathcal{S}}_T)$ be a refined triangulation of $\mathcal{S}$ with vertex set $\widetilde{\mathcal{S}}_V = \mathcal{S}_V \cup f(\mathcal{R}_V)$. Note that $f(\widetilde{\mathcal{R}}_V) = \widetilde{\mathcal{S}}_V$, and let $f_0 = f_{|\widetilde{\mathcal{R}}_V}$ be the $\widetilde{\mathcal{R}}_V$-$\widetilde{\mathcal{S}}_V$ bijection and call it the *vertex map*. Additionally, let $f_1 = f_{|\widetilde{\mathcal{R}}_E}$ be the *edge map*, which maps each edge $e = (u, u') \in \widetilde{\mathcal{R}}_E$ into a curve in $\mathcal{S}$ with endpoints $f_0(u)$ and $f_0(u')$.

An edge map f_1 describes an embedding of the graph $(\widetilde{\mathcal{R}}_V, \widetilde{\mathcal{R}}_E)$ into the surface $\mathcal{S}$. This embedding is combinatorially equivalent to the embedding of $(\widetilde{\mathcal{R}}_V, \widetilde{\mathcal{R}}_E)$ on $\mathcal{R}$ specified by the triangulation $\widetilde{\mathcal{R}}$. In particular, it preserves the cyclic order of edges around vertices, and it maps boundary points to boundary points. Moreover, being a restriction of f, the Fréchet length of f_1 is at most δ. Hence, the existence of f implies the existence of a combinatorially equivalent embedding of $(\widetilde{\mathcal{R}}_V, \widetilde{\mathcal{R}}_E)$ into $\mathcal{S}$ that is at Fréchet distance at most δ of $(\widetilde{\mathcal{R}}_V, \widetilde{\mathcal{R}}_E)$. We show that a stronger statement holds: the existence of an $\mathcal{R}$-$\mathcal{S}$ homeomorphism of Fréchet length δ implies the existence of refinements $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{S}}$, and an edge map f_1 of Fréchet length at most δ that has all the following properties.

(1) The edge map f_1 gives a combinatorially equivalent embedding of $(\widetilde{\mathcal{R}}_V, \widetilde{\mathcal{R}}_E)$ into $\mathcal{S}$.

(2) For each $e \in \widetilde{\mathcal{R}}$, $f_1(e)$ is a piecewise linear curve on $\mathcal{S}$. Moreover, the intersection of $f_1(e)$ with any triangle $t \in \widetilde{\mathcal{S}}_T$ is a set of line segments with endpoints on the boundary of t.

(3) Each $s \in \widetilde{\mathcal{S}}_E$ is crossed by $f_1(\widetilde{\mathcal{R}}_E)$ at most $2^{O(|\mathcal{R}_V|+|\mathcal{S}_V|)}$ times.

We call an edge map with all these properties an *scaffold map* and show that an scaffold map of Fréchet length δ extends to a homeomorphism of Fréchet length arbitrary close to δ. Therefore, our decision problem is reduced to deciding if a scaffold map of Fréchet length at most δ exists.

In searching for a scaffold map of Fréchet length δ, our algorithm proceeds in two high-level steps. First, it builds a list L of combinatorial descriptions that contains the description of a scaffold map of Fréchet at most δ (if any scaffold map of Fréchet length at most δ exists). Next, for each description $S \in L$, our algorithm builds a system of polynomials that is feasible if and only if a scaffold map with description S and Fréchet length at most δ exists.

The combinatorial description of a scaffold map is composed of (1) a combinatorial vertex map that specifies for each vertex $u \in \mathcal{R}_V$ which triangle of $\mathcal{S}_T$ it maps to, and for each vertex $v \in \mathcal{S}_V$ which triangle of $\mathcal{R}_T$ it maps to, (2) combinatorial specifications of the refinements $(\widetilde{\mathcal{R}}, \widetilde{\mathcal{S}})$, and (3) the crossing sequence of the image of every edge $e \in \widetilde{\mathcal{R}}_E$: the sequence of edges of $\mathcal{S}_E$ that the image of e crosses. The crossing bound for scaffold maps (property (3)) guarantees a finite number of choices for the combinatorial descriptions that we need to consider.

For each combinatorial description, our algorithm constructs a system of polynomial inequalities. The variables are the exact locations of the images of vertices and crossing points. The constraints are to ensure that (i) the refinements are valid with respect to the exact locations of vertices, (ii) the images of different edges do not cross, and (iii) the pairwise distances between corresponding vertices or crossing points are at most δ. The number of variables and the number of constraints in our system is $2^{\Theta(|\mathcal{R}_V|+|\mathcal{S}_V|)}$ because of the exponential number of crossing points and their related constraints. Finally, we will use Canny's result (Lemma 1) to test if each of the polynomial systems is feasible. If any one of them is feasible our algorithm accepts, otherwise it rejects.

For the case that $\mathcal{S}$ is a triangle, we show that the image of each $e \in \widetilde{\mathcal{R}}_E$ is a shortest homotopic path, thus, it is composed of a set of straight line segments between vertices of $\widetilde{\mathcal{S}}_V$. Since, the image of different edges of $\widetilde{\mathcal{R}}_E$ form a set of noncrossing paths in $\mathcal{S}$, we can assume (after modifying $\widetilde{\mathcal{S}}$) that the image of every edge $e \in \widetilde{\mathcal{R}}_E$ is a piecewise linear curve whose pieces coincide with edges of $\widetilde{\mathcal{S}}_E$. In particular, all crossing points are located on the vertices of $\widetilde{\mathcal{S}}_V$. Based on this observation, we can show that most of the variables representing crossing points and most of their related constraints are redundant, and can be disregarded from the system without affecting its feasibility condition. Consequently, we show that the Fréchet distance between a surface and a triangle can be decided in PSPACE.

2 Preliminaries

Maps. Let $f : A \to B$ be a function. For any $U \subseteq A$, we define $f(U) = \{f(u) | u \in A\}$. The function $f|_U : U \to B$, called the **restriction** of f to the subset U, is defined as for all $u \in U$, $f|_U(u) = f(u)$. In this case, we also say, that f is an **extension** of $f|_U$ to A. If A and B are topological space, f is a **homeomorphism** if (1) it is a bijection, (2) it is continuous, and (3) its inverse is continuous.

Surfaces. A **surface** $\mathcal{Q}$ (or a 2-manifold) is a space, in which every point has a neighborhood that is homeomorphic to the plane or half-plane. The set of points that are homeomorphic to half-plane form the **boundary** of $\mathcal{Q}$. An **embedding** $\Phi : \mathcal{Q} \to \mathbb{R}^3$ is a continuous one-to-one map. An **immersion** $\varphi : \mathcal{Q} \to \mathbb{R}^3$ is a continuous map, such that for any $x \in \mathcal{Q}$ there is a neighborhood N_x of x, on which f is an embedding.

A **piecewise linear surface.** is a surface $\mathcal{Q}$ that is constructed from a set of Euclidean triangles by identifying pairs of equal-length edges. We denote the constituent vertices, edges, and triangles of $\mathcal{Q}$ by $\mathcal{Q}_V$, $\mathcal{Q}_E$, and $\mathcal{Q}_T$, in order. In short, we write $\mathcal{Q} = (\mathcal{Q}_V, \mathcal{Q}_E, \mathcal{Q}_T)$. In this paper, we consider **locally isometric immersions**, those that map each triangle to a congruent triangle in $\mathbb{R}^3$.

Embedded curves and graphs. Let $\mathcal{Q}$ be a surface and let $\alpha, \beta : [0,1] \to \mathcal{Q}$ be curves embedded on $\mathcal{Q}$ with the same endpoints, $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$. A **homotopy** $h : [0,1] \times [0,1] \to \mathcal{Q}$ is a continuous map such that $h[x,0] = \alpha$, $h[x,1] = \beta$, $h[0,t] = \alpha(0) = \beta(0)$, and $h[1,t] = \alpha(1) = \beta(1)$. If such a homotopy exists, we say that α and β are **homotopic**. A **shortest homotpic path** or a **tight path** is the shortest path within a homotopy class.

An **embedding** of a graph $G = (V, E)$ into a surface $\mathcal{Q}$ is a continuous function that maps vertices in V into distinct points in $\mathcal{Q}$ and edges in E into disjoint paths except for their endpoints. The **faces** of the embedding are maximal subsets of $\mathcal{Q}$ that are disjoint from the image of the graph. An embedding is **cellular** if all its faces are topological disks. In particular, each boundary component in a cellular embedding is covered by the image of the graph. A cellular embedding on an orientable surface can be described by a **rotation system**. A rotation system is composed of a cyclic (clockwise) order of edges around vertices. A rotation system is a combinatorial description of the embedding of a graph. Equivalent rotation systems of G on two different surfaces $\mathcal{Q}$ and $\mathcal{Q}'$ induce a one-to-one correspondence between the vertices, edges, and faces of the different embeddings. Therefore, they can be extended to a homeomorphism between $\mathcal{Q}$ and $\mathcal{Q}'$. Note that the homotopy classes of an edge may be different in two combinatorially equivalent embeddings. For example, one can apply Dehn Twists on a cycle that avoids vertices of the embedding to change the homotopy class of edges without affecting the rotation system.

Fréchet distance. Let $\mathcal{R}$ and $\mathcal{S}$ be homeomorphic piecewise linear triangulations, and let $\varphi_{\mathcal{R}} : \mathcal{R} \to \mathbb{R}^3$ and $\varphi_{\mathcal{S}} : \mathcal{S} \to \mathbb{R}^3$ be locally isometric immersions. The Fréchet length (under the ℓ_p norm) of a homeomorphism $\sigma : \mathcal{R} \to \mathcal{S}$, is defined to be $\delta_F(\sigma) = \max_{x \in \mathcal{R}} ||\varphi_{\mathcal{R}}(x) - \varphi_{\mathcal{S}}(\sigma(x))||_p$. The Fréchet distance between (immersed) $\mathcal{R}$ and $\mathcal{S}$ is defined to be $\delta_F(\mathcal{R}, \mathcal{S}) = \inf_\sigma \delta_F(\sigma)$, where σ ranges over all homeomorphisms between $\mathcal{R}$ and $\mathcal{S}$. Note that $\delta_F(\mathcal{R}, \mathcal{S})$ depends on the immersions $\varphi_{\mathcal{R}}$ and $\varphi_{\mathcal{S}}$, which are not explicitly mentioned here and throughout the paper to simplify the notation. Also, note that, a homeomorphism that realizes $\delta_F(\mathcal{R}, \mathcal{S})$ does not necessarily exist as the Fréchet distance is defined as an infimum.

Existential theory of reals. Let $x_1, \dots, x_n$ be variables over reals, and let $F(x_1, \dots, x_n)$ be a quantifier-free formula involving real polynomial equalities and inequalities. The decision problem for the existential theory of reals is to decide if the following formula is true

$$\exists x_1 \cdots \exists x_n F(x_1, \dots, x_n).$$

That is to decide whether real numbers $\overline{x}_1 \dots \overline{x}_n$ exists such that $F(\overline{x}_1, \dots, \overline{x}_n)$ is true. Canny [Can88] shows that existential theory of reals can be decided in PSPACE. Note this is polynomial space under the Turing Machine model, that is a required space is the polynomial function of the number of bits used to specify the problem.

Lemma 1 (Canny [Can88], Theorem 3.3)**.** *The existential theory of the reals is decidable in PSPACE.*

In this paper, we use a special case of existential theory of reals, that the feasibility of a system of polynomial inequalities is decidable in PSPACE.

3 The Fréchet Distance between Two Surfaces

Let $\mathcal{R} = (\mathcal{R}_V, \mathcal{R}_E, \mathcal{R}_T)$ and $\mathcal{S} = (\mathcal{S}_V, \mathcal{S}_E, \mathcal{S}_T)$ be two piecewise linear surfaces, and let $\varphi_\mathcal{R} : \mathcal{R} \to \mathbb{R}^3$ and $\varphi_\mathcal{S} : \mathcal{S} \to \mathbb{R}^3$ be immersions of $\mathcal{R}$ and $\mathcal{S}$ into $\mathbb{R}^3$ respectively. Also, let $m = |\mathcal{R}_V|$, $n = |\mathcal{S}_V|$, and $\delta \geq 0$. In this section, we describe an algorithm to decide if the Fréchet distance between $\mathcal{R}$ and $\mathcal{S}$ is at most δ, i.e. our algorithm accepts if $\delta_F(\mathcal{R}, \mathcal{S}) \leq \delta$, and rejects otherwise. We consider the Fréchet distance under the general ℓ_p norm for any constant $p \geq 1$. To simplify notation we drop p and use $||\cdot||$ to denote the ℓ_p norm whenever our claim holds for all p.

Intuitively, our algorithm searches over a finite state of possible combinatorial descriptions for an $\mathcal{R}$-$\mathcal{S}$ homeomorphism of Fréchet length δ. In Section 3.1, we introduce combinatorial descriptions along with vertex map and scaffold map concepts. For each of these combinatorial descriptions, our algorithm builds a polynomial system of inequalities that is feasible if and only if there is a homeomorphism of Fréchet length δ. In Section 3.2 we explain how to build this system of inequalities. In Section 3.3 we give the proof of the main theorem of this section.

3.1 Scaffold Map and Combinatorial Specifications

In this section, we introduce the notions of vertex map, refinements, and scaffold maps, that are the main ingredients of our algorithm. For each of these concepts, we introduce its geometric can combinatorial variants together. We state and prove several properties of these concepts that we exploit in our algorithm.

3.1.1 Vertex map

Let $\mathcal{R}'_V \subseteq \mathcal{R}$ be a point set with cardinality $|\mathcal{S}_V|$, and let $\mathcal{S}'_V \subseteq \mathcal{S}$ be a point set with cardinality $|\mathcal{R}_V|$. Also, let $\widetilde{\mathcal{R}}_V = \mathcal{R}_V \cup \mathcal{R}'_V$, and $\widetilde{\mathcal{S}}_V = \mathcal{S}_V \cup \mathcal{S}'_V$. For any $\delta \geq 0$, a mapping $f_0 : \widetilde{\mathcal{R}}_V \to \widetilde{\mathcal{S}}_V$ is called a **vertex map** between $\mathcal{R}$ and $\mathcal{S}$ of cost δ if it can be extended to an $\mathcal{R}$-to-$\mathcal{S}$ homeomorphism of Fréchet length δ. A **combinatorial vertex map** (g, h) is composed of two maps: (1) $g : \mathcal{R}_V \to \mathcal{S}_T \cup \mathcal{S}_E$ that maps each internal vertex of $\mathcal{R}_V$ into a triangle of $\mathcal{S}_T$ and each boundary vertex of $\mathcal{R}_V$ into a boundary edge of $\mathcal{S}_E$, and (2) $h : \mathcal{S}_V \to \mathcal{R}_T \cup \mathcal{R}_E$ that maps each internal vertex of $\mathcal{S}_V$ into a triangle of $\mathcal{R}_T$ and each boundary vertex of $\mathcal{S}_V$ into a boundary edge of $\mathcal{R}_E$. Intuitively, a combinatorial vertex map determines for each internal vertex $u \in \mathcal{R}_V$ the triangle of $\mathcal{S}$ that contains u's image, and for each boundary vertex $b \in \mathcal{R}_V$ the boundary edge of $\mathcal{S}$ that contains b's image. Similarly, for each internal vertex $v \in \mathcal{S}_V$, a combinatorial vertex map specifies the triangle of $\mathcal{R}$ that contains the preimage of v, and for each boundary vertex $c \in \mathcal{S}_V$, it determines the boundary edge of $\mathcal{R}$ that contains the preimage of c.

A vertex map f_0 and a combinatorial vertex map (g, h) are **consistent** if for any $u \in \mathcal{R}_V$, $f_0(u) \in g(u)$ and for any $v \in \mathcal{S}_V$, $f_0^{-1}(v) \in h(v)$. The following lemma is immediately implied by the definition of combinatorial vertex maps.

Lemma 2. *There are $(m+n)^{O(m+n)}$ combinatorial vertex maps.*

3.1.2 Refinement

Let $f_0 : \widetilde{\mathcal{R}}_V \to \widetilde{\mathcal{S}}_V$ be a vertex map between $\mathcal{R}$ and $\mathcal{S}$; $\widetilde{\mathcal{R}}_V = \mathcal{R}_V \cup \mathcal{R}'_V$ and $\widetilde{\mathcal{S}}_V = \mathcal{S}_V \cup \mathcal{S}'_V$. Let $\widetilde{\mathcal{R}} = (\widetilde{\mathcal{R}}_V, \widetilde{\mathcal{R}}_E, \widetilde{\mathcal{R}}_T)$ and $\widetilde{\mathcal{S}} = (\widetilde{\mathcal{S}}_V, \widetilde{\mathcal{S}}_E, \widetilde{\mathcal{S}}_T)$ be geometric refinements of $\mathcal{R} = (\mathcal{R}_V, \mathcal{R}_E, \mathcal{R}_T)$ and $\mathcal{S} = (\mathcal{S}_V, \mathcal{S}_E, \mathcal{S}_T)$, respectively. We say that $(\widetilde{\mathcal{R}}, \widetilde{\mathcal{S}})$ and f_0 are **consistent**, as the preimage and image of f_0 are the vertex sets of $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{S}}$, respectively. Note that $\mathcal{R}_E \subseteq \widetilde{\mathcal{R}}_E$ and $\mathcal{S}_E \subseteq \widetilde{\mathcal{S}}_E$. We refer to the edges of $\mathcal{R}_E$ and $\mathcal{S}_E$ as **original** edges, and to the edges in $\widetilde{\mathcal{R}}_E \backslash \mathcal{R}_E$ and $\widetilde{\mathcal{S}}_E \backslash \mathcal{S}_E$ as **refinement** edges. Since, $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{S}}$ are subgraphs of K_{m+n} (the complete graph with $m+n$ vertices), we can bound the number of possible refinements by the number of subgraphs of K_{m+n}.

Lemma 3. *For any vertex map f_0, there are $2^{O((m+n)^2)}$ possible refinements.*

Disregarding the exact location of vertices and edges, $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{S}}$ can be interpreted as combinatorial embeddings of two triangulations. During this paper, sometime we need to work with these combinatorial embeddings. We call them **combinatorial refinements**, and use the same notation $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{S}}$ to refer to them when it is clear from the context.

Let $\widetilde{\mathcal{R}}, \widetilde{\mathcal{S}}$ be combinatorial refinements, and f_0 be a vertex map. We say that $(\widetilde{\mathcal{R}}, \widetilde{\mathcal{S}})$ and f_0 are **consistent** if they give a valid geometric triangulations of every triangle in $\mathcal{R}_T$ and $\mathcal{S}_T$. Specifically, let $t = (a, b, c) \in \mathcal{S}_T$ (a, b, c are in counterclockwise order), $U = g^{-1}(t)$, and $V = f_0(U)$. Let $v \in V \cup \{a, b, c\}$, and let $v'_0, v'_1, \ldots, v'_{k-1}$ be the neighbors of v inside t in counterclockwise order according to $\widetilde{\mathcal{S}}$. We should have:

(1) For any $0 \le i \le k-1$, we have $0 \le \angle v'_i v v'_{i+1} \le \pi$.

(2) We have:

   (a) $\sum_{i=0}^{k-1} \angle v'_i v v'_{i+1} = 2\pi$, if $v \notin \{a, b, c\}$.

   (b) $\sum_{i=0}^{k-1} \angle v'_i a v'_{i+1} = \angle bac$, $\sum_{i=0}^{k-1} \angle v'_i b v'_{i+1} = \angle cba$, and $\sum_{i=0}^{k-1} \angle v'_i c v'_{i+1} = \angle acb$.

Additionally, the same set of conditions must hold for each triangle of $\mathcal{R}$.

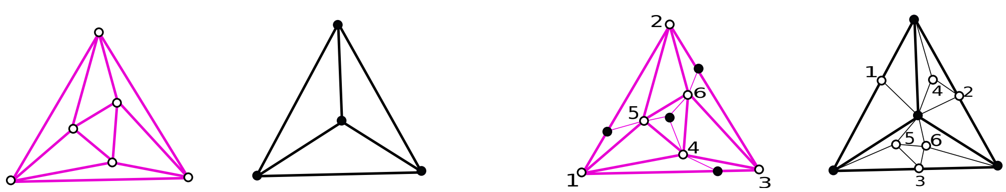

Figure 1: Two surfaces; right: before refinement ($\mathcal{R}$ and $\mathcal{S}$), left: after refinement ($\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{S}}$).

3.1.3 Scaffold map

Let $f_0 : \widetilde{\mathcal{R}}_V \to \widetilde{\mathcal{S}}_V$ be a vertex map. Let $\widetilde{\mathcal{R}} = (\widetilde{\mathcal{R}}_V, \widetilde{\mathcal{R}}_E, \widetilde{\mathcal{R}}_T)$ and $\widetilde{\mathcal{S}} = (\widetilde{\mathcal{S}}_V, \widetilde{\mathcal{S}}_E, \widetilde{\mathcal{S}}_T)$ be refinements of $\mathcal{R}$ and $\mathcal{S}$, respectively, that are consistent with f_0. A **scaffold map** (over refinements $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{S}}$) is a continuous one-to-one map $f_1 : \widetilde{\mathcal{R}}_E \to \widetilde{\mathcal{S}}$ with the following properties.

(1) $f_1(\widetilde{\mathcal{R}}_V) = \widetilde{\mathcal{S}}_V$.

(2) f_1 is a cellular embedding of $(\widetilde{\mathcal{R}}_V, \widetilde{\mathcal{R}}_E)$ on $\widetilde{\mathcal{S}}$.

(3) f_1 maps boundary edges to boundary edges (so, it preserves the cyclic order of boundary edges around boundary components).

(4) f_1 preserves the cyclic order of edges around each vertex: for any $u \in \widetilde{\mathcal{R}}_V$ with neighbors $\{w_1, \ldots, w_k\}$, the cyclic order of the edges $\{(u, w_1), (u, w_2), \ldots, (u, w_k)\}$ around u is identical to the cyclic order of curves $\{f_1(u, w_1), \ldots, f_1(u, w_k)\}$ around $f_1(u)$.

(5) For each $e \in \widetilde{\mathcal{R}}_E$ and each $t \in \widetilde{\mathcal{S}}_T$, $f_1(e) \cap t$ is a collection of straight line segments that intersect $\partial(t)$ at their endpoints.

A scaffold map can be viewed as a collection of maps from the edges in $\widetilde{\mathcal{R}}_E$ to the underlying surface of $\widetilde{\mathcal{S}}$. The **crossing points set** $\widetilde{\mathcal{S}}_X$ is the set of all crossing points $f_1(\widetilde{\mathcal{R}}_E)$ and $\widetilde{\mathcal{S}}_E$. Each element $y \in \widetilde{\mathcal{S}}_X$ is the crossing point of $f_1(e)$ and s for an $e \in \widetilde{\mathcal{R}}_E$ and $s \in \widetilde{\mathcal{S}}_E$; note that two edges may have multiple crossing points. The preimage of y, $x = f_1^{-1}(y)$, is a crossing point between e and $f_1^{-1}(s)$ on $\widetilde{\mathcal{R}}$ that corresponds to y. The preimages of all points in $\widetilde{\mathcal{S}}_X$ is the set of crossing points $\widetilde{\mathcal{R}}_X$ on $\widetilde{\mathcal{R}}$ that is in one-to-one correspondence with $\widetilde{\mathcal{S}}_X$.

For each $s \in \widetilde{\mathcal{S}}_E$, its crossing number $\chi(s)$ is the number of crossing points on s. The **crossing number** of a scaffold map f_1 is the maximum crossing number of the edges of $\widetilde{\mathcal{S}}_E$, denoted

$$\chi(f_1) = \max_{s \in \widetilde{\mathcal{S}}_E} \chi(f_1(s)).$$

The **Fréchet length of the scaffold map** f_1 is the maximum Fréchet length of all its restrictions to edges $e \in \widetilde{\mathcal{R}}_E$, denoted

$$\delta_F(f_1) = \max_{e \in \widetilde{\mathcal{R}}_E} \delta_F(f_{1|e}).$$

The following lemma implied by the authors' previous work [NX16] (Lemmas 7, 8, and 9) reduces the problem of deciding whether the Fréchet distance between $\mathcal{R}$ and $\mathcal{S}$ is at most δ to determining if there is a scaffold map of Fréchet length δ and crossing number $O(2^{m+n})$.

Lemma 4. *For any $\delta \geq 0$, the Fréchet distance between $\mathcal{R}$ and $\mathcal{S}$ is at most δ if and only if there is a scaffold map of Fréchet length at most δ and crossing number at most 2^{m+n}.*

Finally, we show that the Fréchet length of a scaffold map is determined by a pair of crossing points or vertices, hence, checking a finite set of pairs is all we need to determine this length.

Lemma 5. *Let f_1 be a scaffold map over refinements $\widetilde{\mathcal{R}} = (\widetilde{\mathcal{R}}_V, \widetilde{\mathcal{R}}_E, \widetilde{\mathcal{R}}_T)$ and $\widetilde{\mathcal{S}} = (\widetilde{\mathcal{S}}_V, \widetilde{\mathcal{S}}_E, \widetilde{\mathcal{S}}_T)$. There is a scaffold map f_1' with the following properties:*

(1) For each $e \in \widetilde{\mathcal{R}}_E$, we have $f_1(e) = f_1'(e)$, the images are identical.

(2) $\delta_F(f_1') \leq \delta_F(f_1)$.

(3) $\delta_F(f_1') = \max_{x \in \widetilde{\mathcal{R}}_V \cup \widetilde{\mathcal{R}}_X} ||x - f_1(x)||$.

Proof. We define f_1' by defining its restriction $\sigma_e : e \to \mathcal{S}$ on every edge $e \in \widetilde{\mathcal{R}}_E$. For each $e \in \widetilde{\mathcal{R}}_E$, we show that (1) $\sigma_e(e) = f_1(e)$, and (2) $\delta_F(\sigma_e) = \max_{x \in (\widetilde{\mathcal{R}}_V \cup \widetilde{\mathcal{R}}_X) \cap e} ||x - f_1(x)|| \leq f_1(e)$. The statement of lemma will follow from these two properties as $f_1 = \bigcup_{e \in \widetilde{\mathcal{R}}_E} \sigma_e$.

By the definition of a scaffold map, for each $(a, b) = e \in \widetilde{\mathcal{R}}_E$, $f_1(e)$ is a piecewise linear curve whose intersection with each $t \in \widetilde{\mathcal{S}}_T$, $f_1(e) \cap t$ is a finite set of line segments that intersect $\partial(t)$ at crossing points. Let $y_1, y_2, \ldots, y_k$ be the set of these crossing points on $f_1(e)$. Extend this sequence by setting $y_0 = a$ and $y_{k+1} = b$. For each $0 \leq i \leq k+1$, let x_i be the preimage of y_i under f_1, that is $f_1(x_i) = y_i$. We define σ_e as follows. For any $0 \leq i \leq k$ and $p \in e[x_i, x_{i+1}]$, if $p = \lambda \cdot x_i + (1-\lambda) \cdot x_{i+1}$ $(0 \leq \lambda \leq 1)$, then $\sigma_e(p) = \lambda \cdot f_1(x_i) + (1 - \lambda) f_1(x_{i+1})$.

The images $f_1(e[x_i, x_{i+1}])$ and $\sigma_e(e[x_i, x_{i+1}])$ are both line segments between $f_1(x_i)$ and $f_1(x_{i+1})$ for all $0 \leq i \leq k$. Hence, $f_1(e) = \sigma_e(e)$.

For any $0 \leq i \leq k$ and $p \in e[x_i, x_{i+1}]$ we have:

$$\begin{aligned} ||p - \sigma_e(p)|| &= ||\,(\lambda \cdot x_i + (1-\lambda) \cdot x_{i+1}) - (\lambda \cdot f_1(x_i) + (1-\lambda) f_1(x_{i+1}))\,|| \\ &= ||\lambda \cdot (x_i - f_1(x_i)) + (1-\lambda) \cdot (x_{i+1} - f_1(x_{i+1}))\,|| \\ &\leq \max\,(x_i - f_1(x_i), x_{i+1} - f_1(x_{i+1})), \end{aligned}$$

where the last inequality holds as $||\cdot||$ is a norm, so, a convex function. Consequently, we have:

$$\delta_F(\sigma_e) = \max_{0 \leq i \leq k+1} ||x_i - f_1(x_i)|| = \max_{x \in (\widetilde{\mathcal{R}}_V \cup \widetilde{\mathcal{R}}_X) \cap e} ||x - f_1(x)|| \qquad \square$$

3.1.4 Combinatorial Scaffold Maps and Normal Coordinates

Now, we are ready to define combinatorial descriptions of scaffold maps, which our algorithm use to limit its search space to a finite set.

Let f_1 be a scaffold map, and let $e \in \widetilde{\mathcal{R}}_E$. The **crossing sequence** of $f_1(e)$ is the sequence of edges $(s_1, s_2, \ldots, s_k)$ of $\widetilde{\mathcal{S}}_E$ that $f_1(e)$ crosses in order. This crossing sequence gives a combinatorial description of $f_1(e)$, as it determines $f_1(e)$ up to an isotopy in $\widetilde{\mathcal{R}} \backslash \widetilde{\mathcal{R}}_V$. The combinatorial description can be defined by the crossing sequence of all edges in addition to a combinatorial vertex map and combinatorial refinements. In their previous work [NX16], the authors describe how to use normal coordinates to compactly describe the crossing sequence for all edges. Following them, we use normal coordinates to compactly present the crossing sequences of all edges.[1] We remark that to obtain the decidability of the Fréchet distance between two surfaces we do not need this compact representatoin. However, in our second result, that deciding the Fréchet distance between a surface and a triangle is in PSPACE, using this compact representation is crucial.

Normal coordinates. Let $f_1 : \widetilde{\mathcal{R}}_E \to \widetilde{\mathcal{S}}$ be a scaffold map, and let $t = (s_1, s_2, s_3) \in \widetilde{\mathcal{S}}_T$, where $s_1, s_2, s_3 \in \widetilde{\mathcal{S}}_E$. The intersection of $f_1(\widetilde{\mathcal{R}}_E)$ with t is a collection of **elementary segments**: straight line segments with endpoints on $\partial(t)$. The intersection pattern of $f_1(\widetilde{\mathcal{R}}_E) \cap t$ can be presented (up to continuous deformation) with three numbers, one per edge. For each edge $s \in \mathcal{S}_E$ we define its **normal coordinate**, denoted by $N(s)$, as follows: (1) $N(s) = -1$ if $s \in f_1(\widetilde{\mathcal{R}}_E)$, and (2) $N(s)$ is the number of elementary segments intersecting the interior of e, otherwise. See Figure 3 for examples of normal coordinates in triangles.

The **set of normal coordinates** of $f_1(\widetilde{\mathcal{R}}_E)$ is a vector of $|\widetilde{\mathcal{S}}_E| = m + n$ numbers, one per edge in $\widetilde{\mathcal{S}}_E$. Each of these numbers is lower bounded by zero and upper bounded by the crossing number

[1] See references [SSS02, Ste05, EN12] for a detailed exposition of normal curves, the two dimensional variant of standard normal surfaces introduced by Haken [Hak61].

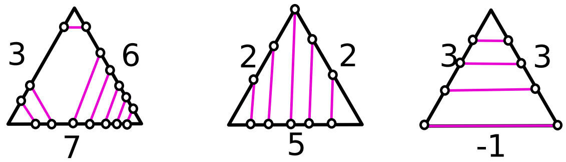

Figure 2: Examples for normal coordinates in triangles.

of f_1, $\chi(f_1)$. Provided the normal coordinates, there is a unique way of locating the elementary segments inside each $t \in \widetilde{\mathcal{S}}_T$ (up to a continuous deformation) so that they do not cross. Hence, the normal coordinates specify, for every $e \in \widetilde{\mathcal{R}}_E$, the crossing sequence of $f_1(e)$.

A **combinatorial scaffold map** is a triple $\langle (g,h), (\widetilde{\mathcal{R}}, \widetilde{\mathcal{S}}), N \rangle$ where (g,h) is a combinatorial vertex map, $(\widetilde{\mathcal{R}}, \widetilde{\mathcal{S}})$ is a combinatorial refinement over (g,h), and N is a set of normal coordinates over $(\widetilde{\mathcal{R}}, \widetilde{\mathcal{S}})$ specifying the crossing sequence for the image of every edge in $\widetilde{\mathcal{R}}_E$ in $\widetilde{\mathcal{S}}$. A combinatorial scaffold map $\langle (g,h), (\widetilde{\mathcal{R}}, \widetilde{\mathcal{S}}), N \rangle$ **extends** a scaffold map f_1 if (i) $f_0 = f_{1|\widetilde{\mathcal{R}}_V \cup f_1^{-1}(\widetilde{\mathcal{S}}_V)}$ is consistent with (g,h), (ii) f_0 is consistent with $(\widetilde{\mathcal{R}}, \widetilde{\mathcal{S}})$, and (iii), for every edge $e \in \widetilde{\mathcal{R}}_E$, the crossing sequence of $f_1(e)$ is consistent with the one implied by the normal coordinates N. The following corollary immediately follows from Lemma 4.

Corollary 1. *For any pair of piecewise linear surfaces, $\mathcal{R}$ and $\mathcal{S}$, and any $\delta \geq \delta_F(\mathcal{R}, \mathcal{S})$, there is a list of combinatorial scaffold maps L of size $2^{O((m+n)^2)}$ that can be computed in $2^{O((m+n)^2)}$ time that has the following properties:*

(1) There is $\langle (g,h), (\widetilde{\mathcal{R}}, \widetilde{\mathcal{S}}), N \rangle \in L$ that extends to a scaffold map of Fréchet length at most δ.

(2) For any $\langle (g,h), (\widetilde{\mathcal{R}}, \widetilde{\mathcal{S}}), N \rangle \in L$, every coordinate of N is at most 2^{m+n}.

3.2 System of Polynomial Inequalties

Recall our decision problem: given two surfaces of genus zero $\mathcal{R}$ and $\mathcal{S}$, and $\delta \geq 0$, we want to decide whether $\delta_F(\mathcal{R}, \mathcal{S}) \leq \delta$.

The algorithm of Corollary 1 builds a list of combinatorial scaffold maps L that is guaranteed to contain one that extends to a scaffold map of Fréchet length at most δ, if and only if $\delta \geq \delta_F(\mathcal{R}, \mathcal{S})$. To decide $\delta_F(\mathcal{R}, \mathcal{S}) \leq \delta$, we check every element of L to see if such an extendable map exists. Therefore, we need an algorithm to decide if a combinatorial scaffold map is extendable. To that end, given $S = \langle (g,h), (\widetilde{\mathcal{R}}, \widetilde{\mathcal{S}}), N \rangle \in L$, we show how to build a system of inequalities that is feasible if and only if S extends to a scaffold map of Fréchet length at most δ.

During the rest of this section, for each triangle t fix an ordering of its vertices, and for each edge e fix an ordering of its endpoints. Hence, we can unambiguously denote a triangle by an ordered triple of vertices, such as (a,b,c), and unambiguously denote an edge by an ordered pair of vertices, such as (a,b).

Vertex Variables. For each vertex $u \in \mathcal{R}_V$, we specify $f_0(u)$ by two variables α_u, and β_u, in addition to $g(u)$. Specifically, if $g(u)$ is a triangle (a,b,c) (with $a,b,c \in \mathcal{S}_V$) then $f_0(u) = a + \alpha_u \overrightarrow{ab} + \beta_u \overrightarrow{ac}$. We can guarantee that $f_0(u) \in g(u)$ by enforcing $\alpha_u \geq 0$, $\beta_u \geq 0$ and $\alpha_u + \beta_u \leq 1$. Additionally, the distance between u and $f_0(u)$ must be at most δ. Hence, for each $u \in \mathcal{R}_V$ and $g(u) = (a,b,c) \in \mathcal{S}_T$, we have the following constraints.

$$\begin{aligned} &\alpha_u, \beta_u \geq 0, \ \alpha_u + \beta_u \leq 1 \\ &||u - f_1(u)|| = ||u - a + \alpha_u \overrightarrow{ab} + \beta_u \overrightarrow{ac}|| \leq \delta \end{aligned} \tag{1}$$

Similarly, for each $v \in \mathcal{S}_V$, we specify $f_0^{-1}(v)$ with two variables α'_v and β'_v, in addition to $h(v)$. By a similar argument, for each $v \in \mathcal{S}_V$ and $h(v) = (a', b', c') \in \mathcal{R}_T$, we have the following constraints.

$$\begin{aligned} &\alpha'_v, \beta'_v \geq 0,\ \alpha'_v + \beta'_v \leq 1 \\ &||v - f_1^{-1}(v)|| = ||v - a' + \alpha'_v \overrightarrow{a'b'} + \beta'_v \overrightarrow{a'c'}|| \leq \delta \end{aligned} \tag{2}$$

In each case, the first three constraints are linear and the last one is of degree p under ℓ_p norm, in particular it is linear for the ℓ_1 norm and quadratic for the ℓ_2 norm.

Crossing Point Variables. Let $e \in \widetilde{\mathcal{R}}_E$, let $s \in \widetilde{\mathcal{S}}_E$, let x be the crossing point between e and $f_1^{-1}(s)$, and let y be the corresponding crossing point between $f_1(e)$ and s. Note that $f_1(x) = y$. Now, let $e = (a_e, b_e)$ and let $s = (a_s, b_s)$. We specify (the location of) x by one variable α_x, that is $x = a_e + \alpha_x \cdot \overrightarrow{a_e b_e}$. We can guarantee that $f_1(x) \in (a, b)$ by enforcing $0 \leq \alpha_x \leq 1$. Similarly, we specify $y = a_s + \alpha_y \cdot \overrightarrow{a_s b_s}$, and enforce $0 \leq \alpha_y \leq 1$. Finally, as $y = f_1(x)$, the x-to-y distance must be at most δ. In summary, for a pair of corresponding crossing points $x \in (a_e, b_e) \in \mathcal{R}_E$, and $y \in (a_s, b_s) \in \mathcal{S}_E$, we obtain the following set of constraints.

$$\begin{aligned} &0 \leq \alpha_x, \alpha_y \leq 1 \\ &||x - y|| = ||(a_e + \alpha_x \cdot \overrightarrow{a_e b_e}) - (a_s + \alpha_y \cdot \overrightarrow{a_s b_s})|| \leq \delta \end{aligned} \tag{3}$$

The first two constraints are linear, and the third constraint is of degree $2p$ for the ℓ_p norm. Note a_e, b_e, a_s, and b_s be described by variables themselves. In particular the last equation is at most quadratic for the ℓ_1 norm and at most quartic for the ℓ_2 norm.

Valid Refinements. We check the consistency of the combinatorial refinements $(\widetilde{\mathcal{R}}, \widetilde{\mathcal{S}})$ and the vertex map f_0. Conditions (1) and (2) of consistency must be verified for each triangle of $\mathcal{S}_T$ and each triangle of $\mathcal{R}_T$. Here we explain how to build a system of inequalities to verify these conditions for a triangle $t \in \mathcal{S}_T$; the other case is symmetric. Let $t = (a, b, c) \in \mathcal{S}_T$ and let V be the set of vertices of $\widetilde{\mathcal{S}}_V$ that are in t. Let $v \in (a, b, c) \cup V$ and let $V' = \{v'_0, v'_2, \ldots, v'_{k-1}\} \subseteq \{a, b, c\} \cup V$ be the set of its neighbors in cyclic (counterclockwise) order according to $\widetilde{\mathcal{S}}$. We restate the required properties for the consistency of the combinatorial refinements are f_0.

(1) For any $0 \leq i \leq k-1$, we have $0 \leq \angle v'_i v v'_{i+1} \leq \pi$.

(2) We have:

   (a) $\sum_{i=0}^{k-1} \angle v'_i v v'_{i+1} = 2\pi$, if $v \notin \{a, b, c\}$.

   (b) $\sum_{i=0}^{k-1} \angle v'_i a v'_{i+1} = \angle bac$, $\sum_{i=0}^{k-1} \angle v'_i b v'_{i+1} = \angle cba$, and $\sum_{i=0}^{k-1} \angle v'_i c v'_{i+1} = \angle acb$.

Condition (1) is equivalent to the following constraint:

$$(v'_i - v) \times (v'_{i+1} - v) \geq 0,$$

where $\times$ is the cross product. Recall, from vertex variables that locations of each vertex in the image of U under f_0 is specified by two variables. For any $0 \leq i \leq k-1$, let

$$v'_i = a + \alpha'_i \cdot \overrightarrow{ab} + \beta'_i \cdot \overrightarrow{ac}.$$

Also, let

$$v = a + \alpha \cdot \overrightarrow{ab} + \beta \cdot \overrightarrow{ac}.$$

Therefore, the cross product mentioned above can be written as follows.

$$(v'_i - v) \times (v'_{i+1} - v) = \big((\alpha'_i - \alpha)(\beta'_{i+1} - \beta) - (\beta'_i - \beta)(\alpha'_{i+1} - \alpha)\big) \cdot (\overrightarrow{ab} \times \overrightarrow{ac})$$

Therefore, if $\overrightarrow{ab} \times \overrightarrow{ac} > 0$, then

$$(v'_i - v) \times (v'_{i+1} - v) \geq 0 \Leftrightarrow (\alpha'_i - \alpha)(\beta'_{i+1} - \beta) - (\beta'_i - \beta)(\alpha'_{i+1} - \alpha) \geq 0 \tag{4}$$

Otherwise,

$$(v'_i - v) \times (v'_{i+1} - v) \geq 0 \Leftrightarrow (\alpha'_i - \alpha)(\beta'_{i+1} - \beta) - (\beta'_i - \beta)(\alpha'_{i+1} - \alpha) \leq 0. \tag{5}$$

In either case, we add a quadratic constraint to our system of constraints. Note, that if any of v, v'_i, or v'_{i+1} correspond to a, b, or c, the constrain becomes simpler because these vertices have fixed locations, so, less variables will be involved in our constraints.

Next, we show that if Condition (1) holds for every vertex of $\{a, b, c\} \cup V$, then Condition (2) must hold for all these vertices. Hence, we do not need to explicitly check Condition(2).

Lemma 6. *If Condition (1) is satisfied, then*

(1) Condition (2-b) is satisfied, and

(2) for any $v \notin \{a, b, c\}$, there is a $k \in \mathbb{N}$ such that $\sum_{i=0}^{k-1} \angle v'_i v v'_{i+1} = 2k\pi$.

Proof. First, we show Condition (2-b) for vertex a; the arguments for vertices b and c are symmetric. Let $v'_0, v'_2, \ldots, v'_{k-1}$ be the set of a's neighbors in counterclockwise order. In particular, we have, $v'_0 = b$ and $v'_{k-1} = c$. Therefore, the set of angles $\{\angle v'_i a v'_{i+1} | 0 \leq i \leq k-2\}$ cover the angle $\angle bac$. Also, for every $0 \leq i \leq k-1$, we have $\angle v'_i a v'_{i+1} \geq 0$ by Condition (1). Therefore, for any $0 \leq i \leq k-3$, $\angle v'_i a v'_{i+1}$ and $\angle v'_{i+1} a v'_{i+2}$ are internally disjoint. Hence, we conclude $\sum_{i=0}^{k-1} \angle v'_i a v'_{i+1} = \angle bac$.

Next, consider an internal $v \notin \{a, b, c\}$. Note that v and V' are in the same plane, and $\angle v'_i v v'_{i+1} \geq 0$ by Condition (1). Therefore, $v'_0, v'_1, \ldots, v'_{k-1}, v'_0$ give a traversal around v that always goes counterclockwise, and starts and ends at the same vertex. We conclude that $\sum_{i=0}^{k-1} \angle v'_i v v'_{i+1}$ is a positive multiple of 2π. □

Lemma 7. *Condition (1) implies Condition (2).*

Proof. By Lemma 6 Condition (1) implies Condition (2-b). It remains to show that condition (1) implies condition (2-a).

To that end, let $\tau = (\tau_V, \tau_E, \tau_T)$ be the combinatorial refinement of t according to $\widetilde{\mathcal{S}}$. Further, let $\widetilde{t}$ be the piecewise linear surface that is obtained by identifying sides of geometric triangles according to τ. Therefore, for each combinatorial triangle $(z_1, z_2, z_3) \in \tau_T$ we have a geometric triangle with side lengths $||\overline{z_1} - \overline{z_2}||$, $||\overline{z_1} - \overline{z_3}||$, and $||\overline{z_2} - \overline{z_3}||$, where $\overline{z}$ is used to denote the location of the combinatorial vertex z determined by f_0 if z is internal, or as part of the input of z is a vertex of t. The (dual of the) combinatorial description τ specifies how to identify the sides of these triangles to obtain $\widetilde{\tau}$.

For the topological disk $\widetilde{\tau}$, a discrete form of the Gauss-Bonnet theorem implies $\sum_{v \in \tau_V} \angle v = 2\pi(|\tau_V| - 3) + \pi$, where $\angle v$ denotes the total angle around v on the surface $\widetilde{\tau}$. We include a short proof based on Euler's formula.

Since $\widetilde{\tau}$ is a topological disk, by the Euler formula we have: $|\tau_V| - |\tau_E| + |\tau_T| = 1$. Additionally, as τ has exactly three boundary edges, we have: $|\tau_E| = \frac{3(|\tau_T|+1)}{2}$. It follows that

$$|\tau_V| - \frac{3(|\tau_T| + 1)}{2} + |\tau_T| = 1 \Rightarrow |\tau_V| - \frac{|\tau_T|}{2} = \frac{5}{2} \Rightarrow 2|\tau_V| - |\tau_T| = 5 \Rightarrow |\tau_T| = 2|\tau_V| - 5,$$

Consequently,

$$\sum_{v \in \tau_V} \angle v = \pi \cdot |\tau_T| = 2\pi|\tau_V| - 5\pi = 2\pi(|\tau_V| - 3) + \pi.$$

As condition (1) holds $\angle a + \angle b + \angle c = \pi$, therefore,

$$\sum_{v \in \tau_V \setminus \{a,b,c\}} \angle v_i = 2\pi(|\tau_V| - 3).$$

But, since condition (1) holds, we have $\angle v_i \geq 2\pi$ for all $v \in \tau_V \setminus \{a,b,c\}$. It follows that, $\angle v_i$ must be exactly 2π, for all $v \in \tau_V \setminus \{a,b,c\}$, and the proof is complete. □

Non-crossing images. Let $e \in \widetilde{\mathcal{R}}_E$, and let $x_1, x_2, \ldots, x_k$ be the set of crossing points on $s = (a,b)$ in order deduced from the normal coordinated N. Also, for each $1 \leq i \leq k$, let $x_i = a + \alpha_i \cdot \overrightarrow{ab}$, as specified above for crossing points. To ensure that the images of the edges of $\widetilde{\mathcal{R}}_E$ under f_1 do not cross, it is sufficient to force that x_i's appear in order on s; that is

$$\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_k. \tag{6}$$

Similarly, for each $e = (c,d) \in \widetilde{\mathcal{S}}_E$, if $y_1, y_2, \ldots, y_\ell$ is the sequence of crossing points on e deduced from N, and $y_i = c + \beta_i \cdot \overline{cd}$ (for each $1 \leq i \leq \ell$), the following condition must hold:

$$\beta_1 \leq \beta_2 \leq \ldots \leq \beta_\ell. \tag{7}$$

All the constraints under this last category are linear.

Now, we are ready to state the main lemma of this part, that given a combinatorial description we can verify if a low-cost scaffold map with that description exists.

Lemma 8. *Let $\mathcal{R}$ and $\mathcal{S}$ be piecewise linear surfaces with m and n vertices, respectively, and let $\delta > 0$. Also, let $S = \langle (g,h), (\widetilde{\mathcal{R}}, \widetilde{\mathcal{S}}), N \rangle$ be a combinatorial scaffold map, such that the value of every coordinates of N is at most 2^{m+n}. In $2^{O(m+n)}$ time, a system of polynomial constraints of size $2^{O(m+n)}$ can be computed that is feasible if and only if S extends to a scaffold map of Fréchet length at most δ.*

Proof. Our algorithm builds variables and constraints for vertices, crossing points, refinements, and non-crossing images in order as detailed above. First, it builds vertex variables and constraints from (g,h) in $O(m+n)$ time (all constraints of type (1) and (2). Next, it expands the normal coordinates to compute the crossing sequence of the image of each $e \in \widetilde{\mathcal{R}}_E$. This crossing sequence identifies all the crossing points on e together with their pairs on $\widetilde{\mathcal{S}}$. As the maximum coordinate is at most 2^{m+n}, the crossing sequence of e's image can be computed by just tracing it based on the normal coordinates in $2^{O(m+n)}$ time. Since, there are $O(m+n)$ edges in $\widetilde{\mathcal{R}}_E$, the crossing sequence of all of them can be computed in $2^{O(m+n)}$ time. After computing all pairs of crossing sequences we introduce $2^{O(m+n)}$ constrains of type (3). Additionally, for each edge we include constraints of type (6) or (7) to ensure the images of edges do not cross. Finally, to ensure that the refinements are consistent with all feasible solutions of our system, we introduce constrains of type (4) or (5) for each vertex and its neighbors.

Any feasible solution of our system can be extended to a scaffold map of Fréchet length at most δ by interpolating the map between consecutive crossing points. On the other hand, by Lemma 5 if a scaffold map f_1 of Fréchet length at most δ that is consistent with S exists, then our system has a feasible solution. □

3.3 Summing up

Now, we are ready to prove the main theorem of this section that is the Fréchet distance between two surfaces is decidable. Our result follows from Corollary 1, Lemma 8, and Lemma 1.

Proof of Theorem 1. By Corollary 1 a list L of $2^{O((m+n)^2)}$ combinatorial scaffold maps can be build in $2^{O((m+n)^2)}$ time so that at least one of them extends to a scaffold map of Fréchet length δ if and only if $\delta_F(\mathcal{R}, \mathcal{S}) \leq \delta$. By Lemma 8, for any $S \in L$ a system M_S of $2^{O(m+n)}$ number of polynomial inequalities can be build in $2^{O(m+n)}$ time that is feasible if and only if S extends to a scaffold map of Fréchet length δ. Finally, by Lemma 1 the feasibility of M_S can be checked in $2^{O(m+n)}$ space. □

4 A Surface and a Triangle

In this section, we show that the special case of deciding the Fréchet distance between a surface and a triangle is in PSPACE. This special case has been studied by Godau [God99], and it is proved to be NP-hard. Even for this special case, no exact algorithm was known that guarantees the correct decision in finite time previous to this paper.

Let $\mathcal{R} = (\mathcal{R}_V, \mathcal{R}_E, \mathcal{R}_T)$ be a piecewise linear surface, and let $\mathcal{S} = (\mathcal{S}_V, \mathcal{S}_E, \mathcal{S}_T)$ be a triangle both immersed into $\mathbb{R}^3$ (by immersions $\varphi_\mathcal{R} : \mathcal{R} \to \mathbb{R}^3$ and $\varphi_\mathcal{S} : \mathcal{S} \to \mathbb{R}^3$). In particular, $|\mathcal{S}_V| = |\mathcal{S}_E| = 3$, and $|\mathcal{S}_T| = 1$. Also, let $m = |\mathcal{R}_V|$, and let $\delta \geq 0$. We describe a PSPACE algorithm to decide whether $\delta_F(\mathcal{R}, \mathcal{S}) \leq \delta$.

4.1 Tight Images

We introduce tight scaffold maps and detailed normal coordinates. We show that enumerating over tight scaffold maps is sufficient when deciding the Fréchet distance between a surface and a triangle. We use detailed normal coordinates to enumerate combinatorial descriptions of tight scaffold maps.

4.1.1 Tight Edge Images

We show that for any $\delta \leq \delta_F(\mathcal{R}, \mathcal{S})$ there is a scaffold map f_1 of Fréchet length at most δ that maps each edge $e \in \mathcal{R}_E$ into a homotopic shortest path in $\mathcal{S} \backslash f_1(\mathcal{R}_V)$. This property enables us to reduce the number of constraints for crossing points to a polynomial, hence, facilitates our PSPACE result. We discuss two auxiliary lemmas before stating our main lemma.

The following lemma is implicit in the work of Colin de Verdière and Erickson [dVE10], and it is follows from Hass and Scott [HS85].

Lemma 9 (Colin de Verdière and Erickson [dVE10], Hass and Scott [HS85])**.** *Let γ_1 and γ_2 be two non-crossing paths on a surface of genus zero with boundary components, and let γ_1', and γ_2' be the shortest homotopic paths in the homotopy classes of γ_1 and γ_2, respectively. If γ_1 does not cross γ_2 then γ_1' does not cross γ_2'.*

Buchin et al. [BBW08] observe that shortcutting a curve along a line segment cannot increase its Fréchet distance to a line segment. Hass and Scott [HS85] show if a curve γ on a surface of genus zero with boundary components is not the shortest path in its homotopy class, then there is an empty bigon whose one side is a subpath of γ and the other side is a global shortest path. Exploiting this property, Nayyeri and Sidiropoulos [NS15] show that each curve in a planar domain can be modified to its homotopic shortest path via a finite sequence of shortcuttings along line segments. Taking the observation of Buchin et al. into account they conclude the following lemma.

Lemma 10 (Nayyeri and Sidiropoulos [NS15], Corollary 3.8)**.** *Let $t \subseteq \mathbb{R}^3$ be a triangle with point punctures, let $\gamma \in t$ be a path, let γ' be the shortest path homotopic to γ, and let $s \in \mathbb{R}^3$ be a line segment. We have, $\delta_F(\gamma, s) \leq \delta_F(\gamma', s)$.*

Now, we are ready to prove the following lemma.

Lemma 11. *Let $\mathcal{R}$ be a piecewise linear surface and let $\mathcal{S}$ be a triangle. Let f_0 be a vertex map between them, and let $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{S}}$ be refinements that are consistent with f_0. Finally, let f_1 be a scaffold map over $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{S}}$. There exists a scaffold map f_1' over $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{S}}$ with the following properties:*

(1) The maps f_1 and f_1' have the same set of normal coordinates.

(2) For any $e \in \mathcal{R}_E$, $f_1'(e)$ is the shortest homotopic path in $\mathcal{S} \backslash \widetilde{\mathcal{S}}_V$.

(3) $\delta_F(f_1') \leq \delta_F(f_1)$.

Proof. We obtain f_1' by iteratively modifying f_1 as follows. For each $e \in \mathcal{R}_E$, we replace $\gamma = f_1(e)$ with the shortest path γ' in γ's homotopy class in $\mathcal{S} \backslash \widetilde{\mathcal{S}}_V$. We modify f_1 such that (i) $f_1(e) = \gamma'$, and (ii) $\delta_F(f_{1|e}) = \delta_F(e, \gamma')$.

Lemma 10 implies for every e, $\delta_F(f'_{1|e}) \leq \delta_F(f_{1|e})$, hence, $\delta_F(f_1') \leq \delta_F(f_1)$. Lemma 9 implies that for each $e, e' \in \mathcal{R}_E$ their images are still non-crossing. In particular, for each vertex $u \in \mathcal{R}_V$ the cyclic order of edges around $f_1(u)$ and $f_1'(u)$ are the same. Hence, f_1' is a valid scaffold map. Therefore, properties (2) and (3) hold.

Additionally, for each $e \in \mathcal{R}_E$, and $\gamma = f_1(e)$ and γ', its homotopic shortest path, γ and γ' are homotopic. Moreover, both γ and γ' are normal. Therefore, they must have the same set of normal coordinates. □

4.1.2 Tight Scaffold Maps

Let f_0 be a vertex map, $\widetilde{\mathcal{R}} = (\widetilde{\mathcal{R}}_V, \widetilde{\mathcal{R}}_E, \widetilde{\mathcal{R}}_T)$ and $\widetilde{\mathcal{S}} = (\widetilde{\mathcal{S}}_V, \widetilde{\mathcal{S}}_E, \widetilde{\mathcal{S}}_T)$ be refinements consistent with f_0, and f_1 an scaffold map consistent with f_0, $\widetilde{\mathcal{R}}$, and $\widetilde{\mathcal{S}}$. We say that f_1 is **tight** over $(\widetilde{\mathcal{R}}, \widetilde{\mathcal{S}})$ if it has the following two properties.

(1) For each $e \in \widetilde{\mathcal{R}}_E$, $f_1(e)$ is the shortest homotopic path in $\mathcal{S} \backslash \mathcal{S}_V$.

(2) For each $e \in \widetilde{\mathcal{R}}_E$, $f_1(e)$ is composed of edges of $\widetilde{\mathcal{S}}_E$.

Lemma 12. *For any $\delta \geq 0$, the Fréchet distance between $\mathcal{R}$ and $\mathcal{S}$ is at most δ if and only if there are refinements $\widetilde{\mathcal{R}} = (\widetilde{\mathcal{R}}_V, \widetilde{\mathcal{R}}_E, \widetilde{\mathcal{R}}_T)$ and $\widetilde{\mathcal{S}} = (\widetilde{\mathcal{S}}_V, \widetilde{\mathcal{S}}_E, \widetilde{\mathcal{S}}_T)$ and a tight scaffold map of Fréchet length at most δ over $(\widetilde{\mathcal{R}}, \widetilde{\mathcal{S}})$ that has crossing number at most $2m2^m$.*

Proof. By Lemma 11, there are refinements $\widetilde{\mathcal{R}}' = (\widetilde{\mathcal{R}}'_V, \widetilde{\mathcal{R}}'_E, \widetilde{\mathcal{R}}'_T)$ and $\widetilde{\mathcal{S}}' = (\widetilde{\mathcal{S}}'_V, \widetilde{\mathcal{S}}'_E, \widetilde{\mathcal{S}}'_T)$, and a scaffold map f_1 such that (1) for each $e \in \widetilde{\mathcal{R}}'_E$, $f_1(e)$ is a homotopic shortest path, (2) the normal coordinates of f_1 are bounded by 2^m, and (3) $\delta_F(f_1) \leq \delta$. Therefore, Condition (1) of tight scaffold maps already holds. We modify $\widetilde{\mathcal{S}}'$ to satisfy Condition (2).

Since Condition (1) holds, for each $e \in \widetilde{\mathcal{R}}'_E$, $f_1(e)$ is a sequence of segments $(f_1(x), f_1(x'))$, where $x, x' \in \widetilde{\mathcal{R}}'_V$. Let T be the set of all segments $s = (f_1(x), f_1(x'))$ such that $s \in f_1(e)$ for at least one $e \in \widetilde{\mathcal{R}}'_E$. Since, the images of edges of $\widetilde{\mathcal{R}}'_E$ are non-crossing, T is a noncrossing set of segments over $\mathcal{S}'_V$. Complete it to a triangulation $\widetilde{\mathcal{S}} = (\widetilde{\mathcal{S}}'_V, \widetilde{\mathcal{S}}_E, \widetilde{\mathcal{S}}_T)$ of $\mathcal{S}$ by adding more segments.

By the construction of $\widetilde{\mathcal{S}}$, Condition (2) holds for f_1 and $\widetilde{\mathcal{S}}_E$. It remains to bound the crossing number of each segment in $\widetilde{\mathcal{S}}_E$.

Let $s \in \widetilde{\mathcal{S}}_E$, and let n_s be the number of times (with multiplicity) that the images of edges of $\widetilde{\mathcal{R}}'_E$ use s. We show that $n_s \leq 2^m$. If $s \in \widetilde{\mathcal{S}}'_E$ then the statement follows from the bound on the crossing number according to $\widetilde{\mathcal{S}}'$. Otherwise, s crosses at least one edge $\ell \in \widetilde{\mathcal{S}}'_E$, therefore, $n_S \leq \chi_{\widetilde{\mathcal{S}}'}(\ell) \leq 2^m$, as any traversal of s crosses ℓ.

Now, let $s' = (y, y') \in \widetilde{\mathcal{S}}_E$, and note that any subpath that crosses s' must use an edge that is adjacent to y or y'. But, there are at most $2m$ such edges, and each one is used at most 2^m times by the images of $\widetilde{\mathcal{R}}'_E$ as proved above. □

4.1.3 Detailed Normal Coordinates

Because of Lemma 12, we can assume that crossings between images of $\widetilde{\mathcal{R}}_E$ and any edge $s \in \widetilde{\mathcal{S}}_E$ only happen at endpoints of s, we call these endpoints **portals**. For each $e \in \widetilde{\mathcal{R}}_E$, in addition to its edge crossing sequence, we define its **portal crossing sequence**, that is the sequence of portals in order that $f_1(e)$ crosses. We refine the normal coordinates to include two numbers for each edge $s \in \widetilde{\mathcal{S}}_E$, $N_1(s)$ and $N_2(s)$: the number of crossings in each endpoint. If $N(s) = -1$ then we set $N_1(s) = N_2(s) = -1$. Otherwise, for each edge $(a, b) \in \widetilde{\mathcal{S}}_E$, $N_1(s)$ and $N_2(s)$ are the number of crossings of s at a and b, respectively. The set of **detailed normal coordinates** of $f_1(\widetilde{\mathcal{R}}_E)$ is composed of two vectors N_1 and N_2 each with $|\widetilde{\mathcal{S}}_E|$ numbers, one per edge in $\widetilde{\mathcal{S}}_E$. Each of these numbers is lower bounded by zero and upper bounded by the crossing number of f_1, $\chi(f_1)$. Provided the normal coordinates, there is a unique way of locating the elementary segments inside each $t \in \widetilde{\mathcal{S}}_T$. Note, that many of these segments may overlap, but no pair of them cross.

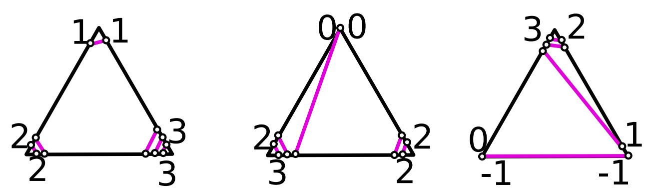

Figure 3: Detailed normal coordinates; note in reality the segments intersect each edge only in its endpoints; the figures are slightly modified for demonstration.

A **combinatorial scaffold map with detailed coordinates** is a triple $\langle (g, h), (\widetilde{\mathcal{R}}, \widetilde{\mathcal{S}}), (N_1, N_2) \rangle$ where (g, h) is a combinatorial vertex map, $(\widetilde{\mathcal{R}}, \widetilde{\mathcal{S}})$ is a combinatorial refinement over (g, h), and (N_1, N_2) is a set of detailed normal coordinates over $(\widetilde{\mathcal{R}}, \widetilde{\mathcal{S}})$ specifying the crossing sequence of edges and portals for the image of every edge in $\widetilde{\mathcal{R}}_E$. A combinatorial scaffold map $\langle (g, h), (\widetilde{\mathcal{R}}, \widetilde{\mathcal{S}}), (N_1, N_2) \rangle$ **extends** a scaffold map f_1 if (i) $f_0 = f_{1|\widetilde{\mathcal{R}}_V \cup f_1^{-1}(\widetilde{\mathcal{S}}_V)}$ is consistent with (g, h), (ii) f_0 is consistent with $(\widetilde{\mathcal{R}}, \widetilde{\mathcal{S}})$, and (iii), for every edge $e \in \widetilde{\mathcal{R}}_E$, the portal crossing sequence of $f_1(e)$ is the same as the one implied by the detailed normal coordinates (N_1, N_2). The following corollary immediately follows from Lemma 12.

Corollary 2. *Let $\mathcal{R}$ be a piecewise linear surface with m vertices, $\mathcal{S}$ a triangle, and and $\delta \geq \delta_F(\mathcal{R}, \mathcal{S})$. There is a list of combinatorial scaffold maps (with detailed normal coordinates) L of size $2^{O(m^2)}$ that can be computed in $2^{O(m^2)}$ time and be enumerated in $O(m^2)$ space, and that has the following properties:*

(1) There is $\langle (g, h), (\widetilde{\mathcal{R}}, \widetilde{\mathcal{S}}), (N_1, N_2) \rangle \in L$ that extends to a tight scaffold map of Fréchet length at most δ.

(2) For any $\langle (g,h), (\widetilde{\mathcal{R}}, \widetilde{\mathcal{S}}), (N_1, N_2) \rangle \in L$, every coordinate of N_1 or N_2 is at most $2m2^m$.

4.2 A System of Polynomial Size

If f_1 is a tight scaffold map over $(\widetilde{\mathcal{R}}, \widetilde{\mathcal{S}})$ then the crossing points on $\mathcal{S}$ co-locate with vertices $\widetilde{\mathcal{S}}_V$. The following lemma uses this property to reduce the number of required constraints in our systems.

Lemma 13. *Let f_1 be a tight scaffold map over refinements $\widetilde{\mathcal{R}} = (\widetilde{\mathcal{R}}_V, \widetilde{\mathcal{R}}_E, \widetilde{\mathcal{R}}_T)$ and $\widetilde{\mathcal{S}} = (\widetilde{\mathcal{S}}_V, \widetilde{\mathcal{S}}_E, \widetilde{\mathcal{S}}_T)$. Let $e \in \widetilde{\mathcal{R}}_E$, $s = (p_1, p_2) \in \widetilde{\mathcal{S}}_E$. Finally, let $x_1, \ldots, x_k$ be the set of all corresponding points on e in order such that $f_1(x_1) = \ldots = f_1(x_k) = p_1$. For any $1 \le i \le k$, we have $||x_i - f_1(x_i)|| \le \max(||x_1 - f_1(x_1)||, ||x_k - f_1(x_k)||)$.*

Proof. Let $x_i = (1-\lambda)x_1 + \lambda x_k$ for a $0 \le \lambda \le 1$. We have:

$$\begin{aligned} ||x_i - f_1(x_i)|| &= ||(1-\lambda)x_1 + \lambda x_k - p_1|| \\ &= ||(1-\lambda)(x_1 - p_1) + \lambda(x_k - p_1)|| \\ &\le \max(||x_1 - p_1||, ||x_k - p_1||) \\ &= \max(||x_1 - f_1(x_1)||, ||x_k - f_1(x_k)||) \end{aligned}$$

The inequality follows from the convexity of the norm function. □

Lemma 13 implies that we can disregard all constraints of type (3) except $O(m^2)$ of them: two per each choice of $e \in \widetilde{\mathcal{R}}_E$ and a portal in $\widetilde{\mathcal{S}}$. Now, we are ready to show how to build our system of inequalities in polynomial space.

Lemma 14. *Let $\mathcal{R}$ be a piecewise linear surface with m vertices, $\mathcal{S}$ a triangle, and and $\delta \ge \delta_F(\mathcal{R}, \mathcal{S})$. Also, let $S = \langle (g,h), (\widetilde{\mathcal{R}}, \widetilde{\mathcal{S}}), (N_1, N_2) \rangle$ be a combinatorial scaffold map (with detailed normal coordinates), such that the value of every coordinates of N_1 and N_2 is $2^{O(m)}$. In $2^{O(m)}$ time, a system of polynomial constraints of size $O(m^2)$ can be computed that is feasible if and only if S extends to a tight scaffold map of Fréchet length at most δ.*

Proof. Our algorithm builds variables and constraints similar to the algorithm of Lemma 8. The number of variables and constraints for vertices, and refinements are polynomial (all constraints of type (1), (2), (4), and (5)). We show that in the new case that $\mathcal{S}$ is a triangle we can reduce the number of variables and constraints for crossing points into $O(m^2)$. Each crossing point of $\widetilde{\mathcal{S}}_X$ is co-located with a vertex, therefore, given the final location of vertices and the detailed normal coordinates the location of all crossing points are uniquely determined. Hence, all constraints of type (6) and (7) can be disregarded. Finally, by Lemma 13, only $O(m^2)$ constraints of type (3) can represent all constraints of this type; all others are redundant. □

4.3 Summing up

Now, we are ready to prove the main theorem of this section that deciding the Fréchet distance between a surface and a triangle is in PSAPCE. Our result follows from Corollary 2, Lemma 14, and Lemma 1.

Proof of Theorem 2. By Corollary 2, there is a PSPACE algorithm that enumerates a sequence of $2^{O(m^2)}$ combinatorial scaffold maps with detailed normal that at least one of them extends to a scaffold map of Fréchet length at most δ if and only if $\delta_F(\mathcal{R}, \mathcal{S}) \le \delta$. By Lemma 14, for any combinatorial scaffold map S (with detailed normal coordinates) a system of a polynomial number of inequalities can be build in PSPACE time that is feasible if and only if S extends to a scaffold map of Fréchet length at most δ. Finally, Lemma 1 ensures that the feasibility of S can be checked in PSPACE. □

References

[AB09] Helmut Alt and Maike Buchin. Can we compute the similarity between surfaces? *Discrete Comput. Geom.*, 43(1):78–99, December 2009.

[AG95] Helmut Alt and Michael Godau. Computing the Fréchet distance between two polygonal curves. *Int. J. Comput. Geometry Appl.*, 5:75–91, 1995.

[AHPK+06] Boris Aronov, Sariel Har-Peled, Christian Knauer, Yusu Wang, and Carola Wenk. Fréchet distance for curves, revisited. In *Proceedings of the 14th Conference on Annual European Symposium - Volume 14*, ESA'06, pages 52–63, London, UK, UK, 2006. Springer-Verlag.

[BBS10] Kevin Buchin, Maike Buchin, and André Schulz. Fréchet distance of surfaces: Some simple hard cases. In *Proceedings of the 18th Annual European Conference on Algorithms: Part II*, ESA'10, pages 63–74, Berlin, Heidelberg, 2010. Springer-Verlag.

[BBW08] Kevin Buchin, Maike Buchin, and Carola Wenk. Computing the Fréchet distance between simple polygons. *Comput. Geom. Theory Appl.*, 41(1-2):2–20, October 2008.

[BOS15] Kevin Buchin, Tim Ophelders, and Bettina Speckmann. *Computing the Similarity Between Moving Curves*, pages 928–940. Springer Berlin Heidelberg, Berlin, Heidelberg, 2015.

[BOS17] Kevin Buchin, Tim Ophelders, and Bettina Speckmann. Computing the Fréchet distance between real-valued surfaces. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '17, pages 2443–2455, Philadelphia, PA, USA, 2017. Society for Industrial and Applied Mathematics.

[Can88] John Canny. Some algebraic and geometric computations in PSPACE. In *Proceedings of the Twentieth Annual ACM Symposium on Theory of Computing*, STOC '88, pages 460–467, New York, NY, USA, 1988. ACM.

[CDHP+11] Atlas F. Cook, IV, Anne Driemel, Sariel Har-Peled, Jessica Sherette, and Carola Wenk. Computing the Fréchet distance between folded polygons. In *Proceedings of the 12th International Conference on Algorithms and Data Structures*, WADS'11, pages 267–278, Berlin, Heidelberg, 2011. Springer-Verlag.

[DHW12] Anne Driemel, Sariel Har-Peled, and Carola Wenk. Approximating the Fréchet distance for realistic curves in near linear time. 48:94–127, 2012.

[dVE10] Éric Colin de Verdière and Jeff Erickson. Tightening nonsimple paths and cycles on surfaces. *SIAM J. Comput.*, 39(8):3784–3813, December 2010.

[EM94] Thomas Eiter and Heikki Mannila. Computing discrete Fréchet distance. Tech. Report CD-TR 94/64, Christian Doppler Lab. Expert Sys., TU Vienna, Austria, 1994.

[EN12] Jeff Erickson and Amir Nayyeri. Tracing compressed curves in triangulated surfaces. In *Proceedings of the Twenty-eighth Annual Symposium on Computational Geometry*, SoCG '12, pages 131–140, New York, NY, USA, 2012. ACM.

[God99] Michael Godau. *On the Complexity of Measuring the Similarity Between Geometric Objects in Higher Dimensions.* PhD thesis, Free University of Berlin, 1999.

[Hak61] Wolfgang Haken. Theorie der Normalflächen: Ein Isotopiekriterium für den Kreisknoten. *Acta Mathematica*, 105:245–375, 1961.

[HS85] Joel Hass and Peter Scott. Intersections of curves on surfaces. *Israel Journal of Mathematics*, 51(1-2):90–120, 1985.

[NS15] Amir Nayyeri and Anastasios Sidiropoulos. Computing the fréchet distance between polygons with holes. In *Automata, Languages, and Programming - 42nd International Colloquium, ICALP 2015, Kyoto, Japan, July 6-10, 2015, Proceedings, Part I*, pages 997–1009, 2015.

[NX16] Amir Nayyeri and Hanzhong Xu. On computing the Fréchet distance between surfaces. In *32nd International Symposium on Computational Geometry, SoCG 2016, June 14-18, 2016, Boston, MA, USA*, pages 55:1–55:15, 2016.

[SSS02] Marcus Schaefer, Eric Sedgwick, and Daniel Stefankovic. Algorithms for normal curves and surfaces. In *Proceedings if the 8th Annual International Computing and Combinatorics Conference*, pages 370–380, 2002.

[Ste05] Daniel Stefankovic. *Algorithms for Simple Curves on Surfaces, String Graphs, and Crossing Numbers*. PhD thesis, Chicago, IL, USA, 2005. AAI3168396.